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The degenerate four-wave mixer: SU(1,1) symmetry and photon statistics

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Abstract. The degenerate four-wave mixer is a nonlinear optical device capable of being configured to provide an extraordinarily broad range of photon statistics. We examine these statistics at its output in terms of the matrix representation of the process, which belongs to the SU(1,1) group of second-order unimodular matrices. The connection between this group and that of proper Lorentz transformations in two space dimensions and one time dimension permits the field density operators at the input and output ports of the device to be related by means of unitary transformations. This, in turn, provides the joint output photon number distribution for any joint input state. The formalism also applies to the degenerate parametric amplifier when the Lorentz boost parameter is interpreted with different nonlinear interaction constants. When both of the input states are vacuums the marginal output photon number distribution is Bose–Einstein, in accordance with the well known result for the squeezed vacuum. When one of the inputs is a photon number state and the other is the vacuum, the marginal output photon number is described by the negative-binomial distribution, indicating that each input photon behaves as a classical particle in a pure birth amplification process. If both inputs are number states, the marginal output photon number distributions can be expressed in terms of the hypergeometric functions and calculated via a three-term recursion relation. These distributions are photon number squeezed, or sub-Poissonian, when the nonlinear interaction is sufficiently weak. Quadrature-squeezed light can be generated for arbitrary input states by using a suitable combination of the output beams, when the interaction is sufficiently strong.

1. Introduction

The degenerate four-wave mixer has been used for the generation of non-classical light, in particular squeezed states [1–5]. The usual theoretical treatments for the photon statistics of such devices rely on the assumption that the input states are coherent. This is, indeed, the configuration in which experiments are usually conducted. We determine the photon statistics at the output of the device for arbitrary states at the input, treating the pump waves classically and the two input waves quantum mechanically. Following Yurke *et al* [6] we find it useful to analyse the device

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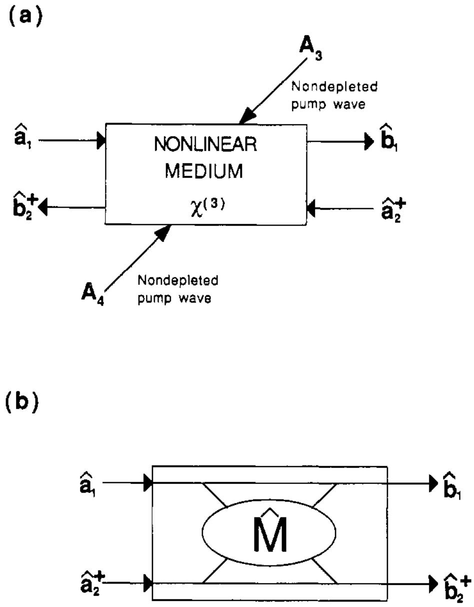

Figure 1. (a) Possible four-wave mixer geometry and (b) its diagrammatic representation as an optical element with two input ports and two output ports. The transformation is governed by a unitary operator $\hat{M}$.

in terms of group theory. The analysis is applicable for any device that has the same group symmetry, such as the degenerate parametric amplifier.

In section 2 we consider the connection between the SU(1,1) field transformation matrices obeyed by this device and the Lorentz group in three dimensions. In section 3 we derive the output photon number probability distributions for a general joint input state with particular emphasis on the case of number states at the input. In section 4 we consider the superposed outputs for any input state and show that they can be quadrature squeezed under certain conditions.

2. Four-wave mixer matrix transformation

To make the connection between the degenerate four-wave mixer and the SU(1,1) group we first derive the general matrix representations of the device. We take all four fields of the four-wave mixer to have the same frequency ω and the same polarization. We assume that the two pump waves are non-depleted such that their field amplitudes A_3 and A_4 may be treated as c-numbers. Thus, the four-wave mixer can be represented as a device with two input ports and two output ports (figure 1). The quantities $\hat{a}_1$, $\hat{a}_2$ and $\hat{b}_1$, $\hat{b}_2$ represent the boson annihilation operators of the two quantized input and output fields, respectively. These operators and their conjugates, the boson creation operators, satisfy the commutation relations

$$[\hat{a}_1, \hat{a}_j^\dagger] = \delta_{ij} \qquad i, j = 1, 2 \tag{1}$$

where δ_{ij} is the Kronecker delta function.

The output operators are related to the input operators by [6, 7]

$$\begin{pmatrix} \hat{b}_1 \\ \hat{b}_2^{\dagger} \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2^{\dagger} \end{pmatrix}. \tag{2}$$

The transformation matrix **M** has elements M_{ij} that are in general complex, that is

$$M_{ij} = |M_{ij}| \exp(\mathrm{i}\varphi_{ij}). \tag{3}$$

Since the output operators must also satisfy the commutation relations of equation (1), we are led to the following conditions:

$$|M_{11}|^2 - |M_{12}|^2 = 1 \tag{4a}$$

$$|M_{22}|^2 - |M_{21}|^2 = 1 \tag{4b}$$

$$M_{11}M_{21}^* - M_{12}M_{22}^* = 0. \tag{4c}$$

Equation (4c) can be decomposed into separate conditions on the magnitude and phase:

$$|M_{11}||M_{21}| = |M_{12}||M_{22}| \tag{4d}$$

$$\varphi_{11} - \varphi_{21} = \varphi_{12} - \varphi_{22}. \tag{4e}$$

Combining equation (4*d*) with equations (4*a*) and (4*b*) leads to

$$\begin{aligned} |M_{11}|^2 = |M_{22}|^2 &\equiv \cosh^2(\beta/2) \\ |M_{12}|^2 = |M_{21}|^2 &= \sinh^2(\beta/2). \end{aligned} \tag{5}$$

Thus all the magnitudes are governed by the single parameter β, which can have any value greater than or equal to zero. This parameter is related to the non-depleted pump field amplitudes A_3 and A_4 by [7]

$$\cosh(\beta/2) = \sec\left(\frac{2\pi\omega}{nc}\chi^{(3)}A_3A_4L\right) \tag{6a}$$

where $\chi^{(3)}$ is the third-order nonlinear susceptibity of the medium, n is its index of refraction, L is its length and c is the speed of light in vacuum.

This formalism also applies to the degenerate parametric amplifier, which involves the mixing of three waves in a crystal with a second-order nonlinear susceptibility $\chi^{(2)}$. With one of the waves assumed to have a constant field amplitude A, the parametric amplifier can be regarded as a device with two input fields and two output fields (see figure 1(*b*)). It is then formally equivalent to the degenerate four-wave mixer with β given by

$$\cosh\left(\frac{\beta}{2}\right) = \sec\left(\frac{2\pi\omega}{nc}\chi^{(2)}AL\right). \qquad (6b)$$

Combining equation (4*e*) with the convenient phase redefinitions

$$\begin{aligned} \varphi_0 &\equiv \tfrac{1}{2}(\varphi_{11} + \varphi_{22}) \\ \varphi_1 &\equiv \tfrac{1}{2}(\varphi_{11} - \varphi_{22}) \\ \varphi_2 &\equiv \tfrac{1}{2}(\varphi_{12} - \varphi_{21}) \end{aligned} \qquad (7)$$

allows us to write the most general four-wave mixer transformation matrix as

$$\mathbf{M} = \exp(\mathrm{i}\varphi_0)\begin{pmatrix} \cosh(\beta/2)\exp(\mathrm{i}\varphi_1) & \sinh(\beta/2)\exp(\mathrm{i}\varphi_2) \\ \sinh(\beta/2)\exp(-\mathrm{i}\varphi_2) & \cosh(\beta/2)\exp(-\mathrm{i}\varphi_1) \end{pmatrix}. \qquad (8)$$

The determinant of $\mathbf{M}$ is $\exp(\mathrm{i}2\varphi_0)$. $\mathbf{M}$ has the general form of a matrix belonging to the U(1,1) group of transformations, which leave the Hermitian form

$$V^{\dagger}\sigma_3 V = V_1^{\dagger}V_1 - V_2^{\dagger}V_2 \qquad (9)$$

invariant, where V is a column vector with elements V_1 and V_2, and σ_3 is the Pauli matrix

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (10)$$

Setting $V_1 = \hat{a}_1$ and $V_2 = \hat{a}_2^{\dagger}$ and using the commutations relations (1) we find that the conserved quantity is the difference in photon number,

$$\hat{N}_1 - \hat{N}_2 = \hat{n}_1 - \hat{n}_2 \qquad (11)$$

where $\hat{n}_1$, $\hat{n}_2$, and $\hat{N}_1$, $\hat{N}_2$, are the photon number operators at the input and output ports, respectively, defined as

$$\hat{n}_j = \hat{a}_j^{\dagger}\hat{a}_j \qquad \hat{N}_j = \hat{b}_j^{+}\hat{b}_j \qquad j = 1, 2. \qquad (12)$$

If we impose the additional condition that the determinant of $\mathbf{M}$ be unity by eliminating the global phase factor φ_0,

$$\varphi_0 = 0 \qquad (13)$$

then we restrict the transformation to the subgroup SU(1,1). Now the elements of $\mathbf{M}$ involve only three independent quantities.

The group SU(1,1) is isomorphic to the group SO(2,1) of proper Lorentz transformations in two space dimensions and one time dimension. This implies that the complex two-dimensional matrices $\mathbf{M}$ correspond to real matrices in three dimensions, the transformations of which may be more readily visualized in terms of rotations and Lorentz boosts [8].

The four-wave mixer may also be characterized by a unitary operator $\hat{M}$, which is related to the transformation matrix $\mathbf{M}$ by

$$\begin{pmatrix} \hat{b}_1 \\ \hat{b}_2^\dagger \end{pmatrix} = \hat{M} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2^\dagger \end{pmatrix} \hat{M}^\dagger = \mathbf{M} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2^\dagger \end{pmatrix}. \tag{14}$$

This operator may be written in the general form

$$\hat{M}(\alpha, \beta, \gamma) = \exp(-\mathrm{i}\alpha \hat{K}_i) \exp(-\mathrm{i}\beta \hat{K}_j) \exp(-\mathrm{i}\gamma \hat{K}_k) \qquad \begin{matrix} i, j, k = 1, 2, 3 \\ i \neq j, j \neq k \end{matrix} \tag{15}$$

where the operators $\hat{K}_{1,2,3}$ are the generators of the group SO(2,1) [9, 10]. They satisfy the commutator algebra

$$[\hat{K}_1, \hat{K}_2] = -\mathrm{i}\hat{K}_3 \qquad [\hat{K}_2, \hat{K}_3] = \mathrm{i}\hat{K}_1 \qquad [\hat{K}_3, \hat{K}_1] = \mathrm{i}\hat{K}_2 \tag{16}$$

and may be written in terms of the field annihilation and creation operators as

$$\begin{aligned} \hat{K}_1 &= \tfrac{1}{2}(\hat{a}_1^\dagger \hat{a}_2^\dagger + \hat{a}_1 \hat{a}_2) \\ \hat{K}_2 &= -\tfrac{1}{2}\mathrm{i}(\hat{a}_1^\dagger \hat{a}_2^\dagger - \hat{a}_1 \hat{a}_2) \\ \hat{K}_3 &= \tfrac{1}{2}(\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2 \hat{a}_2^\dagger). \end{aligned} \tag{17}$$

The standard form of equation (15) is

$$\hat{M}(\alpha, \beta, \gamma) = \exp(-\mathrm{i}\alpha \hat{K}_3) \exp(-\mathrm{i}\beta \hat{K}_2) \exp(-\mathrm{i}\gamma \hat{K}_3). \tag{18}$$

The parameters α and γ are group constants in the range $(0, 2\pi)$ and are similar to the Euler angles of the rotation group, while β is a Lorentz boost parameter in the range $(0, \infty)$. The null boost ($\beta = 0$) is not of interest in this paper.

The exponentiated $\hat{K}_3$ operator performs a rotation about the 3-axis:

$$\exp(-\mathrm{i}\gamma \hat{K}_3) \begin{pmatrix} \hat{K}_1 \\ \hat{K}_2 \\ \hat{K}_3 \end{pmatrix} \exp(\mathrm{i}\gamma \hat{K}_3) = \begin{pmatrix} \cos(\gamma) & \sin(\gamma) & 0 \\ -\sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{K}_1 \\ \hat{K}_2 \\ \hat{K}_3 \end{pmatrix} \tag{19}$$

while the exponentiated $\hat{K}_2$ operator performs a Lorentz boost along the 1-axis:

$$\exp(-\mathrm{i}\beta \hat{K}_2) \begin{pmatrix} \hat{K}_1 \\ \hat{K}_2 \\ \hat{K}_3 \end{pmatrix} \exp(\mathrm{i}\beta \hat{K}_2) = \begin{pmatrix} \cosh(\beta) & 0 & \sinh(\beta) \\ 0 & 1 & 0 \\ \sinh(\beta) & 0 & \cosh(\beta) \end{pmatrix} \begin{pmatrix} \hat{K}_1 \\ \hat{K}_2 \\ \hat{K}_3 \end{pmatrix}. \tag{20}$$

This implies that the boson operators $\hat{a}_1$ and $\hat{a}_2^\dagger$ are transformed according to

$$\exp(-\mathrm{i}\beta \hat{K}_2) \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2^\dagger \end{pmatrix} \exp(\mathrm{i}\beta \hat{K}_2) = \begin{pmatrix} \cosh(\beta/2) & \sinh(\beta/2) \\ \sinh(\beta/2) & \cosh(\beta/2) \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2^\dagger \end{pmatrix} \tag{21a}$$

$$\exp(-\mathrm{i}\gamma\hat{K}_3)\begin{pmatrix}\hat{a}_1\\ \hat{a}_2^\dagger\end{pmatrix}\exp(\mathrm{i}\gamma\hat{K}_3)=\begin{pmatrix}\mathrm{e}^{\mathrm{i}\gamma/2} & 0\\ 0 & \mathrm{e}^{-\mathrm{i}\gamma/2}\end{pmatrix}\begin{pmatrix}\hat{a}_1\\ \hat{a}_2^\dagger\end{pmatrix}. \tag{21b}$$

The overall effect of $\hat{M}(\alpha,\beta,\gamma)$ on the vector

$$\begin{pmatrix}\hat{a}_1\\ \hat{a}_2^\dagger\end{pmatrix}$$

can then be written as the matrix

$$\mathbf{M}=\begin{pmatrix}\mathrm{e}^{\mathrm{i}\gamma/2} & 0\\ 0 & \mathrm{e}^{-\mathrm{i}\gamma/2}\end{pmatrix}\begin{pmatrix}\cosh(\beta/2) & \sinh(\beta/2)\\ \sinh(\beta/2) & \cosh(\beta/2)\end{pmatrix}\begin{pmatrix}\mathrm{e}^{\mathrm{i}\alpha/2} & 0\\ 0 & \mathrm{e}^{-\mathrm{i}\alpha/2}\end{pmatrix}$$

$$=\begin{pmatrix}\cosh(\beta/2)\,\mathrm{e}^{\mathrm{i}(\gamma+\alpha)/2} & \sinh(\beta/2)\,\mathrm{e}^{\mathrm{i}(\gamma-\alpha)/2}\\ \sinh(\beta/2)\,\mathrm{e}^{-\mathrm{i}(\gamma-\alpha)/2} & \cosh(\beta/2)\,\mathrm{e}^{-\mathrm{i}(\gamma+\alpha)/2}\end{pmatrix} \tag{22}$$

which is precisely in the form of equation (8) with equation (13) and the associations

$$\varphi_1=\tfrac{1}{2}(\gamma+\alpha), \qquad \varphi_2=\tfrac{1}{2}(\gamma-\alpha). \tag{23}$$

The general four-wave mixer operator in equation (18) is therefore rewritten as

$$\hat{M}(\beta,\varphi_1,\varphi_2)=\exp[-\mathrm{i}(\varphi_1-\varphi_2)\hat{K}_3]\exp(-\mathrm{i}\beta\hat{K}_2)\exp[-\mathrm{i}(\varphi_1+\varphi_2)\hat{K}_3]. \tag{24}$$

Introducing raising and lowering operators,

$$\hat{K}_\pm=\hat{K}_1\pm\mathrm{i}\hat{K}_2=\{\hat{a}_1^\dagger\hat{a}_2^\dagger,\ \hat{a}_1\hat{a}_2\} \tag{25}$$

and using equation (20), we can recast equation (24) in terms of the two-mode squeezing operator $\hat{S}(z)$ [11, 12] as

$$\hat{M}(\beta,\varphi_1,\varphi_2)=\hat{S}^\dagger(z)\exp(-\mathrm{i}2\varphi_1\hat{K}_3) \tag{26}$$

where

$$\hat{S}(z)\equiv\exp(z\hat{K}_+-z^*\hat{K}_-) \qquad z\equiv\tfrac{1}{2}\beta\exp[-\mathrm{i}(\varphi_1-\varphi_2)] \tag{27}$$

It is interesting to note that an SU(1,1) device couples the annihilation operator of one input field to the creation operator of the other. On the other hand, SU(2) devices such as the beam splitter couple the annihilation operators of both input fields [6, 13]. This crucial difference results in different conservation laws for the two groups of devices. While it is the total number of photons (energy) that is preserved by SU(2) devices it is the difference in photon number that is conserved by SU(1,1) devices. An SU(1,1) device can function as an amplifier by drawing from the classical pump reservoir and energy need not be conserved.

3. Photon number probability distributions

In section 2 we considered the transformation of field operators by the four-wave mixer. A complementary approach that is useful in evaluating the full photon number probability distribution at the output of the device is to consider the transformation of the joint input state according to

$$\hat{\rho}_{\mathrm{out}}=\hat{M}^\dagger(\beta,\varphi_1,\varphi_2)\hat{\rho}_{\mathrm{in}}\hat{M}(\beta,\varphi_1,\varphi_2)=\exp(\mathrm{i}2\varphi_1\hat{K}_3)\hat{S}(z)\hat{\rho}_{\mathrm{in}}\hat{S}^\dagger(z)\exp(-\mathrm{i}2\varphi_1\hat{K}_3) \tag{28}$$

where $\hat{\rho}$ denotes a joint density operator. At the input, it can be written in the number state representation as

$$\hat{\rho}_{\text{in}} = \sum_{n_1, n_2=0}^{\infty} \sum_{n_1', n_2'=0}^{\infty} \rho_{\text{in}}(n_1, n_2; n_1', n_2')|n_1, n_2\rangle\langle n_2', n_1'| \tag{29}$$

where the primed variables admit the possibility of off-diagonal elements.

We write the density operator at the output in terms of the basis states $|N_1\rangle$ and $|N_2\rangle$ at the output ports:

$$\hat{\rho}_{\text{out}} = \sum_{N_1, N_2=0}^{\infty} \sum_{N_1', N_2'=0}^{\infty} \rho_{\text{out}}(N_1, N_2; N_1', N_2')|N_1, N_2\rangle\langle N_2', N_1'|. \tag{30a}$$

The matrix elements of this density operator are given by

$$\rho_{\text{out}}(N_1, N_2; N_1', N_2') = \sum_{n_1, n_2} \sum_{n_1', n_2'} M_{N_1, N_2}^{(n_1, n_2)} M_{N_1', N_2'}^{*(n_1', n_2')} \rho_{\text{in}}(n_1, n_2; n_1', n_2'). \tag{30b}$$

The complex coefficients M depend on the four-wave mixer parameters β, φ_1 and φ_2, and are related to the four-wave mixer operator of equation (24) by

$$\begin{aligned} M_{N_1, N_2}^{(n_1, n_2)} &\equiv \langle N_1, N_2|\hat{M}^{\dagger}(\beta, \varphi_1, \varphi_2)|n_1, n_2\rangle \\ &= D_{N_1, N_2}^{(n_1, n_2)} \exp\{\mathrm{i}[\varphi_1(N_1 + n_2 + 1) + \varphi_2(N_1 - n_1)]\}. \end{aligned} \tag{30c}$$

Its magnitude is provided by the four-wave mixer operator matrix element

$$D_{N_1, N_2}^{(n_1, n_2)} \equiv |M_{N_1, N_2}^{(n_1, n_2)}| = \langle N_1, N_2|\exp(\mathrm{i}\beta\hat{K}_2)|n_1, n_2\rangle \tag{30d}$$

which represents the output photon number probability amplitude for input number states, and which will subsequently be evaluated. Conservation of the photon number difference is mandated by the SU(1,1) symmetry of the four-wave mixer operator for each matrix element of equation (30c) in accordance with equation (11). For fixed N_1 and N_2 with $N_2 \geqslant N_1$, we sum over the range $n_1 = (0, \ldots, \infty)$ in equation (30*b*) and throughout this section. A second sum is not necessary, as $n_2 = N_2 - N_1 + n_1$. The same applies to the primed variables. If $N_1 \geqslant N_2$, we would again have only one sum over the range $n_2 = (0, \ldots, \infty)$.

The joint probability of observing N_1 and N_2 photons at the first and second output ports, respectively (see figure 1), is given by the diagonal elements of the output density operator equation (30*a*),

$$P_{\text{out}}(N_1, N_2) \equiv \langle N_1, N_2|\hat{\rho}_{\text{out}}|N_1, N_2\rangle = \sum_{n_1=0}^{\infty} \sum_{n_1'=0}^{\infty} M_{N_1, N_2}^{(n_1, n_2)} M_{N_1, N_2}^{*(n_1', n_2')} \rho_{\text{in}}(n_1, n_2; n_1', n_2'). \tag{31a}$$

The marginal photon number output distributions are therefore given by

$$P_{\text{out}}(N_1) = \sum_{N_2=0}^{\infty} P_{\text{out}}(N_1, N_2) \tag{31b}$$

$$P_{\text{out}}(N_2) = \sum_{N_1=0}^{\infty} P_{\text{out}}(N_1, N_2). \tag{31c}$$

For the case of pure number states at the input ports, the joint probability of having N_1 and N_2 photons at the first and second output ports, respectively, is given by the square of the amplitude function of equation (30*d*),

$$P_{\mathrm{out}}(N_1, N_2|n_1, n_2) = (D_{N_1, N_2}^{(n_1, n_2)})^2. \tag{32}$$

Since we have the condition of photon number difference conservation (equation (11)) and a definite number of photons at the input, there is only one term in each of the sums in equations (31*b*) and (31*c*), so the marginal probabilities are equal to the joint probabilities.

To evaluate the photon number probability amplitude (equation (30*d*)) we begin by expanding the input state in terms of the vacuum as

$$|n_1, n_2\rangle = \frac{(\hat{a}_1^\dagger)^{n_1}(\hat{a}_2^\dagger)^{n_2}}{(n_1!\, n_2!)^{1/2}}|0, 0\rangle. \tag{33}$$

Using the unitarity of the operator $\hat{K}_2$ together with equation (21*a*) yields

$$\begin{aligned}\exp(\mathrm{i}\beta\hat{K}\ )|n\ , n\ \rangle \\ &= \frac{[\cosh(\beta/2)\hat{a}_1^\dagger - \sinh(\beta/2)\hat{a}_2]^{n_1}[\cosh(\beta/2)\hat{a}_2^\dagger - \sinh(\beta/2)\hat{a}_1]^{n_2}}{(n_1!\, n_2!)^{1/2}} \\ &\quad\times \exp(\mathrm{i}\beta\hat{K}_2)|0, 0\rangle. \end{aligned} \tag{34}$$

To evaluate the effect of the operator $\exp(\mathrm{i}\beta\hat{K}_2)$ in equation (34) on the vacuum state, we use the identity [14]

$$\begin{aligned}\exp(\mathrm{i}\beta\hat{K}_2) &= \exp(\tfrac{1}{2}\beta\hat{K}_+ - \tfrac{1}{2}\beta\hat{K}_-) \\ &= \exp[\tanh(\tfrac{1}{2}\beta)\hat{K}_+]\exp\{-2[\ln\cosh(\tfrac{1}{2}\beta)]\hat{K}_3\} \\ &\quad\times \exp[-\tanh(\tfrac{1}{2}\beta)\hat{K}_-]\end{aligned} \tag{35}$$

so that

$$\exp(\mathrm{i}\beta\hat{K}_2)|0, 0\rangle = \mathrm{sech}(\tfrac{1}{2}\beta)\sum_{n=0}^{\infty}[\tanh(\tfrac{1}{2}\beta)]^n|n, n\rangle. \tag{36}$$

This is the well known two-mode squeezed vacuum state [15]. Inserting this result in equation (34) and expanding, we finally obtain

$$\begin{aligned}D_{N_1, N_2}^{(n_1, n_2)} \\ &= \sum_{k=0}^{n_1}(-1)^{n_1-k}\left[\binom{n_1}{k}\binom{N_1-k+n_2}{n_1-k}\binom{N_1-k+n_2}{n_2}\binom{N_1}{k}\right]^{1/2} \\ &\quad\times[\mathrm{sech}(\beta/2)]^{n_2-n_1+1}[\tanh(\beta/2)]^{-2k+N_1+n_1}\end{aligned} \tag{37}$$

where we have used $N_2 = N_1 - n_1 + n_2$ from equation (11) and

$$\binom{n}{k}$$

denotes the binomial coefficients $n!/(n-k)!k!$.

This result can be expressed in terms of the hypergeometric function [9, 10]

$$D^{(n_1;\,n_2)}_{N_1,\,N_2} = (-1)^{n_1}\left[\binom{N_1+n_2}{n_2}\binom{N_1+n_2}{n_1}\right]^{1/2}[\sinh(\tfrac{1}{2}\beta)]^{n_1-n_2-1}[\tanh(\tfrac{1}{2}\beta)]^{N_1+n_2+1}$$

$$\times\,{}_2F_1[-N_1, -n_1; -N_1-n_2; \coth^2(\tfrac{1}{2}\beta)] \qquad n_2>n_1. \tag{38}$$

For $n_1>n_2$ the indices 1 and 2 are interchanged in equation (38).

When the vacuum fills both input ports

$$D^{(0,\,0)}_{N_1,\,N_2} = \delta_{N_1,\,N_2}\,\mathrm{sech}(\tfrac{1}{2}\beta)[\tanh(\tfrac{1}{2}\beta)]^{N_1} \tag{39}$$

and

$$P(N_1, N_2|0,0) = \delta_{N_1,\,N_2}\,\mathrm{sech}^2(\tfrac{1}{2}\beta)[\tanh^2(\tfrac{1}{2}\beta)]^{N_1}. \tag{40}$$

As noted by Yurke *et al* [6, 15] the photons are emitted in pairs, as required by the conservation of photon number difference, with a Bose–Einstein distribution at each port (figure 2).

When a number state is incident at either port and the remaining input port is in the vacuum state, the marginal probability distributions at each port are given by

$$P(N_1|0, n_2) = \binom{N_1+n_2}{n_2}\left[\mathrm{sech}^2\left(\frac{\beta}{2}\right)\right]^{n_2+1}\left[\tanh^2\left(\frac{\beta}{2}\right)\right]^{N_1} \tag{41a}$$

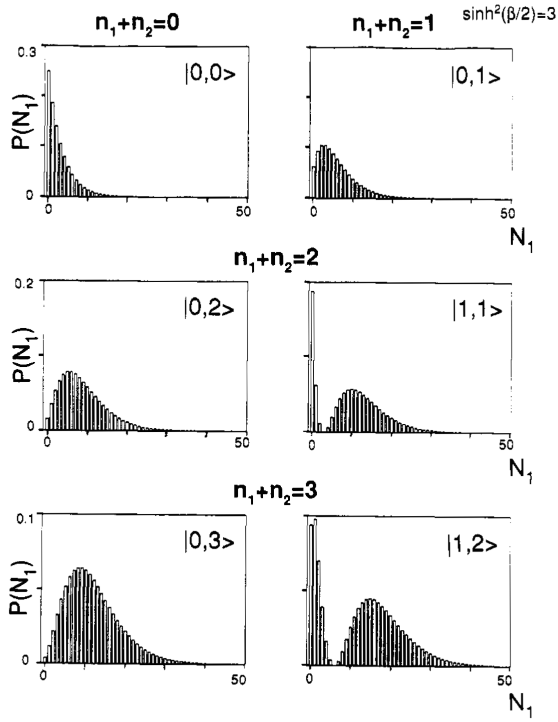

Figure 2. Four-wave mixer output marginal photon number probability distributions for a total of zero, one, two and three photons at the input ports, when $\sinh^2(\beta/2)=3$.

$$P(N_2|0, n_2) = \binom{N_2}{n_2} \left[\operatorname{sech}^2\left(\frac{\beta}{2}\right)\right]^{n_2+1} \left[\tanh^2\left(\frac{\beta}{2}\right)\right]^{N_2-n_2} \tag{41b}$$

$$P(N_1|n_1, 0) = \binom{N_1}{n_1} \left[\operatorname{sech}^2\left(\frac{\beta}{2}\right)\right]^{n_1+1} \left[\tanh^2\left(\frac{\beta}{2}\right)\right]^{N_1-n_1} \tag{41c}$$

$$P(N_2|n_1, 0) = \binom{N_2+n_1}{n_1} \left[\operatorname{sech}^2\left(\frac{\beta}{2}\right)\right]^{n_1+1} \left[\tanh^2\left(\frac{\beta}{2}\right)\right]^{N_2}. \tag{41d}$$

The photon number distribution at the output port directly opposite to the input port with the vacuum is a negative-binomial distribution with mean photon number $(1+n)\sinh^2(\beta/2)$, where n is the number of input photons (equations (41*a*) and (41*d*)). The distribution at the remaining output port is a negative binomial with mean $[n+(1+n)\sinh^2(\beta/2)]$ so it is shifted in photon number space by the number of input photons (equations (41*b*) and (41*c*)). The negative binomial is an important distribution that can arise from a number of classical mechanisms such as pure birth processes [16].

When number states fill both input ports we use a three-term recursion relation for the output photon number probability amplitude [17]. We first rewrite the last transformation of equation (20) as

$$\exp(i\beta\hat{K}_2)\hat{K}_3 = \cosh(\beta)\hat{K}_3\exp(i\beta\hat{K}_2) - \sinh(\beta)\left(\frac{\hat{K}_+ + \hat{K}_-}{2}\right)\exp(i\beta\hat{K}_2). \tag{42}$$

Bracketing both sides of this equation with the input and output number states $|n_1, n_2\rangle$ and $\langle N_1, N_2|$ then gives the recursion relation

$$D^{(n_1,n_2)}_{N_1,N_2} = a_{N_1} D^{(n_1,n_2)}_{N_1-1,N_2-1} - b_{N_1} D^{(n_1,n_2)}_{N_1-2,N_2-2} \qquad N_1 \geqslant 2 \qquad n_2 \geqslant n_1 \tag{43}$$

with the initial condition

$$D^{(n_1,n_2)}_{1,n_2-n_1+1} = a_1 D^{(n_1,n_2)}_{0,n_2-n_1}.$$

The coefficients a_{N_1} and b_{N_1} are given by

$$a_{N_1} = \frac{\cosh(\frac{1}{2}\beta)[2N_1-(n_1-n_2)-1]-\frac{1}{2}(n_1+n_2+1)}{\sinh(\frac{1}{2}\beta)\{N_1[N_1-(n_1-n_2)]\}^{1/2}} \tag{44}$$

$$b_{N_1} = \left(\frac{(1-N_1)(n_1-n_2+1-N_1)}{N_1[N_1-(n_1-n_2)]}\right)^{1/2}$$

and the seed value for the recursion process can be obtained directly from equation (37):

$$D^{(n_1,n_2)}_{0,n_2-n_1} = (-1)^{n_1}\binom{n_2}{n_1}^{1/2}\left[\sinh\left(\frac{\beta}{2}\right)\right]^{n_1}\left[\cosh\left(\frac{\beta}{2}\right)\right]^{-(n_2+1)}. \tag{45}$$

When $n_1 > n_2$ the subscripts 1 and 2 are interchanged in equations (43)–(45). The marginal photon number distribution at the first output port of the four-wave mixer is shown in figures 2, 3 and 4 for several joint input states.

To gain further insight into the photon number probability distributions we focus on its first two moments. The annihilation operators at the four-wave mixer output ports are, from equations (8), (13) and (14),

$$\hat{b}_1 = \cosh(\tfrac{1}{2}\beta)\exp(i\varphi_1)\hat{a}_1 + \sinh(\tfrac{1}{2}\beta)\exp(i\varphi_2)\hat{a}_2^\dagger \tag{46a}$$

$$\hat{b}_2 = \sinh(\tfrac{1}{2}\beta)\exp(i\varphi_2)\hat{a}_1^\dagger + \cosh(\tfrac{1}{2}\beta)\exp(i\varphi_1)\hat{a}_2. \tag{46b}$$

From the definition of the output number operators (equation (12)) and equations (46*a*, *b*) we find that for number state inputs the average photon number at the output ports is

$$\langle \hat{N}_1 \rangle \equiv \langle n_1, n_2 | \hat{N}_1 | n_1, n_2 \rangle = \cosh^2(\tfrac{1}{2}\beta) n_1 + \sinh^2(\tfrac{1}{2}\beta)(1+n_2) \tag{47a}$$

$$\langle \hat{N}_2 \rangle = \cosh^2(\tfrac{1}{2}\beta) n_2 + \sinh^2(\tfrac{1}{2}\beta)(1+n_1). \tag{47b}$$

As can be seen from equations (47*a*, *b*) $\sinh^2(\tfrac{1}{2}\beta)$ is the gain of the conversion process. Note that for non-zero β there is a finite mean output even when only the vacuum state is present at both the input ports. This contribution arises from the non-commutativity of the field operators and represents the quantum noise of the conversion process.

The variance may be obtained by evaluating

$$\langle (\Delta \hat{N}_i)^2 \rangle \equiv \langle \hat{N}_i^2 \rangle - \langle \hat{N}_i \rangle^2 = \langle \hat{b}_i^\dagger \hat{b}_i \hat{b}_i^\dagger \hat{b}_i \rangle - \langle \hat{b}_i^\dagger \hat{b}_i \rangle^2 \qquad i = 1, 2. \tag{48}$$

Substituting equations (46*a*, *b*) into equation (48) we find that for number state inputs

$$\langle (\Delta \hat{N}_1)^2 \rangle = \langle (\Delta \hat{N}_2)^2 \rangle = \cosh^2(\tfrac{1}{2}\beta)\sinh^2(\tfrac{1}{2}\beta)(1 + n_1 + n_2 + 2n_1 n_2). \tag{49}$$

The photon number variance-to-mean ratio Fano factor (at the first port) is

$$F = \frac{\cosh^2(\tfrac{1}{2}\beta)\sinh^2(\tfrac{1}{2}\beta)(1 + n_1 + n_2 + 2n_1 n_2)}{\cosh^2(\tfrac{1}{2}\beta) n_1 + \sinh^2(\tfrac{1}{2}\beta)(1+n_2)}. \tag{50}$$

and it is the same at the second port with n_1 and n_2 interchanged. The input number states are perfectly squeezed in photon number, that is have a Fano factor of zero. This property is destroyed by the amplification of the four-wave mixer. For given n_1, n_2 the output will be sub-Poissonian (it exhibits $F < 1$) if

$$\sinh^2\left(\frac{\beta}{2}\right) < \frac{(n_1^2 n_2^2 + 2n_1 n_2 + n_1 + n_2 + 1)^{1/2} - n_1 n_2}{2n_1 n_2 + n_1 + n_2 + 1} \tag{51}$$

that is for a sufficiently weak interaction. The Fano factor increases with increasing $\sinh^2(\beta/2)$ while the dips in the distribution become more pronounced and the distribution extends out to higher photon number (figure 3). In figures 2 and 4 we see that for a given total photon number the number of dips in the distribution increases with decreasing photon number difference at the input and there is a corresponding increase in the photon number variance.

The single-mode squeezed number state has been discussed by Kim *et al* [18]. They also observed large-scale oscillations in the photon number distribution. They discuss these oscillations in terms of interference effects in phase space, as Schleich and Wheeler [19] have done for the single-mode squeezed coherent state.

In addition to the large-scale oscillations, the distribution for the total number of photons at the output ($N = N_1 + N_2$) exhibits pairwise oscillations with only an even or only an odd number of photons allowed when the difference $n_1 - n_2$ is even or odd, respectively. An example with $n_1 - n_2$ even is shown in figure 5. This is due to the fact that the four-wave mixer operator creates or destroys photons in the two modes in pairs as can be seen from equations (25), (27) and (28).

One major difference between the photon number distributions of the SU(1,1) devices and the SU(2) devices is that the distributions for the SU(2) devices terminate

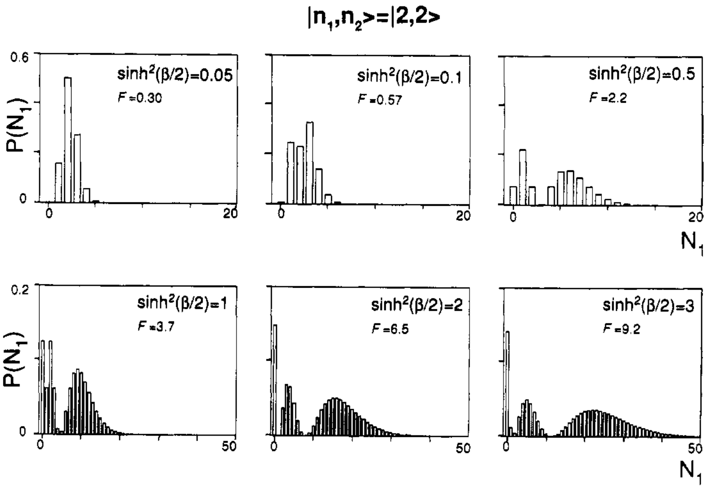

Figure 3. Comparison of four-wave mixer output marginal photon number probability distributions for various values of $\sinh^2(\beta/2)$ with two photons at each input port. F is the Fano factor of the distribution.

after a definite photon number since the total number of photons is conserved [13]. Both groups of devices produce particle-like statistics at the output ports when there is a vacuum at one of the input ports; in the SU(2) case it is binomial and in the SU(1,1) case it is a negative binomial. For number state inputs at both ports the output distributions of both groups of devices exhibit oscillatory behaviour.

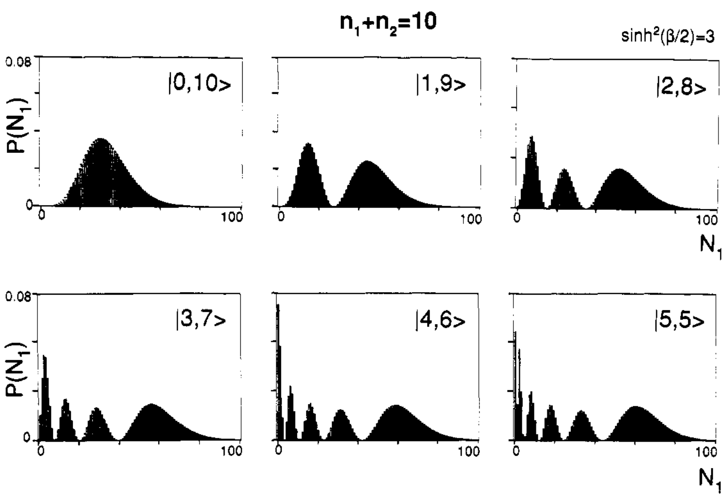

Figure 4. Comparison of four-wave mixer output marginal photon number probability distributions for a total of ten photons at the input ports, when $\sinh^2(\beta/2) = 3$.

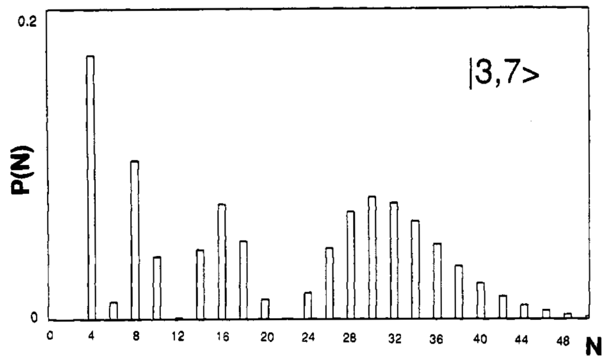

Figure 5. Four-wave mixer total output photon number probability distribution for ten photons at the input ports with $|n_1 - n_2| = 4$, when $\sinh^2(\beta/2) = 0.5$. In addition to the large-scale fluctuations, the distribution exhibits pairwise oscillation; only even numbers of photons are allowed.

4. Quadrature squeezing in the combined output

The quadrature components of the operator $\hat{o}_i$ are defined by

$$\begin{aligned} \hat{X}_{\hat{o}_i} &\equiv (\hat{o}_i + \hat{o}_i^\dagger)/2 \\ \hat{Y}_{\hat{o}_i} &\equiv (\hat{o}_i - \hat{o}_i^\dagger)/2\mathrm{i}. \end{aligned} \tag{52}$$

The quadrature variances of the input number states are

$$\langle(\Delta \hat{X}_{\hat{a}_i})^2\rangle = \langle(\Delta \hat{Y}_{\hat{a}_i})^2\rangle = \tfrac{1}{4}(2n_i + 1) \qquad i = 1, 2. \tag{53}$$

The output states also have equal uncertainties in each field quadrature with variances given by

$$\langle(\Delta \hat{X}_{\hat{b}_1})^2\rangle = \langle(\Delta \hat{Y}_{\hat{b}_1})^2\rangle = \tfrac{1}{4}[\cosh^2(\tfrac{1}{2}\beta)(1 + 2n_1) + \sinh^2(\tfrac{1}{2}\beta)(1 + 2n_2)] \tag{54a}$$

$$\langle(\Delta \hat{X}_{\hat{b}_2})^2\rangle = \langle(\Delta \hat{Y}_{\hat{b}_2})^2\rangle = \tfrac{1}{4}[\cosh^2(\tfrac{1}{2}\beta)(1 + 2n_2) + \sinh^2(\tfrac{1}{2}\beta)(1 + 2n_1)]. \tag{54b}$$

The output quadrature fluctuations of both modes are increased by the four-wave mixing process, as we would expect for an amplification process. It is only when we consider the two correlated modes together that we observe quadrature squeezing.

Yuen and Shapiro [4] have shown that for coherent state inputs, a superposition of the two output modes using a 50/50 beam splitter gives rise to quadrature-squeezed states. Using this same technique to combine the output modes of a four-wave mixer with arbitrary states at the input leads to annihilation operators given by

$$\hat{c}_1 = (\hat{b}_1 + \mathrm{e}^{\mathrm{i}\theta}\hat{b}_2)/2^{1/2} \tag{55a}$$

and

$$\hat{c}_2 = (-\mathrm{e}^{-\mathrm{i}\theta}\hat{b}_1 + \hat{b}_2)/2^{1/2} \tag{55b}$$

where θ is the phase shift imparted by the beam splitter. The quadrature components of $\hat{c}_1$ are, from equations (55), (52) and (46),

$$\hat{X}_{\hat{c}_1} = \frac{1}{2^{1/2}}\{\tfrac{1}{2}[\cosh(\tfrac{1}{2}\beta)\exp(i\varphi_1) + \sinh(\tfrac{1}{2}\beta)\exp[-i(\varphi_2+\theta)]]\hat{a}_1$$

$$+\tfrac{1}{2}[\cosh(\tfrac{1}{2}\beta)\exp[i(\varphi_1+\theta)] + \sinh(\tfrac{1}{2}\beta)\exp(-i\varphi_2)]\hat{a}_2^{\dagger}\} + \mathrm{cc} \quad (56)$$

$$\hat{Y}_{\hat{c}_1} = \frac{1}{2^{1/2}}\{\tfrac{1}{2}[\cosh(\tfrac{1}{2}\beta)\exp(i\varphi_1) - \sinh(\tfrac{1}{2}\beta)\exp[-i(\varphi_2+\theta)]]\hat{a}_1$$

$$+\tfrac{1}{2}[\cosh(\tfrac{1}{2}\beta)\exp[i(\varphi_1+\theta)] - \sinh(\tfrac{1}{2}\beta)\exp(-i\varphi_2)]\,\hat{a}_2^{\dagger}\} - \mathrm{cc}.$$

The squeezing effect becomes obvious when the phase shift θ imparted by the beam splitter is adjusted so that the fluctuations in $\hat{X}_{\hat{c}_1}$ and $\hat{Y}_{\hat{c}_1}$ are, respectively, maximized and minimized, that is when

$$\theta + \varphi_1 + \varphi_2 = 0 \quad (57)$$

In that case equation (56) can be written as

$$\hat{X}_{\hat{c}_1} = \frac{1}{2^{1/2}}\,e^{\beta/2}(\hat{X}_{\hat{a}_1} + \hat{X}_{\hat{a}_2})$$

$$\hat{Y}_{\hat{c}_1} = \frac{1}{2^{1/2}}\,e^{-\beta/2}(\hat{Y}_{\hat{a}_1} + \hat{Y}_{\hat{a}_2}) \quad (58)$$

and the variances are

$$\langle(\Delta\hat{X}_{\hat{c}_1})^2\rangle = \tfrac{1}{2}e^{\beta}\langle[\Delta(\hat{X}_{\hat{a}_1} + \hat{X}_{\hat{a}_2})]^2\rangle \quad (59)$$

$$\langle(\Delta\hat{Y}_{\hat{c}_1})^2\rangle = \tfrac{1}{2}e^{-\beta}\langle[\Delta(\hat{Y}_{\hat{a}_1} + \hat{Y}_{\hat{a}_2})]^2\rangle.$$

The combination mode $\hat{c}_1$ does not necessarily represent a minimum-uncertainty state, for which the product of the variances is $\frac{1}{16}$. However, as β grows, the uncertainty in $\hat{Y}_{\hat{c}_1}$ is reduced at the expense of an increased uncertainty in $\hat{X}_{\hat{c}_1}$. When the variance of $\hat{Y}_{\hat{c}_1}$ falls below what it would be for a minimum-uncertainty state with equal uncertainties in each quadrature (namely, the coherent state), the state is said to be quadrature squeezed. We can see that when both input states are coherent states the output will be squeezed for any non-zero β. When the input states are anything other than the coherent states the output can be squeezed provided the interaction is strong enough that β is above a certain threshold value.

For number states at the input equations (59) become

$$\langle(\Delta\hat{X}_{\hat{c}_1})^2\rangle = \tfrac{1}{4}e^{\beta}(1+n_1+n_2)$$

$$\langle(\Delta\hat{Y}_{\hat{c}_1})^2\rangle = \tfrac{1}{4}e^{-\beta}(1+n_1+n_2). \quad (60)$$

Unlike the $\hat{a}$ and $\hat{b}$ modes, the mode $\hat{c}_1$ has quadratures with unequal uncertainties and will exhibit quadrature squeezing when

$$\beta > \ln(1+n_1+n_2) \quad (61)$$

that is in the strong interaction regime. Note that the enhancement of quadrature squeezing with increasing β comes at the expense of increased fluctuations in the photon number (see equation (49)).

The roles of $\hat{X}$ and $\hat{Y}$ are reversed when

$$\theta+\varphi_1+\varphi_2=\pi. \tag{62}$$

Similarly, the combination mode $\hat{c}_2$ will be squeezed when the condition

$$\theta-\varphi_1-\varphi_2=0, \pi \tag{63}$$

is met.

The beam splitter transformation matrices belong to the group SU(2), which is isomorphic to the three-dimensional rotation group [6, 13]. The groups SU(2) and SU(1,1) are subgroups of the Lorentz group O(3,2) with three space-like coordinates and two time-like coordinates [20]. We have shown that the cascade of an SU(1,1) device and an SU(2) device can lead to squeezing of arbitrary input states for sufficiently large values of the parameter β (see equation (58)).

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