Optimum photon detection with a simple counting processor

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It is shown that, for an arbitrary discrete process embedded in independent additive discrete noise, the classical binary detection problem using a likelihood-ratio test reduces to a simple comparison of the number of events with a single threshold. Only a weak condition on the noise distribution is required. Our results are appropriate for the analysis of photocounting optical communications and photocounting radar systems as well as neural counting in auditory psychophysics. We specifically apply our method to a signal-detection theory model of the human visual system and draw a comparison to the analysis of Hecht, Shlaer, and Pirenne [J. Gen. Physiol. **25**, 819 (1942)].

The relevance of the statistical nature of light to detection by the human visual system was first recognized by Hecht, Shlaer, and Pirenne[1] (HSP) in 1942. These authors assumed that the photon-counting distribution arising from a thermal (chaotic) source of light was Poisson and explicitly considered a noiseless model of the visual system in which the number of quanta detected had to exceed a fixed threshold in order for a subject to perceive a dim flash of light. They were able to show that the frequency-of-seeing data obtained from psychophysical experiments conducted at the threshold of seeing were fit satisfactorily by cumulative Poisson curves, in accordance with their model. More recently, Teich and Prucnal[2] have shown that, although the proper photon-counting distribution for chaotic light is the negative binomial,[3,4] the Poisson is indeed the appropriate limit in the regime in which HSP operated their experiment.

Subsequent to the original work of HSP, a number of researchers recognized that there was inherent noisiness in the visual system[5–11] and advocated the use of signal detection theory (SDT) in visual psychophysics.[8–11] In contrast to the simple intuitive model proposed by HSP, however, most formulations using SDT in psychophysics are implausible, inasmuch as they assume that the observer carries out complex computations of the likelihood ratio based on sensory information and on noise statistics.[12] In this Letter we demonstrate that, for light of arbitrary statistical properties, the optimum counting processor prescribed by SDT leads to a simple threshold-comparison test of the kind first imagined by HSP. Only a weak condition is required on the equivalent noise distribution at the retina. The hallmark of our work is the use of discrete counting distributions; indeed our effort was inspired by a classic paper of McGill[13] in which a similar problem for the processing of neural counts in audition was considered. Our results extend the work of Barlow[10,11] and Sakitt,[14,15] who have previously emphasized the importance of internal (dark) noise and false-alarm rate in reconciling theory and experiment in the case of Poisson signal embedded in Poisson noise.

Though our discussion is couched primarily in the language of visual science, the results are quite general. In fact, they are applicable to the classical binary detection problem using a likelihood-ratio test for an arbitrary discrete counting process embedded in independent additive noise. They are therefore expected to be useful in the analysis of photocounting optical communications[16,17] and radar systems[18,19] and neural counting in auditory psychophysics.[13,20] We will consider the optical communications problem in greater detail elsewhere.

Let hypothesis "zero" (H_0) represent the situation in which no light is transmitted to the subject (noise only) and hypothesis "one" (H_1) represent the transmission to the subject of a pulse of light of specified characteristics (signal plus noise). We wish to determine the optimum rule for deciding which hypothesis is true on the basis of a single observation. This is a simple binary hypothesis-testing problem. It is often convenient to use the Neyman–Pearson criterion, which maximizes the probability of detection P_D with the probability of false alarm P_F constrained to a particular value α. The general solution is most readily obtained using the method of Lagrange multipliers[21] and yields the well-known likelihood-ratio test

$$\Lambda(n) = \frac{p(n|H_1)}{p(n|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda, \tag{1}$$

where $\Lambda(n)$ represents the likelihood ratio, $p(n|H_i)$ is the probability of obtaining n counts given that H_i is true, and λ is the threshold. The constraint $P_F = \alpha$ is satisfied by choosing λ appropriately. It should be noted that our results are also valid for other criteria that lead to the likelihood-ratio test (e.g., Bayes's).[21]

Equation (1) indicates that optimum processing can be implemented by using the received data n to compute $\Lambda(n)$, which is then compared with λ, indicating which decision is appropriate. In its existing form, the calculation of $\Lambda(n)$ may be rather involved. However, processing of the received data n can be greatly simplified if $\Lambda(n)$ is a monotonic nondecreasing function of n.

We use the notation $p_S(n)$ to represent the arbitrary (discrete) signal counting distribution, and $p_D(n)$ to represent the (discrete) noise counting distribution, where n is a nonnegative integer. The signal and noise processes are assumed to be independent and additive, so that the overall distribution resulting from signal plus noise is simply the convolution sum of the individual signal and noise distributions. The noise we consider (e.g., detector dark noise) is assumed not to interfere with the signal[17]; interfering superposed ratiation[20] is accounted for in the statistics of $p_S(n)$. The likelihood ratio, defined by Eq. (1), can then be explicitly written as

$$\Lambda(n) = \left[\sum_{k=0}^{n} p_S(k)p_D(n-k)\right] \Big/ p_D(n). \qquad (2)$$

Though it is possible to deal with the noise in the discrete form given above, it is somewhat easier to consider a continuous extension of the noise statistics. Let $f_D(x)$ be a continuous extension of $p_D(n)$ obtained by the substitutions $x = n$ and $\Gamma(x+1) = n!$, with x real and nonnegative, so that $f_D(n) = p_D(n)$ for n integer (other continuous extensions are also possible). We proceed to demonstrate that if the logarithm of $f_D(x)$ is concave downward, i.e., if

$$\mathrm{d}^2[\log f_D(x)]/\mathrm{d}x^2 \le 0, \qquad (3)$$

then the likelihood ratio $\Lambda(n)$ is monotonic nondecreasing. This condition is sufficient but not necessary.

From Eq. (3), it is clear that

$$\mathrm{d}[\log f_D(x)]/\mathrm{d}x \le \mathrm{d}[\log f_D(x-k)]/\mathrm{d}x, \qquad (4)$$

where $x \ge k \ge 0$. We consider only the case where k is an integer. If we compute the derivative of the logarithm and cross-multiply, Eq. (4) becomes

$$f_D(x)[\mathrm{d}f_D(x-k)/\mathrm{d}x] - f_D(x-k)[\mathrm{d}f_D(x)/\mathrm{d}x] \ge 0. \qquad (5)$$

Multiplying Eq. (5) by the nonnegative quantity $p_S(k)/f_D{}^2(x)$ leads to

$$\mathrm{d}[p_S(k)f_D(x-k)/f_D(x)]/\mathrm{d}x \ge 0, \qquad (6)$$

so that the quantity in square brackets in Eq. (6) is a nondecreasing function of x. We now use Eq. (2) to obtain the likelihood-ratio difference

$$\Lambda(n+1) - \Lambda(n) = p_S(n+1)p_D(0)/p_D(n+1) + \sum_{k=0}^{n} p_S(k)p_D(n+1-k)/p_D(n+1) - \sum_{k=0}^{n} p_S(k)p_D(n-k)/p_D(n), \qquad (7)$$

which is equivalent to

$$\Lambda(n+1) - \Lambda(n) = p_S(n+1)p_D(0)/p_D(n+1) + \sum_{k=0}^{n} [p_S(k)f_D(n+1-k)/f_D(n+1) - p_S(k)f_D(n-k)/f_D(n)]. \qquad (8)$$

The first term on the right-hand side of Eq. (8) is clearly nonnegative. Furthermore, since each of the n terms in the summation is nonnegative [see the remark following Eq. (6)], then

$$\Lambda(n+1) - \Lambda(n) \ge 0, \qquad (9)$$

which indicates that $\Lambda(n)$ is monotonic nondecreasing. The monotonicity of $\Lambda(n)$ implies that for each value of λ in Eq. (1) there exists a unique integer n_t such that the test

$$n \underset{H_0}{\overset{H_1}{\gtrless}} n_t \qquad (10)$$

is equivalent to the test specified in Eq. (1). Therefore, if the condition of Eq. (3) is satisfied, optimum processing can be implemented by the simple test given in Eq. (10), which defines a simple counting processor. Here n is called a proper decision variable.[22] Thus, in contrast to the usual *ad hoc* procedures[22] that can become rather cumbersome for complex signal distributions, Eq. (3) provides a standard sufficiency condition, *independent of the signal distribution,* for determining whether n is a proper decision variable.

Let us now consider the specific signal and noise statistics for an HSP-type experiment. For chaotic radiation (e.g., incandescent, gas discharge, LED, below-threshold laser, or fluorescent light) such as that used by HSP, the signal distribution arising from the presence of the stimulus alone is given approximately by the negative binomial photon-counting distribution,[2–4]

$$p_S(n) = \frac{\Gamma(n+M)}{n!\Gamma(M)}\left(1+\frac{M}{\langle n_0\rangle}\right)^{-n} \times \left(1+\frac{\langle n_0\rangle}{M}\right)^{-M}. \qquad (11)$$

Here $p_S(n)$ represents the probability that exactly n photons are detected during the sampling time T,[4,19] and $\langle n_0\rangle$ represents the mean number of quanta (or mean light energy in units of $h\nu$) detected in this time. The parameter M is the number of modes ($M \ge 1$); it contains information relative to the spatiotemporal coherence and polarization properties of the light, the flash duration and area, and the detector integration time and area.[2]

The equivalent noise distribution, which may be considered to arise from spontaneous counts indistinguishable from quantum absorptions, is considered always to be present. Following Barlow,[10] we assume that the noise-counting distribution is Poisson with mean $\langle n_D\rangle$,

$$p_D(n) = p(n|H_0) = \langle n_D\rangle^n \exp(-\langle n_D\rangle)/n!; \qquad (12)$$

other noise distributions can be dealt with as easily.

Using Eqs. (2), (11), (12), and some rather complex algebra, we can demonstrate by means of a direct calculation that n is a proper decision variable in this particular case.[22,23] We can obtain the same result more simply, however, by dealing with the noise distribution alone in the manner described above. The continuous extension of Eq. (12) is $f_D(x) = \langle n_D\rangle^x \times \exp(-\langle n_D\rangle)/\Gamma(x+1)$, $x \ge 0$. Using the series representation for $\mathrm{d}^2[\ln \Gamma(x+1)]/\mathrm{d}x^2$ provided by Gradshteyn and Ryzhik,[24] we obtain

$$d^2[\ln f_D(x)]/dx^2 = -\sum_{n=1}^{\infty}(x+n)^{-2} < 0, \tag{13}$$

thereby satisfying Eq. (3). It is important to note that Eq. (11) for the signal distribution did not enter our calculation, indicating the general nature of our method. Thus the optimum processor for chaotic radiation embedded in Poisson noise is the simple counting processor given by Eq. (10). Here the threshold count n_t is chosen to satisfy the constraint on P_F and may be obtained as the smallest-integer solution to

$$\begin{aligned} P_F &= \sum_{n=n_t}^{\infty} p(n|H_0) \\ &= \exp(-\langle n_D\rangle)\sum_{n=n_t}^{\infty}\langle n_D\rangle^n/n! \le \alpha. \end{aligned} \tag{14}$$

Once n_t is determined, we may compute the probability of detection (psychometric function) P_D from the relation

$$\begin{aligned} P_D &= \sum_{n=n_t}^{\infty} p(n|H_1) \\ &= \sum_{n=n_t}^{\infty}\sum_{k=0}^{n}\left[\frac{\langle n_D\rangle^{n-k}\exp(-\langle n_D\rangle)}{(n-k)!}\right. \\ &\quad \left.\times\frac{\Gamma(k+M)}{k!\Gamma(M)}\left(1+\frac{M}{\langle n_0\rangle}\right)^{-k}\left(1+\frac{\langle n_0\rangle}{M}\right)^{-M}\right]. \end{aligned} \tag{15}$$

A psychophysical experiment to examine the validity of Eqs. (14) and (15) could be simply realized by using a single-mode gas laser below threshold[25,26] or a pseudothermal light generator,[27,28] both of which are chaotic sources. In particular, dramatically different psychometric functions would be expected as M is changed with P_F constant.[29] Equations (14) and (15) are also applicable to certain photocounting optical communications[16,17] and optical radar systems[18,19] and to neural counting in auditory psychophysics.[13,20]

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