Fractal renewal processes as a model of charge transport in amorphous semiconductors

Steven B. Lowen and Malvin C. Teich
Columbia Radiation Laboratory, Columbia University, New York, New York 10027
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We construct two relatively simple fractal renewal processes that provide a framework for understanding charge transport in amorphous semiconductors. These processes exhibit fractal behavior, with power spectral densities that vary as $1/f^D$, deriving from interevent-time probability density functions that themselves decay in a power-law fashion.

Fractal renewal processes and their generalizations provide useful models for a wide variety of physical phenomena, since power-law behavior and serial independence are ubiquitous. We focus here on charge transport through amorphous semiconductors, although other applications include electronic burst noise, zero crossings of Brownian motion, the decay of ordered systems, movement in systems with fractal boundaries, the digital generation of $1/f^D$ noise, cache misses, self-organized criticality, and ionic currents in cell membranes.

A standard renewal process (SRP) $N(t)$ consists of a series of events separated by random intervals that are independent and identically distributed. If the common probability density function $p(t)$ describing these intervals decays as a power law, a standard fractal renewal process (SFRP) results.

Possibly the simplest interevent-time density with a power-law form is the abrupt-cutoff power-law density

$$p(t)=\frac{D}{A^{-D}-B^{-D}}\times\begin{cases}t^{-(D+1)}, & A<t<B\\ 0 & \text{otherwise},\end{cases} \tag{1}$$

where A and B are the lower and upper cutoffs, respectively, and $D>0$. The associated moments (for $n\neq D$) are given by

$$\langle T^n\rangle=\frac{D}{n-D}\frac{B^{n-D}-A^{n-D}}{A^{-D}-B^{-D}}, \tag{2}$$

and the characteristic function,[1] upon which many of the statistics of a SRP depends, is

$$\begin{aligned}Q(j\omega)&=\frac{D}{A^{-D}-B^{-D}}\int_A^B e^{j\omega t}t^{-(D+1)}dt\\&=\frac{D(-j\omega)^D}{A^{-D}-B^{-D}}[\Gamma(-D,-j\omega B)\\&\qquad-\Gamma(-D,-j\omega A)],\end{aligned} \tag{3}$$

where $\Gamma(\ ,\ )$ is the incomplete Γ function defined by $\Gamma(a,x)\equiv\int_x^\infty e^{-t}t^{a-1}dt$. If $B^{-1}\ll\omega\ll A^{-1}$, then Eq. (3) approaches a simpler form

$$Q(j\omega)\approx 1-(-j\omega A)^D\Gamma(1-D). \tag{4}$$

The power spectral density $S_N(\omega)$ of a stationary SRP is given by[2]

$$S_N(\omega)=\langle T\rangle^{-2}\delta(\omega/2\pi)+\langle T\rangle^{-1}\,\mathrm{Re}\left\{\frac{1+Q(j\omega)}{1-Q(j\omega)}\right\}. \tag{5}$$

Simple expressions for the power spectral density exist in the low- and high-frequency limits. For low frequencies, a Taylor series expansion of $Q(j\omega)$ in Ref. 3 provides

$$\lim_{\omega\to 0}S_N(\omega)=\langle T\rangle^{-3}\,\mathrm{Var}\{T\}, \tag{6}$$

where Var is the variance. For large frequencies, the characteristic functions approach zero, and the power spectral density reduces to $\langle T\rangle^{-1}$. For the abrupt-cutoff power-law density with $0<D<1$, and in the medium-frequency limit $B^{-1}\ll\omega\ll A^{-1}$, then

$$\begin{aligned}S_N(\omega)&\approx 2\langle T\rangle^{-1}\,\mathrm{Re}\{(j\omega A)^{-D}/\Gamma(1-D)\}\\&=2D^{-1}(B/A)^D B^{-1}[\Gamma(1-D)]^{-1}\\&\quad\times\cos(\pi D/2)(\omega A)^{-D}.\end{aligned} \tag{7}$$

Figure 1 shows the resulting power spectral density, normalized to unity at the low-frequency limit, where the asymptotes are given by Eqs. (6) and (7), and $S_N(\omega)=\langle T\rangle^{-1}$. Note that the power spectral density

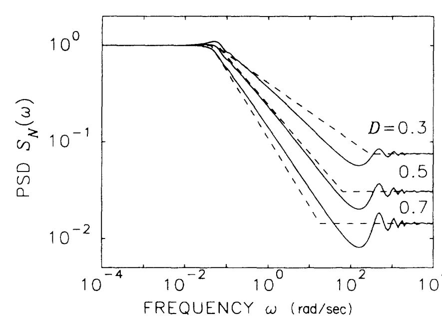

FIG. 1. Double logarithmic plot of the normalized power spectral density for the SFRP, with an abrupt-cutoff power-law probability density, for three values of the exponent D: 0.3, 0.5, and 0.7 ($A=10^{-2}$, $B=10^2$). Asymptotic forms from Eqs. (6) and (7), and the line $S_N(\omega)=\langle T\rangle^{-1}$ are included for comparison. The abrupt cutoff in the interevent-time probability density function gives rise to oscillations in the frequency domain.

$S_N(\omega)$ does indeed vary as $1/f^D$ over a substantial range of frequencies $f=2\pi\omega$, where D corresponds to the power-law exponent in the interevent-time density. Figure 1 also exhibits significant oscillations that arise from the abrupt nature of the interevent-time density $p(t)$. Power-law densities with smoother cutoffs yield power spectral densities that lack these oscillations, and that follow the $1/f^D$ asymptotes more closely.[3] Finally, we note that the SFRP may be filtered, resulting in a kind of generalized shot noise.[2] For a deterministic filter, the resulting overall power spectral density, from linear systems theory,[1] is

$$S(\omega)=|H(\omega)|^2 S_N(\omega)$$
$$=\langle T\rangle^{-1}|H(\omega)|^2 \,\mathrm{Re}\left\{\frac{1+Q(-j\omega)}{1-Q(-j\omega)}\right\}, \tag{8}$$

where $H(\omega)$ is the Fourier transform of the linear system impulse response function.

Extending the range of the power-law exponent D yields no new behavior in the power spectral density. Asymptotic forms for the power spectral density in the limit $B^{-1} \ll |\omega| \ll A^{-1}$, for all values of $D>0$, are

$$S_N(\omega)\to\langle T\rangle^{-1}\times\begin{cases}2\Gamma(1-D)\cos(\pi D/2)(\omega A)^{-D}, & 0<D<1\\ \pi[\ln(\omega A)]^{-2}(\omega A)^{-1}, & D=1\\ 2D^{-2}(D-1)\Gamma(2-D)[-\cos(\pi D/2)](\omega A)^{D-2}, & 1<D<2\\ \frac{1}{2}[-\ln(\omega A)], & D=2\\ D^{-1}(D-2)^{-1}(D-1)^2, & D>2.\end{cases} \tag{9}$$

The power spectral density power-law exponents range between zero and unity, and no new exponents are introduced by considering $D>1$. Presumably this behavior depends on D rather than the nature of the cutoffs, and thus the SFRP will generate $1/f^D$ noise only in the range $0<D<1$.

The coincidence rate $G_N(\tau)$ for a stationary point process $N(t)$ is a measure of the correlation between events with a specified time delay between them, regardless of intervening events, and is the inverse Fourier transform of the power spectral density. It is also proportional to the renewal function $u(t)$, defined by

$$u(t)=\sum_{n=1}^{\infty} p^{*n}(t), \tag{10}$$

where

$$p^{*n}(t)\equiv p*p*\cdots*p(t), \tag{11}$$

with $p(t)$ appearing n times, is the n-fold convolution of the interevent-time density $p(t)$ with itself. Considering the SFRP, for small delays τ, the first term dominates the infinite sum in Eq. (10), yielding $G_N(\tau)\approx\langle T\rangle^{-1}p(|\tau|)$ for $0<|\tau|\ll A$, while for large delays τ, the probabilities of events near t and $t+\tau$ are essentially independent, so that $G_N(\tau)\approx\langle T\rangle^{-2}$ for $|\tau|\gg B$. Obtaining the coincidence rate for the abrupt-cutoff power-law process in the medium-delay limit proves more problematic, since the oscillations in the power spectral density are difficult to approximate accurately. However, using the medium-frequency approximation for the power spectral density in Eq. (7) yields power-law behavior in the coincidence rate

$$G_N(\tau)=\mathcal{F}^{-1}\{S_N(\omega)\}$$
$$=(\pi D)^{-1}B^{D-1}A^{-2D}\sin(\pi D)|\tau|^{D-1} \tag{12}$$

for $A\ll|\tau|\ll B$.

For a stationary SRP $N(t)$, the mean rate of events is $\langle T\rangle^{-1}$, so that $E\{N(t)\}=t/\langle T\rangle$, where $E\{\ \}$ represents averaging taken over the realizations of the process. Using Fourier and Z transforms, we obtain expressions for a type of factorial moment, which in the range $A\ll t\ll B$, leads to

$$E\left\{\frac{[N(t)+k]!}{[N(t)-1]!}\right\}=E\{N(t)[N(t)+1]\cdots[N(t)+k]\}$$
$$=\langle T\rangle^{-1}(k+1)!\int_0^t(t-v)u^{*k}(v)dv$$
$$\sim\langle T\rangle^{-1}(k+1)!\int_0^t(t-v)v^{kD-1}dv$$
$$\sim t^{kD+1} \tag{13}$$

for the SFRP, where we set $1/(-1)!=0$ for completeness. From these factorial moments, all other moments and cumulants may be calculated. In particular, substituting $k=1$ leads to $\mathrm{Var}\{N(t)\}\sim t^{D+1}$.

A different point process will result when several SFRP's are superposed, but the overall power spectral density still will be $1/f^D$. Consider M identical and independent SFRP's, where $G_{N,nm}(\tau)$ represents the coincidence rates between two individual SFRP's indexed by n and m, respectively. The total coincidence rate $G_{N,T}(\tau)$, due to all M individual SFRP's, is given by

$$G_{N,T}(\tau)=\sum_n\sum_m G_{N,nm}(\tau)$$
$$=\sum_n\sum_{m\neq n} G_{N,nm}(\tau)+\sum_n G_{N,nn}(\tau)$$
$$=\sum_n\sum_{m\neq n}\langle T\rangle^{-2}+\sum_n G_N(\tau)$$
$$=(M^2-M)\langle T\rangle^{-2}+MG_N(\tau). \tag{14}$$

As the number M of independent SFRP's increases, the impulsive term grows as M^2-M while the smooth term increases only as M; thus for large M the impulsive term dominates, as it would for most point processes with high rates. The corresponding power spectral density is then

$$S_{N,T}(\omega)=(M^2-M)\langle T\rangle^{-2}\delta(\omega/2\pi)+MS_N(\omega)\ , \qquad (15)$$

which exhibits a $1/f^D$ character in the range $B^{-1}\ll|\omega|\ll A^{-1}$ independent of M.

For stationary (equilibrium) SFRP's constructed from infinite-tail power-law distributions with $D\leq 1$, the mean interevent time is infinite, and with probability 1 no events will be observed in any finite interval. However, a segment of a SRP that begins just after an event may contain a positive number of events. For the abrupt-cutoff power-law density, if the outer cutoff B is set to infinity, then the probability of observing no events in a segment of length C can still be made vanishingly small as the ratio C/A increases, where A is the inner cutoff. Thus when a SFRP with a zero mean rate begins on an event, the resulting process has a nonzero effective rate for all finite times. Therefore, any experiment will, of necessity, measure a process with positive expected rate, and the results derived above also will apply to that process.

An alternating fractal renewal process (AFRP) $X(t)$ may be constructed from the same abrupt-cutoff power-law interevent-time density. This process has two states, and switches alternately between $X=0$ and $X=1$ at times identical to the events in the SFRP. The event number statistics are thus identical to those of the SFRP, but other measures exhibit new behavior. In particular, the power spectral density for a stationary alternating renewal process is given by[4,5]

$$S_X(\omega)=\tfrac{1}{2}\delta(\omega/2\pi)+\langle T\rangle^{-1}\omega^{-2}\,\mathrm{Re}\left\{\frac{1-Q(j\omega)}{1+Q(j\omega)}\right\}\ . \qquad (16)$$

Substituting the fractal interevent-time probability density function, and considering the medium-frequency limit $B^{-1}\ll\omega\ll A^{-1}$, leads to the following result for the AFRP:

$$S_X(\omega)\to(4\langle T\rangle)^{-1}\times\begin{cases}2\Gamma(1-D)\cos(\pi D/2)A^D\omega^{D-2}\ , & 0<D<1\\ \pi A\omega^{-1}\ , & D=1\\ 2(D-1)^{-1}\Gamma(2-D)[-\cos(\pi D/2)]A^D\omega^{D-2}\ , & 1<D<2\\ 2A^2[-\ln(\omega A)]\ , & D=2\\ D(D-2)^{-1}A^2\ , & D>2\ .\end{cases} \qquad (17)$$

The AFRP generates $1/f^D$ noise in the full range $0<D<2$ over a substantial range of frequencies $B^{-1}\ll\omega=2\pi f\ll A^{-1}$, that may be made as large as desired. Figure 2 shows the power spectral density, normalized to unity at the low-frequency limit, for three choices of D. Geometrically, a realization of the AFRP consists of points in the plane rather than on a line, and thus the associated fractal dimension may extend to two. The SFRP and other point processes in $\mathbb{R}^1$, by definition, must have a dimension no greater than unity. Finally, a superposition of a number of AFRP's will converge to a Gaussian process with power spectral density given by a scaled version of Eq. (17).

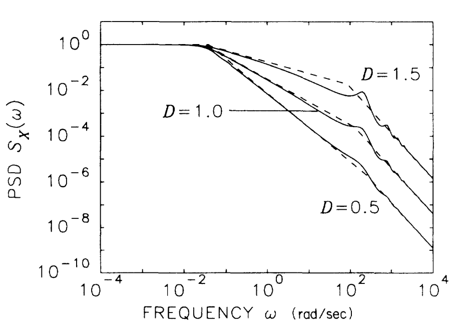

FIG. 2. Double logarithmic plot of the normalized power spectral density for the AFRP, with an abrupt-cutoff power-law probability density, for three values of the exponent D: 0.5, 1.0, and 1.5 ($A=10^{-2}$, $B=10^{2}$). Asymptotic forms are included for comparison. The abrupt cutoff in the interevent-time probability density function gives rise to oscillations in the frequency domain.

The multiple trapping model, as developed by Orenstein, Kastner, and Vaninov,[6] Kastner,[7] and Tiedje and Rose,[8] shows how exponentially distributed traps over a large range of energies lead to a power-law decay of current in an amorphous semiconductor. If a pulse of light strikes such a semiconductor, the many carriers excited out of their traps will be available to carry current until they are recaptured by a trap, which happens relatively quickly. At some point each carrier will be released from its trap by thermal excitation and become mobile for a time, and then be recaptured by another trap. For exponentially distributed states with identical capture cross sections, the electrons tend to be trapped in shallow states at first, but the probability of being caught in a deep trap increases as time progresses. This leads to a current that decreases as a power-law function of time.

The multiple trapping model may be usefully recast in terms of the SFRP. Consider an amorphous semiconductor with localized states (traps) that are exponentially distributed with parameter E_0 between a minimum energy E_L of the order of $k\mathcal{T}$, where k is Boltzmann's constant and $\mathcal{T}$ is the absolute temperature; and a maximum energy E_H determined by the band gap. For a particular trap with energy $\mathcal{E}$, the corresponding mean waiting time is

$$\tau=\tau_0\exp(\mathcal{E}/k\mathcal{T})\ , \qquad (18)$$

where τ_0 is the average vibrational period of the atoms in the semiconductor. If we define characteristic time cutoffs $A \equiv \tau_0 \exp(E_L/k\mathcal{T})$ and $B \equiv \tau_0 \exp(E_H/k\mathcal{T})$, and the power-law exponent $D \equiv k\mathcal{T}/E_0$, then the mean waiting time τ has a power-law density identical to that in Eq. (1). Each trap holds carriers for times that are exponentially distributed given the conditional parameter τ, and averaging this exponential density over all possible values of τ yields the unconditional trapping time density, which is itself approximately power law

$$p(t) \approx D\Gamma(D+1)A^{D}t^{-(D+1)}, \tag{19}$$

for $A \ll t \ll B$. Thus each carrier will be trapped for a period that is essentially power-law distributed.

Upon escaping from a trap, the carrier can conduct current for a relatively short time until it is again captured by another trap. Thus each carrier executes a series of current-carrying jumps well described by a marked version of the SFRP. If the experiment begins with a pulse of light, then the SFRP begins on an event; if the experiment begins after the transients have died out, then the SFRP is in equilibrium. Assuming that each carrier acts independently of the others, the action of the carriers as a whole can be modeled as the superposition of SFRP's. In particular, the steady-state current should behave as impulsive $1/f^D$ noise. Indeed, experimental[9] and theoretical[10] results show precisely this type of frequency dependence, with $0 < D < 1$.

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