Effects of random deletion and additive noise on bunched and antibunched photon-counting statistics

M. C. Teich

Columbia Radiation Laboratory, Department of Electrical Engineering, Columbia University, New York, New York 10027

B. E. A. Saleh

Columbia Radiation Laboratory, Department of Electrical Engineering, Columbia University, New York, New York 10027, and Department of Electrical and Computer Engineering, University of Wisconsin, Madison, Wisconsin 53706

Received April 19, 1982

We examine the effect of Bernoulli random deletion and additive independent Poisson noise on photon-counting statistics. It is shown that under the action of such deletion and/or noise, both bunched and antibunched distributions move toward the Poisson distribution but never convert from bunched to antibunched or vice versa. Specific calculations are carried out for a number of examples of importance in optics.

Point processes often undergo random deletion. An obvious example of importance in optics is the usual case of photodetection, in which the quantum efficiency is invariably less than unity. Another case in point is optical absorption. It has long been known that the Poisson process, which is probably the most ubiquitous of all point processes, remains Poisson under the action of such Bernoulli selection.[1]

In this Letter, we explicitly consider the effects of random deletion on the photon-counting statistics for a number of cases that appear repeatedly in optics. The doubly stochastic Poisson point process (DSPP),[2] which always gives rise to a bunched counting distribution (count variance greater than count mean)[3] is treated in detail. The results for thermal light[4] and multiplied-Poisson light[5] are presented as special examples. Since antibunched light[6–8] appears to be playing an ever-increasingly important role in optics, we also treat several cases in which the count variance is less than the count mean. Finally we study the combined effects of random deletion and additive independent Poisson-noise counts.

Bernoulli Deletion

Consider a random number n of events, and let each event be multiplied by a discrete multiplication or reduction factor $x_k = 0, 1, 2, \ldots$ for $k = 1, 2, \ldots, n$. Then the total number of multiplied events is

$$m = \sum_{k=1}^{n} x_k. \tag{1}$$

If the multiplication factors $\{x_k\}$ are statistically independent, then the random variable m has a moment-generating function (mgf) $Q_m(s) = \langle \exp(-sm)\rangle$, which is related to the mgf's of n and x by[1]

$$Q_m(s) = Q_n(-\ln Q_x(s)). \tag{2}$$

The means and variances are related by

$$\langle m\rangle = \langle n\rangle\langle x\rangle \tag{3}$$

and

$$\mathrm{Var}(m) = \langle x\rangle^2\,\mathrm{Var}(n) + \langle n\rangle\,\mathrm{Var}(x). \tag{4}$$

Equation (4) is known as the Burgess variance theorem.[9,10] When the multiplication factors $\{x_k\}$ are Bernoulli distributed, i.e., $x_k = \{1, 0\}$ with probabilities $\{\eta, 1-\eta\}$,

$$Q_x(s) = 1 - \eta + \eta e^{-s},$$
$$\langle x\rangle = \eta, \qquad \mathrm{Var}(x) = \eta(1-\eta). \tag{5}$$

Substituting Eqs. (5) into Eqs. (2)–(4), we obtain

$$Q_m(s) = Q_n(-\ln(1-\eta+\eta e^{-s})),$$
$$\langle \mathrm{m}\rangle = \eta\langle \mathrm{n}\rangle,$$
$$\mathrm{Var}(m) = \eta^2\,\mathrm{Var}(n) + \eta(1-\eta)\langle n\rangle. \tag{6}$$

Equations (6) relate the properties of the number of counts after Bernoulli deletion to those before Bernoulli deletion, in terms of the deletion parameter η.

By repeated use of Eqs. (6), it can be easily shown that successive random deletions, with deletion parameters $\eta_1, \eta_2, \ldots$, are equivalent to a single process of random deletion with parameter $\eta = \eta_1\eta_2\ldots$.

Invariance of the Doubly Stochastic Poisson Point Process Counting Distribution

For a DSPP, the number of counts n, registered in the time interval $[0, T]$, has a moment-generating function[3]

$$Q_n(s) = \langle \exp[-(1-e^{-s})W]\rangle, \tag{7}$$

where W is the integrated random rate (or energy) driving the point process over the interval $[0, T]$. Using Eqs. (6), we conclude that, after random deletion, the number of counts m has a mgf

0146-9592/82/080365-03$1.00/0

$$Q_m(s) = \langle \exp[-(1-e^{-s})\eta W]\rangle. \quad (8)$$

Thus the process m remains a DSPP, but with a reduced integrated rate ηW. This is what is meant by the invariance of the DSPP counting distribution to Bernoulli selection. It is important to note that the distribution of m is not necessarily identical with that of n with a reduced mean. The structure of the relationship between the variance and the mean is changed, and the profile of the counting distribution may be significantly altered. This will be elucidated by a number of examples.

The negative-binomial distribution, which characterizes the photon-counting statistics of thermal (chaotic) light over a broad range of conditions,[4] is described by

$$Q_n(s) = \left(1 + \frac{\langle n\rangle}{M} - \frac{\langle n\rangle}{M} e^{-s}\right)^{-M},$$
$$\mathrm{Var}(n) = \langle n\rangle + \frac{\langle n\rangle^2}{M}, \quad (9)$$

where M is the degrees-of-freedom parameter.[3–5] By use of Eqs. (6), we obtain

$$Q_m(s) = \left(1 + \frac{\eta\langle n\rangle}{M} - \frac{\eta\langle n\rangle}{M} e^{-s}\right)^{-M},$$
$$\langle m\rangle = \eta\langle n\rangle,$$
$$\mathrm{Var}(m) = \langle m\rangle + \frac{\langle m\rangle^2}{M}, \quad (10)$$

so that the distribution is seen to remain negative binomial with the same degrees-of-freedom parameter M but with a reduced overall mean count $\langle m\rangle = \eta\langle n\rangle$.

The Neyman Type-A distribution is a DSPP counting distribution that arises when two Poisson point processes are multiplied. It provides a useful description for many processes, including cathodoluminescent emission, scintillation photon counting, radiography, and human vision at threshold.[11–13] It is described by

$$Q_n(s) = \exp\left(\frac{\langle n\rangle}{\alpha}\{\exp[\alpha(e^{-s}-1)]-1\}\right),$$
$$\mathrm{Var}(n) = \langle n\rangle(1+\alpha), \quad (11)$$

where α is the multiplication parameter. Equations (6) can be used to show that the Neyman Type-A distribution reemerges on random deletion but with a reduced mean $\eta\langle n\rangle$ *as well as* a reduced multiplication parameter $\eta\alpha$. Thus

$$\langle m\rangle = \eta\langle n\rangle, \qquad \mathrm{Var}(m) = \langle m\rangle(1+\eta\alpha). \quad (12)$$

Because Bernoulli selection alters the multiplication parameter in this case, it changes the overall shape of the distribution. This is in contrast to the result for the negative-binomial distribution considered earlier.

A more general distribution in which two Poisson processes are multiplied arises in the case of the shot-noise-driven doubly stochastic Poisson point process (SNDP).[14] In this case, random delay accompanies the multiplication, and

$$\mathrm{Var}(n) = \langle n\rangle(1+\alpha/\mathcal{M}). \quad (13)$$

The quantity $\mathcal{M}$ is the degrees-of-freedom parameter for shot-noise light.[5] The randomly deleted distribution has the properties

$$\langle m\rangle = \eta\langle n\rangle, \qquad \mathrm{Var}(m) = \langle m\rangle(1+\eta\alpha/\mathcal{M}), \quad (14)$$

i.e., a reduced mean and a reduced multiplication parameter but an unchanged number of degrees of freedom. We note that random deletion of the primary process (i.e., deletion before multiplication) simply reduces the overall mean count, however.

Sub-Poissonian Distributions

We now consider the effect of random deletion on several representative antibunched (sub-Poissonian) distributions. Consider the counting distribution

$$p(n) = \begin{cases} 1, & n = N \\ 0, & n \neq N \end{cases} \quad (15)$$

associated with the number state (Fock state) $|N\rangle$.[15] This corresponds to

$$Q_n(s) = \exp(-sN),$$
$$\langle n\rangle = N, \qquad \mathrm{Var}(n) = 0. \quad (16)$$

Again using Eqs. (6), we obtain

$$Q_m(s) = (1-\eta+\eta e^{-s})^N,$$
$$\langle m\rangle = \eta N,$$
$$\mathrm{Var}(m) = \eta(1-\eta)N = \langle m\rangle(1-\eta), \quad (17)$$

so that the randomly deleted number-state counting distribution is binomial.[15] It is clear from Eqs. (16) and (17) that both the number state and the binomial distributions are antibunched.

Note that, if a binomial distribution of parameters (η, N) is further Bernoulli selected with a deletion parameter η', the resultant distribution remains binomial, with parameters $(\eta\eta', N)$.

The nonparalyzable dead-time-modified Poisson counting distribution, which often arises in photon, nuclear, and neural counting, is sub-Poissonian. It has a mean and a variance given by[16]

$$\langle n\rangle = \bar{n}/(1+\bar{n}\epsilon), \qquad \mathrm{Var}(n) = \bar{n}/(1+\bar{n}\epsilon)^3, \quad (18)$$

where $\bar{n}$ is the unmodified mean count and ϵ is the ratio of the dead time τ_d to the counting time T. By use of Eqs. (6), we find that the mean and the variance of the Bernoulli-selected distribution are

$$\langle m\rangle = \eta\bar{n}/(1+\bar{n}\epsilon),$$
$$\mathrm{Var}(m) = \eta[\bar{n}/(1+\bar{n}\epsilon)^3] \times [1+(1-\eta)(2\bar{n}\epsilon+\bar{n}^2\epsilon^2)]. \quad (19)$$

It is apparent that random deletion increases the variance-to-mean ratio, thereby bringing it closer to (but never permitting it to exceed) the Poisson value of unity.

Permanence of the State of Bunching under Bernoulli Deletion and Additive Poisson Noise

The degree of photon bunching is determined by the Fano factor[14]

$$F_n = \mathrm{Var}(n)/\langle n\rangle. \quad (20)$$

If $F_n > 1$, the distribution is said to be bunched, overdispersed, or super-Poissonian; if $F_n < 1$, it is said to be antibunched, underdispersed, or sub-Poissonian. For the simple Poisson distribution, $F_n = 1$. Let us rewrite the Burgess variance formula [Eq. (4)] in the form

$$F_m = \langle x \rangle F_n + F_x. \tag{21}$$

For Bernoulli deletion

$$F_m = 1 + \eta(F_n - 1). \tag{22}$$

If we define

$$F_n = 1 + \Delta, \tag{23}$$

then Eq. (22) yields

$$F_m = \mathrm{Var}(m)/\langle m \rangle = 1 + \eta\Delta. \tag{24}$$

We now calculate the bunching properties for a counting distribution that is both Bernoulli selected, in the manner described above, and combined with additive independent Poisson noise counts. Two cases are considered: (1) when the Bernoulli selection occurs before the addition and (2) when the Bernoulli selection occurs after the addition.

Let the Poisson-noise-count mean be $\langle r \rangle$, in which case $\mathrm{Var}(r) = \langle r \rangle$. Combining this with Eqs. (6), we obtain for the signal-plus-noise counting random variable q in case (1)

$$\langle q \rangle = \eta\langle n \rangle + \langle r \rangle,$$
$$\mathrm{Var}(q) = \eta^2 \, \mathrm{Var}(n) + \eta(1 - \eta)\langle n \rangle + \langle r \rangle. \tag{25}$$

The overall Fano factor can then be written as

$$F_q = 1 + \eta(1 + \langle r \rangle/\eta\langle n \rangle)^{-1}(F_n - 1). \tag{26}$$

Recognizing that $\beta \equiv (1 + \langle r \rangle/\eta\langle n \rangle)^{-1}$ is always positive, and approaches zero as $\langle r \rangle$ increases, we find that Eq. (26) becomes

$$F_q = 1 + \eta\beta\Delta, \tag{27}$$

which is similar to Eq. (24) in character.

For case (2) the result is not greatly different; β is simply replaced by $\beta' \equiv (1 + \langle r \rangle/\langle n \rangle)^{-1}$.

We are now in a position to draw the following conclusions:

(1) If n is bunched ($\Delta > 0$), then, under the action of Bernoulli selection and/or the addition of Poisson noise counts, q remains bunched but with a reduced Fano factor.

(2) If n is antibunched ($\Delta < 0$), then, under the action of Bernoulli selection and/or the addition of Poisson noise counts, q remains antibunched but with an increased Fano factor.

Thus neither additive independent Poisson noise events *nor Bernoulli random deletion* alters the state of bunching of the light. Indeed, both effects are quite similar; increasing the amount of either drives the counting distribution toward Poisson. A moment's thought provides the reason: The deletion serves to reduce correlated (or anticorrelated) event occurrences, thereby bringing the distributions closer to the zero-memory Poisson distribution.

We refer to this as the permanence of the state of bunching under Bernoulli deletion and additive independent Poisson noise counts. The result is of particular importance in the current effort to produce antibunched light. It shows that although loss (e.g., low quantum efficiency) and additive Poisson noise serve to reduce the observability of the antibunched character of the light, they do not destroy it.

This research was supported by the Joint Services Electronics Program (U.S. Army, U.S. Navy, and U.S. Air Force) under contract DAAG29-82-K-0080 and by the National Science Foundation under grant ENG-78-26498.

References

1. E. Parzen, *Stochastic Processes* (Holden-Day, San Francisco, 1962).
2. D. R. Cox, "Some statistical methods connected with series of events," J. R. Stat. Soc. **B17**, 129–164 (1955).
3. B. E. A. Saleh, *Photoelectron Statistics* (Springer-Verlag, New York, 1978).
4. L. Mandel, "Fluctuations of photon beams: the distribution of the photo-electrons," Proc. Phys. Soc. **74**, 233–243 (1959).
5. M. C. Teich and B. E. A. Saleh, "Fluctuation properties of multiplied-Poisson light: measurement of the photon-counting distribution for radioluminescence radiation from glass," Phys. Rev. A **24**, 1651–1654 (1981).
6. R. J. Glauber, "The quantum theory of optical coherence," Phys. Rev. **130**, 2529–2539 (1963); "Coherent and incoherent states of the radiation field," Phys. Rev. **131**, 2766–2788 (1963).
7. D. Stoler, "Photon antibunching and possible ways to observe it," Phys. Rev. Lett. **33**, 1397–1400 (1974).
8. L. Mandel, "Sub-Poissonian photon statistics in resonance fluorescence," Opt. Lett. **4**, 205–207 (1979).
9. R. E. Burgess, "Homophase and heterophase fluctuations in semiconducting crystals," Discuss. Faraday Soc. **28**, 151–158 (1959).
10. L. Mandel, "Image fluctuations in cascade intensifiers," Brit. J. Appl. Phys. **10**, 233–234 (1959).
11. M. C. Teich, "Role of the doubly stochastic Neyman Type-A and Thomas counting distributions in photon detection," Appl. Opt. **20**, 2457–2467 (1981).
12. B. E. A. Saleh and M. C. Teich, "Statistical properties of a nonstationary Neyman–Scott cluster process," IEEE. J. Trans. Inf. Theory (submitted for publication).
13. M. C. Teich, P. R. Prucnal, G. Vannucci, M. E. Breton, and W. J. McGill, "Multiplication noise in the human visual system at threshold: 1. Quantum fluctuations and minimum detectable energy," J. Opt. Soc. Am. **72**, 419–431 (1982).
14. B. E. A. Saleh and M. C. Teich, "Multiplied-Poisson noise in pulse, particle, and photon detection," Proc. IEEE **70**, 229–245 (1982).
15. J. Peřina, *Coherence of Light* (Van Nostrand-Reinhold, London, 1972).
16. B. I. Cantor and M. C. Teich, "Dead-time-corrected photocounting distribution for laser radiation," J. Opt. Soc. Am. **65**, 786–791 (1975).