Entanglement, mixedness, and spin-flip symmetry in multiple-qubit systems

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A relationship between a recently introduced multipartite entanglement measure, state mixedness, and spin-flip symmetry is established for any finite number of qubits. It is also shown that, for those classes of states invariant under the spin-flip transformation, there is a complementarity relation between multipartite entanglement and mixedness. A number of example classes of multiple-qubit systems are studied in light of this relationship.

I. INTRODUCTION

State entanglement and mixedness are properties central to quantum information theory. It is therefore important, wherever possible, to relate them. The relationship between these quantities has been previously investigated for the simplest case, that of two-qubits (see, for example, Refs. [1–4]), but has been much less well studied for multiple-qubit states because multipartite entanglement measures [5,6] have only been recently given in explicit form (see, for example, Refs. [7–10]). Here, we find a relationship, for any finite number of qubits, between a multipartite entanglement measure and state mixedness, through the introduction of a multiple-qubit measure of symmetry under the spin-flip transformation. Here, several classes of two, three, and four qubit states are examined to illustrate this relationship. Since quantum decoherence affects the mixedness and entanglement properties of multiple-qubit states, our results provide a tool for investigating decoherence phenomena in quantum information processing applications, which utilize just such states.

The purity $\mathcal{P}$ of a general quantum state can be given in all cases by the trace of the square of the density matrix ρ and the mixedness M by its complement:

$$\mathcal{P}(\rho) = \mathrm{Tr}\, \rho^2,$$

$$M(\rho) = 1 - \mathrm{Tr}\, \rho^2 = 1 - \mathcal{P}(\rho). \qquad (1)$$

Entanglement can be captured in several ways, which may or may not be directly related, with varying degrees of coarseness (see, for example, Ref. [9]), depending on the complexity of the system described (see, for example, Refs. [11–14]). Here, we will measure entanglement by a recently introduced measure of multipartite entanglement, the $\mathrm{SL}(2,\mathbb{C})^{\times n}$-invariant quantity

$$S_{(n)}^2 = \mathrm{Tr}(\rho\widetilde{\rho}), \qquad (2)$$

where the tilde indicates the spin-flip operation [7]. The multiple qubit spin-flip operation is defined as

$$\rho \to \widetilde{\rho} \equiv \sigma_2^{\otimes n} \rho^* \sigma_2^{\otimes n}, \qquad (3)$$

where ρ^* is the complex conjugate of the n-qubit density matrix ρ and σ_2 is the spin-flipping Pauli matrix [15]. Its connection to the n-tangle, another multipartite entanglement measure, defined in Ref. [10], will be discussed below. We now discuss this measure in relation to the well-established, bipartite measures of entanglement [1].

In the simplest case of pure states of two qubits (A and B), entanglement has most commonly been described by the entropy of either of the one-qubit reduced density operators, which is obtained by tracing out the variables of one or the other qubits from the total system state described by the projector $P[|\Psi_{AB}\rangle] \equiv |\Psi_{AB}\rangle\langle\Psi_{AB}|$. For mixed two-qubit states ρ_{AB}, the entanglement of formation, E_f, is given by the minimum average marginal entropy of all possible decompositions of the state as a mixture of subensembles. Alternatively, one can use a simpler measure of entanglement, the concurrence C, to describe two-qubit entanglement [11]. For pure states, this quantity can be written as

$$C(|\Psi_{AB}\rangle) = |\langle\Psi_{AB}|\sigma_2\otimes\sigma_2|\Psi_{AB}^*\rangle| = |\langle\Psi_{AB}|\widetilde{\Psi}_{AB}\rangle|, \quad (4)$$

where $|\widetilde{\Psi}_{AB}\rangle$ is the spin-flipped state vector. It has been shown that the concurrence of a mixed two-qubit state, $C(\rho_{AB})$, can be expressed in terms of the minimum average pure-state concurrence $C(|\Psi_{AB}\rangle)$, where the minimum is taken over all possible ensemble decompositions of ρ_{AB} and that, in general, $C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$, where λ_i are the square roots of the eigenvalues of the product matrix $\rho\widetilde{\rho}$, the "singular values," all of which are non-negative real quantities [11]. It has also been shown that the entanglement of formation of a mixed state ρ of two qubits can be expressed in terms of the concurrence as

$$E_f(\rho) = h(C(\rho)), \qquad (5)$$

where $h(x) = -x\,\log_2 x - (1-x)\,\log_2(1-x)$ (see Ref. [11]).

Here, we measure multipartite entanglement involving n qubits by $S_{(n)}^2$ (see Ref. [7]), which is invariant under the group corresponding to stochastic local operations and classical communications (SLOCC) [16]. For two-qubit pure states, this measure (with $n=2$) coincides with the squared concurrence (or tangle):

$$S_{(2)}^2(P[|\psi\rangle]) = \tau_{(2)}(P[|\psi\rangle]) = C^2(P[|\psi\rangle]), \qquad (6)$$

where $P[|\psi\rangle] \equiv |\psi\rangle\langle\psi|$ is the projector corresponding to its state-vector argument $|\psi\rangle$.

In Sec. II, after providing another expression for $S_{(n)}^2$ for all values of n for both pure and mixed states, we show that for pure states this length coincides with the multipartite generalization of the tangle defined for pure states of any finite number of qubits, n, that is, the pure state n-tangle [10,8]. We discuss the geometrical properties of $S_{(n)}^2$ and of the purity $\mathcal{P}$ that allow us to find the general relationship between multipartite entanglement and mixedness in terms of a spin-flip symmetry measure. In Secs. III and IV, we examine a range of classes of two-, three-, and four-qubit states to illustrate the results of the Sec. II.

II. DEFINITIONS AND THE GENERAL CASE

The multiple-qubit SLOCC invariant given in Eq. (2) is most naturally defined in terms of n-qubit Stokes parameters [17,18],

$$S_{i_1 \ldots i_n} = \text{Tr}(\rho \sigma_{i_1} \otimes \cdots \otimes \sigma_{i_n}), \quad i_1, \ldots, i_n = 0,1,2,3, \tag{7}$$

where $\sigma_\mu^2 = 1$, $\mu = 0,1,2,3$, are the three Pauli matrices together with the identity $\sigma_0 = I_{2\times2}$, and $\frac{1}{2}\text{Tr}(\sigma_\mu \sigma_\nu) = \delta_{\mu\nu}$. These directly observable parameters form an n-particle generalized Stokes tensor $\{S_{i_1,\ldots,i_n}\}$. Under SLOCC transformations [5], density matrices undergo local transformations described by the group $\text{SL}(2,\mathbb{C})$, while the corresponding Stokes parameters undergo local transformations described by the isomorphic group $\text{O}_0(1,3)$, the proper Lorentz group [19], that leave the Minkowskian length unchanged. The Minkowskian squared-norm of the Stokes tensor $\{S_{i_1 \ldots i_n}\}$ provides this invariant length [7] (here renormalized by the factor 2^{-n} for convenience):

$$S_{(n)}^2 \equiv \frac{1}{2^n} \left\{ (S_{0\ldots0})^2 - \sum_{k=1}^n \sum_{i_k=1}^3 (S_{0\ldots i_k \ldots 0})^2 \right.$$
$$+ \sum_{k,l=1}^n \sum_{i_k,i_l=1}^3 (S_{0\ldots i_k \ldots i_l \ldots 0})^2$$
$$\left. - \cdots + (-1)^n \sum_{i_1,\ldots,i_n=1}^3 (S_{i_1 \ldots i_n})^2 \right\}. \tag{8}$$

As mentioned above, this quantity can be compactly expressed in terms of density matrices, as

$$S_{(n)}^2 = \text{Tr}(\rho_{12\ldots n} \tilde{\rho}_{12\ldots n}), \tag{9}$$

where $\rho_{12\ldots n}$ is the multiple-qubit density matrix and $\tilde{\rho}_{1\ldots n} = (\sigma_2^{\otimes n})\rho_{1\ldots n}^*(\sigma_2^{\otimes n})$ is the spin-flipped multiple-qubit density matrix. Here, we consider these positive, Hermitian, trace-one matrices as operators acting in the Hilbert space $(\mathcal{C}^2)^{\otimes n}$ of n qubits. For the remainder of this paper,

subscripts on these matrices either will be suppressed or replaced by more descriptive labels, the number n of qubits being otherwise specified.

Like this measure of entanglement, the state purity $\mathcal{P}$ is also naturally captured in terms of the generalized Stokes parameters. The subgroup of deterministic local operations and classical communications (LOCC) [20] on qubits, namely, the unitary group [SU(2)] of transformations on density matrices, corresponds to the subgroup of ordinary rotations [SO(3)] of Stokes parameters that preserve the Euclidean length derived from these parameters. The purity for a general n-qubit state is the Euclidean length in the space of multiple-qubit Stokes parameters:

$$\mathcal{P}(\rho) = \text{Tr}\,\rho^2 = \frac{1}{2^n} \sum_{i_1,\ldots,i_n=0}^3 S_{i_1 \ldots i_n}^2 \tag{10}$$

(see Ref. [7]).

The multiple-qubit state purity [Eq. (10)] and the entanglement [Eqs. (8) and (9)] can be related by the (renormalized) Hilbert-Schmidt distance in the space of density matrices (arising from the Frobenius norm [21]):

$$D_{\text{HS}}(\rho - \rho') \equiv \sqrt{\tfrac{1}{2}\text{Tr}[(\rho - \rho')^2]}. \tag{11}$$

In particular, the multipartite entanglement and mixedness are related by the square of the Hilbert-Schmidt distance between the state ρ and its corresponding spin-flipped counterpart $\tilde{\rho}$:

$$D_{\text{HS}}^2(\rho - \tilde{\rho}) = \tfrac{1}{2}\text{Tr}[(\rho - \tilde{\rho})^2] = \tfrac{1}{2}[\text{Tr}\,\rho^2 + \text{Tr}\,\tilde{\rho}^2 - 2\text{Tr}(\rho\tilde{\rho})]$$
$$= \text{Tr}\,\rho^2 - \text{Tr}(\rho\tilde{\rho}) = \mathcal{P}(\rho) - S_n^2(\rho). \tag{12}$$

Thus, we have the following relation between the chosen measure of multipartite state entanglement and the state purity:

$$S_n^2(\rho) + D_{\text{HS}}^2(\rho - \tilde{\rho}) = \mathcal{P}(\rho), \tag{13}$$

where $D_{\text{HS}}^2(\rho - \tilde{\rho})$ can be understood as a measure of distinguishability between the n-qubit state ρ and the corresponding spin-flipped state $\tilde{\rho}$ [as defined in Eq. (3)].

The relation of Eq. (13) can be recast as the following simple, entirely general, relation between multipartite entanglement $S_{(n)}^2(\rho)$ and mixedness $M(\rho) = 1 - \mathcal{P}(\rho)$:

$$S_{(n)}^2(\rho) + M(\rho) = I(\rho,\tilde{\rho}), \tag{14}$$

where we have introduced $I(\rho,\tilde{\rho}) \equiv 1 - D_{\text{HS}}^2(\rho - \tilde{\rho})$, which measures the indistinguishability of the density matrix ρ from the corresponding spin-flipped state $\tilde{\rho}$; $I(\rho,\tilde{\rho})$ is clearly also a measure of the spin-flip symmetry of the state.

For pure states, $M(\rho) = M(|\psi\rangle\langle\psi|) = 0$, and we have

$$S_{(n)}^2(|\psi\rangle\langle\psi|) = I(|\psi\rangle\langle\psi|, |\tilde{\psi}\rangle\langle\tilde{\psi}|). \tag{15}$$

In this case, the Minkowskian length $S_{(n)}^2(\rho)$ is also seen to be equal to the n-tangle $\tau_{(n)}$, as mentioned above:

$$S_{(n)}^2(|\psi\rangle\langle\psi|) = \mathrm{Tr}[(|\psi\rangle\langle\psi|)(|\widetilde{\psi}\rangle\langle\widetilde{\psi}|)] = \langle\psi|(|\widetilde{\psi}\rangle\langle\widetilde{\psi}|)|\psi\rangle$$

$$= |\langle\psi|\widetilde{\psi}\rangle|^2 = \tau_{(n)}, \qquad (16)$$

where $|\widetilde{\psi}\rangle \equiv \sigma_2^{\otimes n}|\psi^*\rangle$ and $\tau_{(n)} = |\langle\psi|\widetilde{\psi}\rangle|^2$ is the pure state n-tangle. Thus, we also see that for pure states

$$\tau_{(n)} = I(\rho,\widetilde{\rho}), \qquad (17)$$

that is, the n-tangle coincides with the degree of spin-flip symmetry.

From Eq. (14), one obtains an exact complementarity relation for those classes of n-qubit states, whether pure or mixed, for which $\rho = \widetilde{\rho}$ [and thus $I(\rho,\widetilde{\rho}) = 1$]:

$$S_{(n)}^2(\rho) + M(\rho) = 1, \qquad (18)$$

since then $D_{\mathrm{HS}}(\rho - \widetilde{\rho}) = 0$. For the special case of pure states $M(\rho) = 0$; this result can be understood as expressing the fact, familiar from the set of Bell states, that pure states of more than one qubit invariant under n-qubit spin flipping have full n-qubit entanglement. The Bell state $|\Psi^+\rangle$ is the most obvious example from this class. By contrast, for those states ρ that are fully *distinguishable* from their spin-flipped counterparts, $\widetilde{\rho}$, i.e., for which $I(\rho,\widetilde{\rho}) = 0$, one finds instead

$$S_{(n)}^2(\rho) + M(\rho) = 0, \qquad (19)$$

implying that *both* the entanglement *and* the mixedness are zero, $S_{(n)}^2 = 0$ and $M(\rho) = 0$, since both quantities are non-negative and sum to zero. An arbitrary n-fold tensor product of states $|0\rangle$ and $|1\rangle$ is an n-qubit example from this class. A third noteworthy case is that when the mixedness and indistinguishability are nonzero and equal: $M(\rho) = I(\rho,\widetilde{\rho})$. In that case the entanglement $S_{(n)}^2$ is obviously zero. The fully mixed n-qubit state (described by identity matrix normalized to trace unity) is an example from this last class.

III. TWO-QUBIT SYSTEMS

Relations between entanglement and mixedness have previously been found for limited classes of two-qubit states using the tangle $\tau_{(2)}$ as an entanglement measure [2–4]. In particular, for two important classes of states, the Werner states and the maximally entangled mixed states, it was found analytically that as mixedness increases, entanglement decreases [2]. By exploring more of the Hilbert space of two-qubit systems by numerical methods, it was also found that a range of other states exceed the ratio of entanglement to mixedness present in the class of Werner states including, in particular, the maximally entangled mixed states. We now use our results above to provide further insight into the relationship between entanglement and mixedness in two-qubit systems, before going on to examine larger multiqubit systems.

Mixtures of two Bell states,

$$\rho_2 = w P[|\Phi^+\rangle] + (1-w) P[|\Phi^-\rangle], \qquad (20)$$

where $w \in [0,1]$, and mixtures of three Bell states,

$$\rho_3 = w_1 P[|\Phi^+\rangle] + w_2 P[|\Phi^-\rangle] + w_3 P[|\Psi^+\rangle], \qquad (21)$$

where $w_i \in [0,1]$, and $w_1 + w_2 + w_3 = 1$, were both considered in Ref. [4]. Since both cases fall within the same larger class, the Bell-decomposable states

$$\rho_{BD} = w_1 P[|\Phi^+\rangle] + w_2 P[|\Phi^-\rangle] + w_3 P[|\Psi^+\rangle]$$
$$+ w_4 P[|\Psi^-\rangle], \qquad (22)$$

where $w_i \in [0,1]$ and $w_1 + w_2 + w_3 + w_4 = 1$, we consider them via this general case. For these states, the general relation, Eq. (14), tells us that, within this class, as the entanglement increases, the mixedness must decrease, as expressed by Eq. (18), since all of these states are symmetric under the spin-flip operation: $\rho = \widetilde{\rho}$. That is, $M(\rho_{BD}) = 1 - S_{(n)}^2(\rho_{BD})$. These states can be studied in particular cases by performing Bell-state analysis, say on qubit pairs within pure states of three particles, and can characterize, for example, the ensemble of states transmitted during a session of quantum communication using quantum dense coding. The Werner states [22], such as

$$\rho_{\mathrm{Werner}} = w P[|\Phi^+\rangle] + \frac{1-w}{4} I_2 \otimes I_2, \qquad (23)$$

which describe weighted mixtures of Bell states and fully mixed states, where $w \in [0,1]$, also have the spin-flip symmetry property, that is, $I(\rho_{\mathrm{Werner}}, \widetilde{\rho}_{\mathrm{Werner}}) = 1$. Because of their symmetry, all the above examples obey the exact complementarity relation, Eq. (18): entanglement and mixedness are seen to be strictly complementary. However, not all two-qubit states possess full symmetry under spin flips. Therefore, only the full three-way relation [Eq. (14)] will hold in general.

Let us now proceed further by considering two example classes where the state is *not* invariant under the spin-flip operation, one class of pure states and one class of mixed states. For the two-qubit generalized Schrödinger cat states defined by

$$|\Phi(\alpha)\rangle_{AB} = \alpha|00\rangle + \sqrt{1-\alpha^2}|11\rangle, \qquad (24)$$

where $\alpha \in [0,1]$, considered in Ref. [4], clearly $\widetilde{\rho} \neq \rho$ in general. For them, one has for the state distinguishability $D_{\mathrm{HS}}(\rho - \widetilde{\rho}) = |2\alpha^2 - 1|$, and for the indistinguishability, $I(\rho,\widetilde{\rho}) = 1 - D_{\mathrm{HS}}^2 = 4\alpha^2(1-\alpha^2)$. Since these states are pure, the mixedness $M = 0$ and the relation given by Eq. (14) (with $n = 2$) reduces to an expression for state entanglement [Eq. (15)], that is,

$$S_{(2)}^2(\rho) = \tau_{(2)} = 4\alpha^2(1-\alpha^2), \qquad (25)$$

in accordance with Eq. (17). This shows the entanglement (in this case coinciding with the tangle $\tau_{(2)}$) to be parametrized by α^2, reaching its maximum at the maximally entangled

pure state of this class, $|\Psi^+\rangle$, as it should. We will see below that similar expressions obtain for larger integral-n Schrödinger cat states.

Now consider the class of two-qubit "maximally entangled mixed states," as defined in Ref. [3] as the states possessing the greatest possible amount of entanglement among those states having a given degree of purity. These can be written as

$$\rho_{\text{mems}}(\gamma) = \frac{1}{2}[2g(\gamma) + \gamma]P[|\Phi^+\rangle] + \frac{1}{2}[2g(\gamma) - \gamma]P[|\Phi^-\rangle]$$
$$+ [1 - 2g(\gamma)]P[|01\rangle], \qquad (26)$$

where $g(\gamma) = \frac{1}{3}$ for $\frac{2}{3} > \gamma > 0$ and $g(\gamma) = \gamma/2$ for $\gamma \geq \frac{2}{3}$ [3]. For these states, we find that $I(\rho, \tilde{\rho}) = 1 - D^2_{\text{HS}}(\rho - \tilde{\rho}) = 4g(\gamma)[1 - g(\gamma)]$, so that

$$S^2_{(2)}(\rho_{\text{mems}}) + M(\rho_{\text{mems}}) = 4g(\gamma)[1 - g(\gamma)]. \qquad (27)$$

The formal similarity of the expressions for degree of symmetry $I(\rho, \tilde{\rho})$ in the above two cases, the generalized Schrödinger cat and maximally entangled mixed states, is noteworthy, with the degree of spin-flip symmetry of the state being of the same form but with different arguments, α^2 and $g(\gamma)$, respectively. This suggests that the maximally entangled mixed states of Eq. (26) could substitute for Schrödinger cat states in situations requiring such entanglement and symmetry, but where decoherence is unavoidable.

The above example classes have previously been used to explore the relationship between entanglement and mixedness of states for a given ability to violate the Bell inequality [3,4]. The behavior of these properties for Bell-decomposable states was previously shown [4] to differ in that context from that of the Werner and maximally entanglement mixed states [3]. Therefore, no general conclusion could be drawn about the relationship of entanglement and mixedness for the general two-qubit state for a given amount of Bell inequality violation. That is presumably because Bell inequality violation indicates a deviation from classical behavior (see, for example, Ref. [23]), rather than an inherent property of the state. The results here show instead that the degree of state symmetry governs the relationship of entanglement and mixedness.

IV. THREE-QUBIT AND FOUR-QUBIT SYSTEMS

Let us first use the relation of Eq. (14) with $n = 3$ to study three-qubit states. For pure states, this relation takes the special form given in Eq. (15) and provides the entanglement measure $S^2_{(3)}$ directly in terms of the spin-flip symmetry measure $I(\rho, \tilde{\rho})$. For the generalized W state with complex amplitudes,

$$|W^g\rangle = \alpha|100\rangle + \beta|010\rangle + \gamma|001\rangle, \qquad (28)$$

with $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$, using Eq. (15), we have

$$S^2_{(3)} = \text{Tr}(\rho_{W_g}\tilde{\rho}_{W_g}) = 0, \qquad (29)$$

$|W^g\rangle$ is manifestly distinguishable from $|\tilde{W}^g\rangle$, so the indistinguishability $I(\rho_{W_g}, \tilde{\rho}_{W_g})$ is clearly zero as well, in accordance with Eq. (14) with $M(\rho_{W_g}) = 0$. Since this three-qubit state is pure, this is an instance of Eq. (19), the uniquely three-qubit entanglement $\tau_{(3)}$ is seen to be zero. Furthermore, for the two-qubit subsystems of this three-qubit system, which are in mixed states, the entanglement measures for $n = 2$, $S^2_{(2)}$, calculated using their two-qubit reduced states, $\rho_{(AB)}, \rho_{(BC)}, \rho_{(AC)}$, are

$$S^2_{(AB)}(\rho_{(AB)}) = 4|\alpha|^2|\beta|^2 = C^2(\rho_{AB}),$$

$$S^2_{(BC)}(\rho_{(BC)}) = 4|\beta|^2|\gamma|^2 = C^2(\rho_{BC}),$$

$$S^2_{(AC)}(\rho_{(AC)}) = 4|\alpha|^2|\gamma|^2 = C^2(\rho_{AB}), \qquad (30)$$

each taking the value $\frac{4}{9}$ for the traditional W state [that is, the state of Eq. (28) with $\alpha = \beta = \gamma = \sqrt{\frac{1}{3}}$].

On the other hand, for generalized GHZ (or three-cat) states, defined by

$$|\text{GHZ}^g\rangle = \alpha|000\rangle + \sqrt{1 - \alpha^2}|111\rangle, \qquad (31)$$

we find the entanglement behaves oppositely, that is

$$S^2_{(3)} = \text{Tr}(\rho_{\text{GHZ}g}\tilde{\rho}_{\text{GHZ}g}) = 4|\alpha|^2(1 - |\alpha|^2) = \tau_{(3)} \qquad (32)$$

[from Eq. (16), and analogously for two-cat states], while $C^2_{(AB)} = C^2_{(BC)} = C^2_{(AC)} = 0$ for all values of α and $S^2_{(3)} = \tau_{(3)}$ can reach the maximum value, 1 [for the three-cat state with $\alpha = \sqrt{\frac{1}{2}}$, see Eq. (25)].

Now consider the following class of *mixed* three-qubit states:

$$\rho_{m3} = wP[|000\rangle] + (1 - w)P[|111\rangle], \qquad (33)$$

with $w \in [0,1]$. Being mixed, these states obey the general relation, Eq. (14), rather than its special case, Eq. (15). For these states the mixedness is

$$M(\rho_{m3}) = 1 - \text{Tr}(\rho^2_{m3}) = 2w(1 - w), \qquad (34)$$

and the multipartite entanglement measure takes the value

$$S^2_{(3)}(\rho_{m3}) = \text{Tr}(\rho_{m3}\tilde{\rho}_{m3}) = 2w(1 - w), \qquad (35)$$

as well. In accordance with Eq. (14), we also have

$$I(\rho_{m3}, \tilde{\rho}_{m3}) = 4w(1 - w). \qquad (36)$$

In this case, we see that both M and $S^2_{(3)}$ vary proportionally to the degree of spin-flip symmetry.

Note that the states ρ_{m3} can be viewed as three-qubit reduced states arrived at by partial tracing out one of the qubits from the class of generalized four-qubit Schrödinger cat pure states (taking $\alpha^2 \equiv w$),

$$|4\text{cat}^g\rangle = \alpha|0000\rangle + \sqrt{1 - \alpha^2}|1111\rangle, \qquad (37)$$

with $\alpha \in [0,1]$, for which the entanglement measure $S^2_{(4)}$ takes the values

$$S^2_{(4)} = \mathrm{Tr}(\rho_{4\mathrm{cat}g}\widetilde{\rho}_{4\mathrm{cat}g}) = 4\alpha^2(1-\alpha^2). \qquad (38)$$

Because these states are pure, the mixedness $M(\rho)$ is zero; we have that the four-qubit entanglement is the degree of spin-flip symmetry $S^2_{(4)} = \tau_{(4)}(\rho_{4\mathrm{cat}g}) = I(\rho_{4\mathrm{cat}g},\widetilde{\rho}_{4\mathrm{cat}g})$, and that $S^2_{(3)}(\rho_{m3}) = S^2_{(4)}(\rho_{4\mathrm{cat}g})/2 = \frac{1}{2}\tau_4$ is half that value, as is the three-qubit mixedness.

V. CONCLUSIONS

By considering multipartite entanglement and mixedness together with the degree of symmetry of quantum states under the n-qubit spin-flip transformation, a general relation was found between these fundamental properties. Multipartite entanglement, as described by a recently introduced measure, and state mixedness were seen to be complementary within classes of states possessing the same degree of spin-flip symmetry. For pure states, the value of this multipartite entanglement measure, the degree of spin-flip symmetry, and the n-tangle were seen to coincide. These results can be expected to be particularly useful for the study of quantum states used for quantum information processing in the presence of decoherence.

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