PHYSICAL REVIEW LETTERS

VOLUME 63 | 23 OCTOBER 1989 | NUMBER 17

Fractal Shot Noise

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(Received 8 June 1989)

We define fractal shot noise, which is a stationary continuous-time process that is fundamentally different from fractional Brownian motion. Two applications in physics are considered: the mass distribution of collections of solid-particle aggregates and the electric field at the growing edge of a doped semiconductor quantum wire. For a broad range of parameters, the amplitude probability density function of this process is a Lévy-stable random variable with dimension less than unity; it therefore does *not* converge to Gaussian form.

PACS numbers: 05.40.+j, 02.50.+s

Fractal shot noise $I(t)$ may be expressed as an infinite sum of impulse response functions

$$I(t) \equiv \sum_{j=-\infty}^{\infty} h(t-t_j), \tag{1}$$

where

$$h(t) \equiv \begin{cases} Kt^{-\beta}, & A \le t \le B; \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

and the times t_j are random events from a Poisson point process of rate μ. The amplitude parameter may be either random (denoted by K) or deterministic (denoted by K_0). The parameters μ, A, B, and β are deterministic and fixed. In general, the range of the function may extend down to $A=0$ or up to $B=\infty$, and β may range between 0 and ∞ exclusive. In this paper we consider power-law impulse response functions with $\beta > 1$; the case $0<\beta<1$ is fundamentally different, with applications in semiconductor $1/f$ noise.[1] For all calculations we assume that t is finite, so that the shot-noise process has reached steady state.

All moments of the fractal-shot-noise process may be given in terms of the cumulants. The nth cumulant (semi-invariant) C_n of $I(t)$ is given by[2]

$$C_n = \mu \left\langle \int_{-\infty}^{\infty} h^n(t)\,dt \right\rangle = \mu \langle K^n \rangle \frac{A^{1-n\beta} - B^{1-n\beta}}{n\beta - 1}, \tag{3}$$

where the angular brackets denote expectation over the distribution of K. The nth cumulant will be infinite if $\langle K^n \rangle$ is infinite or if $A=0$. The first three moments and the variance are

$$E[I] = C_1, \quad E[I^2] = C_2 + C_1^2,$$
$$E[I^3] = C_3 + 3C_1C_2 + C_1^3, \quad \mathrm{Var}[I] = C_2, \tag{4}$$

where $E[\cdots]$ denotes expectation over the distribution of I.

To compute the moment generating function of the fractal-shot-noise process $I(t)$, we first consider the case where K_0 is deterministic and fixed, and therefore all impulse response functions are identical. Then $h(t) = K_0 t^{-\beta}$, and the first-order moment generating function $Q_I(s)$ of the shot-noise process I is given by

$$Q_I(s) \equiv E[e^{-sI}] = \exp\left\{ -\mu(B-A) - \frac{\mu(sK_0)^{1/\beta}}{\beta} \left[\Gamma\left(-\frac{1}{\beta}, sK_0 A^{-\beta} \right) - \Gamma\left(-\frac{1}{\beta}, sK_0 B^{-\beta} \right) \right] \right\}, \tag{5}$$

or equivalently,

$$Q_I(s)=\exp\left\{\mu A[1-\exp(-sK_0A^{-\beta})]-\mu B[1-\exp(-sK_0B^{-\beta})]+\mu(sK_0)^{1/\beta}\Gamma\left[1-\frac{1}{\beta},sK_0B^{-\beta}\right]-\mu(sK_0)^{1/\beta}\Gamma\left[1-\frac{1}{\beta},sK_0A^{-\beta}\right]\right\}, \quad (6)$$

where $\Gamma(\cdots,\cdots)$ is the incomplete gamma function defined by $\Gamma(a,x)\equiv\int_x^\infty e^{-t}t^{a-1}dt$. Both Eqs. (5) and (6) are valid for all values of μ, A, B, and β, including the case $\beta<1$.

Returning to the case $\beta>1$, if we let $A\to 0$ and $B\to\infty$ then a much simpler form for $Q_I(s)$ results. After evaluating limits using l'Hôpital's rule, we obtain

$$Q_I(s)=\exp[-\mu(sK_0)^{1/\beta}\Gamma(1-1/\beta)]\,.$$

Defining $D\equiv 1/\beta$, we have $0<D<1$ since $\beta>1$. Furthermore, for $A=0$ and $B=\infty$, we can consider stochastic K by using the equivalent deterministic impulse-response-function method of Gilbert and Pollak,[3] which leads to

$$Q_I(s)=\exp[-\mu\langle K^D\rangle\Gamma(1-D)s^D]\,. \quad (7)$$

This moment generating function is of the form $Q(s)=\exp[-(cs)^D]$, where c is a constant, so that for all μ the shot noise I is a Lévy-stable random variable[4,5] with extreme asymmetry of dimension D: $0<D<1$.

Therefore an infinite-area impulse response function may be used to construct a shot-noise process which is nontrivial and non-Gaussian for all driving rates μ, even in the limits $\mu\to 0$ and $\mu\to\infty$. The conditions of the Gaussian central-limit theorem are violated, and, in particular, all moments of the shot-noise process are infinite. If $B<\infty$, the probability density function will also approach a Lévy-stable form.[6] This is readily understood in the limit $\mu\to\infty$, since the resulting impulse response function is the same as in the $B\to\infty$ case except for the missing tail. Since the missing area is finite, and the total area is infinite, the difference is negligible for large μ.

In the case $\beta<1$, the limit $B\to\infty$ leads to a degenerate probability distribution such that the amplitude of the process is infinite with probability one.[1]

If $C_n<\infty$ for all n, for any β, and for either stochastic or deterministic K, the moment generating function may alternatively be expressed in terms of the cumulants of the process:

$$Q_I(s)=\exp\left[\mu\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\langle K^n\rangle\frac{A^{1-n\beta}-B^{1-n\beta}}{n\beta-1}s^n\right]. \quad (8)$$

Equations (5) and (6) admit $A=0$, $B=\infty$, and arbitrary β, whereas Eq. (8) does not allow $A=0$ for $\beta>1$, or $\beta=1/n$ for any integer n; however, Eq. (8) is valid for stochastic K as well as deterministic K_0.

This Lévy-stable shot-noise process should be contrasted with fractional Brownian motion (FBM), developed by Mandelbrot and Van Ness.[7] Fractional Brownian motion usually has a Gaussian amplitude distribution, but the times between zero crossings have a Lévy-stable time distribution. Our Lévy-stable process, however, has a Lévy-stable amplitude distribution and no zero crossings. In addition, the fractal nature of our Lévy-stable shot-noise process differs from that of FBM, which is self-affine and nonstationary; our Lévy-stable process is strict-sense stationary.

Values for the amplitude probability density function may be obtained from the moment generating function by several methods. If $A=0$ and $B\to\infty$, for $\beta>1$ and for either deterministic or stochastic K, the amplitude probability density function is Lévy stable with dimension $D\equiv 1/\beta$, and an infinite-sum form may be used,[5,8]

$$P(I)=\frac{1}{\pi I}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\Gamma(1+nD)\sin(\pi nD)}{n!}\left[\frac{\mu\Gamma(1-D)\langle K^D\rangle}{I^D}\right]^n. \quad (9)$$

For the particular case $D=\frac{1}{2}$ the exact amplitude probability density function assumes the well-known closed form[4,5]

$$P(I)=\frac{\mu\langle K^{1/2}\rangle}{2}I^{-3/2}\exp\left[-\frac{\mu^2\pi\langle K^{1/2}\rangle^2}{4I}\right]. \quad (10)$$

Figure 1 displays Lévy-stable amplitude probability density functions for three values of the dimension D. All have long power-law tails. Indeed, for $A=0$, $B\to\infty$, $\beta>1$, and $D\equiv 1/\beta$, $P(I)$ approaches a simple asymptotic form in the limit $I\to\infty$. Examining Eq. (9) and using well-known properties of the gamma function,[9] we obtain $\lim_{I\to\infty}P(I)=\mu D\langle K^D\rangle I^{-(1+D)}$.

In all cases with deterministic $h(t)$, including $\beta<1$, the amplitude probability density function of fractal shot noise may be found by evaluating the Fourier integral[2,5]

$$P(I)=\frac{1}{\pi}\mathrm{Re}\int_0^\infty \exp\left[-j\omega I-\mu\int_A^B (1-e^{j\omega K_0 t^{-\beta}})dt\right]d\omega\,, \tag{11}$$

which is, unfortunately, often difficult. However, the amplitude probability density function may alternatively be obtained from an integral equation.[3] We note that if $B<\infty$, then $\mathrm{Prob}\{I=0\}=e^{-\mu(B-A)}>0$, so that the density will have a δ function at $I=0$. The amplitude probability density function is given by[6]

$$P(I)=\begin{cases} 0\,,\ I<0; \\ e^{-\mu(B-A)}\delta(I)\,,\ I=0; \\ 0\,,\ 0<I\le K_0B^{-\beta}; \\ \dfrac{\mu K_0^{1/\beta}}{\beta I}\displaystyle\int_{K_0B^{-\beta}}^{I} P(I-u)u^{-1/\beta}du\,,\ K_0B^{-\beta}<I<K_0A^{-\beta}; \\ \dfrac{\mu K_0^{1/\beta}}{\beta I}\displaystyle\int_{K_0B^{-\beta}}^{K_0A^{-\beta}} P(I-u)u^{-1/\beta}du\,,\ I\ge K_0A^{-\beta}. \end{cases} \tag{12}$$

If $B\to\infty$, Eq. (12) simplifies to

$$P(I)=\frac{\mu K_0^{1/\beta}}{\beta I}\int_0^{\min(I,K_0A^{-\beta})} P(I-u)u^{-1/\beta}du\,, \tag{13}$$

and the integral-equation solution must be multiplied by a scaling constant, determined by requiring $\int_0^\infty P(I)dI=1$. The results obtained from the integral equation for $\beta>1$ are then identical to those given by the Lévy-stable case for small values of I, except for a scaling constant required to normalize the amplitude probability density function to unit area. In that case, the values for the Lévy-stable amplitude probability density function, which are more easily calculated, may be used for values of I between 0 and $K_0A^{-\beta}$.

If $C_n<\infty$ for all n, then the amplitude probability density function $P(I)$ satisfies the conditions of the central-limit theorem, and therefore approaches a Gaussian distribution as $\mu\to\infty$, with the mean and variance given by the first and second cumulants, respectively [see Eq. (4)].

The autocorrelation function $R_I(\tau)$ of the fractal-shot-noise process $I(t)$ is given by

$$R_I(\tau)=\langle I\rangle^2+\mu\left\langle\int_{-\infty}^{\infty}h(t)h(t+|\tau|)dt\right\rangle$$
$$=\langle I\rangle^2+\mu\langle K^2\rangle\int_A^{B-|\tau|}(t^2+|\tau|t)^{-\beta}dt\,. \tag{14}$$

Note that $R_I(\tau)=\langle I\rangle^2$ for $|\tau|\ge B-A$. For $\beta>1$ this integral is infinite and therefore $R_I(\tau)$ does not exist if $A=0$, in which case the power spectral density does not exist either. In addition, the autocorrelation integral is not solvable analytically except for the case when $\beta=n/2$, where n is a positive integer. For $\beta=2$, we ob-

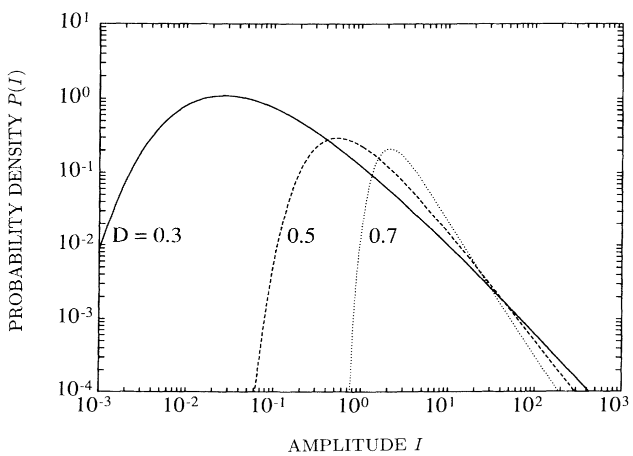

FIG. 1. Double-logarithmic plot of the Lévy-stable amplitude probability density $P(I)$ vs I given in Eqs. (9) and (10) for three values of the fractal dimension D: 0.3, 0.5, and 0.7 ($A=0$, $B=\infty$, $K_0=1$, $\mu=1$). Note the long power-law tails for all values of D.

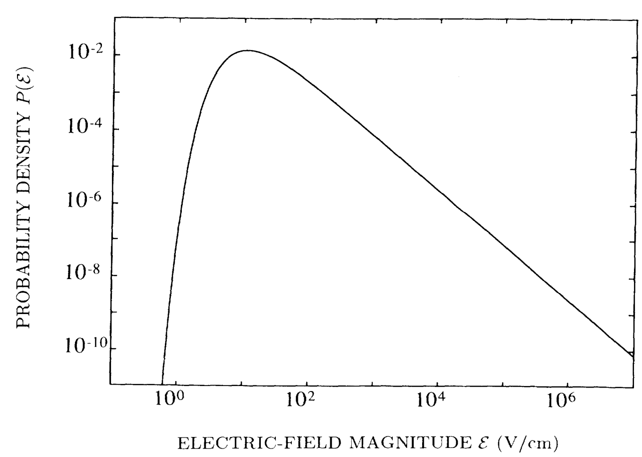

FIG. 2. Double-logarithmic plot of the electric-field-magnitude probability density $P(\mathcal{E})$ vs $\mathcal{E}$ at the edge of a Te-doped GaAs quantum wire with dopant ionic radius $A=0.211$ nm, area $a=400$ nm^2, and dopant concentration $N_D=10^{16}$ cm^{-3}.

tain[6]

$$R_I(\tau)=\begin{cases}\langle I\rangle^2+\dfrac{\mu\langle K^2\rangle}{3}[A^{-3}-B^{-3}], & \tau=0;\\ \langle I\rangle^2+\mu\langle K^2\rangle\left\{\dfrac{2A+|\tau|}{|\tau|^2A(A+|\tau|)}-\dfrac{2B-|\tau|}{|\tau|^2B(B-|\tau|)}+\dfrac{2}{|\tau|^3}\ln\left[\left(1-\dfrac{|\tau|}{B}\right)\left(1+\dfrac{|\tau|}{A}\right)\right]\right\}, & 0<|\tau|<B-A;\\ \langle I\rangle^2, & |\tau|\geq B-A.\end{cases} \tag{15}$$

For $A>0$, $B\to\infty$, and $\beta>1$ the autocorrelation function $R_I(\tau)$ approaches a simpler form[6] in the limit $|\tau|\to\infty$,

$$\lim_{|\tau|\to\infty} R_I(\tau)=\langle I\rangle^2+\mu\langle K^2\rangle\frac{A^{1-\beta}}{1-\beta}|\tau|^{-\beta}=\langle I\rangle^2+\langle I\rangle\frac{\langle K^2\rangle}{\langle K\rangle}|\tau|^{-\beta}, \tag{16}$$

illustrating that it is a power-law function with the same exponent as the impulse response function.

Fractal shot noise has widespread applicability in physics since both Poisson events (e.g., random location of particles) and power-law behavior (e.g., inverse square-law fields) are ubiquitous in physics. We consider two particular applications.

The magnitude of the electric field at the growing edge of a doped semiconductor whisker or quantum wire is precisely described by the fractal-shot-noise process developed here. As growth proceeds, dopant atoms are introduced into the wire in Poisson fashion. Each ionized donor (or acceptor) atom produces an inverse-square electric field that decays as x^{-2}, where x is the distance from the ionized donor to the edge of the quantum wire. The mobile carriers are uniformly distributed throughout the material so that they do not contribute a spatially varying field. Our approach is readily generalized by allowing A or $h(t)$ to be stochastic.

Although our general results apply for random *processes*, for some problems it is sufficient to consider the resulting *distributions* associated with this process. At the edge of a quantum wire of fixed length,[10] for example, the first-order electric-field statistics arising from the ionized impurity atoms (ignoring the constant field contributed by the free carriers) are given by Eq. (13). This is plotted in Fig. 2 for a Te-doped, n-type GaAs quantum wire, for which $A=0.211$ nm as provided by the ionic radius of tellurium; $B=\infty$ for a sufficiently long wire, the Coulomb constant $K_0=q/4\pi\epsilon=1.32\times10^6$ V/cm nm^2, where q is the electronic charge and the permittivity ϵ of GaAs is 9.65×10^{-13} F/cm; $\beta=2$; and $\mu=aN_D=0.004$ nm^{-1} for a wire of cross-sectional area $a=400$ nm^2 and dopant concentration $N_D=10^{16}$ cm^{-3}. This density is proportional to, and essentially coincident with, the Lévy-stable density given in Eq. (10) for fields as high as 2.97×10^7 V/cm. An analogous application is the magnitude of the gravitational field provided by a random distribution of masses.[11] An infinite number of these corresponds to a noncasual power-law form for $h(t)$ and leads to a symmetric Lévy-stable probability density of dimension $D=\frac{1}{2}$.

A particularly important example of our analysis lies in the domain of solid-particle aggregates, including diffusion-limited aggregates, cluster-cluster aggregates, and aerosols. The mass distribution of the aggregated particles often obeys a power law over some range of masses m in these systems, such that[12-14]

$$\text{Prob}\{M\geq m\}=cm^{-D}, \tag{17}$$

where c is a normalizing constant and the power-law exponent D typically falls in the range $0<D<1$. The probability distribution for the individual masses is isomorphic to sampling the time function $M(t)=Kt^{-\beta}$ uniformly over some range of times, where again $\beta=1/D$. The total mass enclosed within a specified region is then isomorphic to the fractal-shot-noise amplitude distribution. In particular the enclosed mass has a moment generating function given by Eqs. (5) and (6), and in the limit by Eq. (7).

This work was supported by the National Science Foundation through the Center for Telecommunications Research and by the Joint Services Electronics Program through the Columbia Radiation Laboratory.

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