INDEPENDENT PHOTON DELETIONS FROM QUANTIZED BOSON FIELDS: THE QUANTUM ANALOG OF THE BURGESS VARIANCE THEOREM

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It is demonstrated that if two boson fields are related by a process of independent random photon deletion, their moment-generating functions are related by an equation identical to that derived using classical arguments. A quantum analog of the Burgess variance theorem is recovered. The results confirm that the super- or sub-poissonian nature of a light beam is conserved, under the influence of independent Bernoulli deletions and/or additive independent Poisson noise. The identity results from the correspondence between classical and normally ordered correlation functions, for which vacuum fluctuations play no role.

The Burgess variance theorem plays an important role in connection with stochastic treatments of various counting processes [1–4]. Consider a random number n of events, and let each event be multiplied by a discrete multiplication factor $x_k = 0, 1, 2, \ldots$ for $k = 1, 2, \ldots, n$. Then the total number of multiplied events is $m = \Sigma_{k=1}^{n} x_k$. If the multiplication factors $\{x_k\}$ are statistically independent, then the random variable m has a moment-generating function $\langle \exp(-\lambda m)\rangle$ which is related to the moment-generating functions of n and x by

$$\langle e^{-\lambda m}\rangle = \langle\langle e^{-\lambda x}\rangle^n\rangle = \langle \exp(n \ln \langle e^{-\lambda x}\rangle)\rangle. \tag{1}$$

If, moreover, the multiplication factors $\{x_k\}$ are Bernoulli distributed ($x_k = \{1, 0\}$ with probabilities $\{\eta, 1-\eta\}$) we easily obtain [3]

$$\langle e^{-\lambda m}\rangle = \langle \exp\{n \ln [1-\eta(1-e^{-\lambda})]\}\rangle$$
$$= \langle [1-\eta(1-e^{-\lambda})]^n\rangle. \tag{2}$$

From eq. (2), it is clear that

$$\langle m\rangle = \left.\frac{\partial\langle e^{-\lambda m}\rangle}{\partial(-\lambda)}\right|_{\lambda=0} = \eta\langle n\rangle, \tag{3a}$$

$$\langle m^2\rangle = \left.\frac{\partial^2\langle e^{-\lambda m}\rangle}{\partial(-\lambda)^2}\right|_{\lambda=0} = \eta(1-\eta)\langle n\rangle + \eta^2\langle n^2\rangle, \tag{3b}$$

and

$$\langle(\Delta m)^2\rangle = \eta(1-\eta)\langle n\rangle + \eta^2\langle(\Delta n)^2\rangle, \tag{3c}$$

which is the classical Burgess variance theorem.

The purpose of this letter is to show that a quantum analog of the Burgess theorem is a simple consequence of the definitions of the number operator and its moments for quantized boson fields, and their relation to normally ordered moments, corresponding to classical quantities (there is no contribution from the vacuum fluctuations).

Consider a volume V whose linear dimensions are much larger than the wavelength of the radiation. Define vector detection boson operators $\hat{A}(x,t)$ and $\hat{B}(x,t)$ at a point x and time t, and the corresponding number operators, as defined in [5,6]

$$\hat{n}_{Vt} = \int_V \hat{B}^\dagger(x,t)\cdot\hat{B}(x,t)\,d^3x, \tag{4a}$$

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$$\hat{m}_{Vt} = \int_V \hat{A}^\dagger(x,t)\cdot\hat{A}(x,t)\,\mathrm{d}^3x. \tag{4b}$$

For simplicity, quasimonochromatic fields are assumed [6]. Now if the space–time regions (x', t') and (V, t) are conjoint [5,7] (they can be connected by a light beam), then

$$[\hat{A}(x',t'), \hat{m}_{Vt}] = \hat{A}(x',t'),$$

$$[\hat{m}_{Vt}, \hat{A}^\dagger(x',t')] = \hat{A}^\dagger(x',t'), \tag{5}$$

in analogy to the one-mode case; similarly for $\hat{V}$ and $\hat{n}_{Vt}$. Assume further that the $\hat{n}$- and $\hat{m}$-processes are related in terms of their normal moments by an efficiency η, such that

$$\langle{:}m^k_{Vt}{:}\rangle = \eta^k\langle{:}n^k_{Vt}{:}\rangle. \tag{6}$$

Here, normal ordering is denoted by $::$. This relation is the quantum analog of the corresponding relation for the classical intensity moments. Using (6), together with (4) and (5), it simply follows that [5,7] (sec. 14.2)

$$\langle\exp(-\lambda\hat{m}_{Vt})\rangle = \langle{:}\exp[-\hat{m}_{Vt}(1-\mathrm{e}^{-\lambda})]{:}\rangle$$

$$= \langle{:}\exp[-\eta\hat{n}_{Vt}(1-\mathrm{e}^{-\lambda})]{:}\rangle$$

$$= \langle[1-\eta(1-\mathrm{e}^{-\lambda})]^{n_{Vt}}\rangle, \tag{7}$$

where we have expanded the normal generating functions in a Taylor Series and made use of the relation (e.g., [7], Sec. 14.2)

$$\langle{:}\exp(-\lambda\hat{n}_{Vt}){:}\rangle = \langle(1-\lambda)^{\hat{n}_{Vt}}\rangle. \tag{8}$$

Eq. (7) is precisely the quantum analog of (the classical) eq. (2). Using the commutation rules (5), this leads directly to the relations between the moments, and to the quantum analog of the Burgess variance theorem (3c):

$$\langle\hat{m}\rangle_{Vt} = \eta\langle\hat{n}_{Vt}\rangle, \tag{9a}$$

$$\langle\hat{m}^2_{Vt}\rangle = \eta\langle\hat{n}_{Vt}\rangle + \eta^2\langle{:}\hat{n}^2_{Vt}{:}\rangle, \tag{9b}$$

$$\langle(\Delta\hat{m}_{Vt})^2\rangle = \eta\langle\hat{n}_{Vt}\rangle + \eta^2[\langle{:}\hat{n}^2_{Vt}{:}\rangle - \langle\hat{n}_{Vt}\rangle^2]$$

$$= \eta(1-\eta)\langle\hat{n}_{Vt}\rangle + \eta^2\langle(\Delta\hat{n}_{Vt})^2\rangle. \tag{9c}$$

Consequently,

$$[\langle(\Delta\hat{m}_{Vt})^2\rangle/\langle\hat{m}_{Vt}\rangle - 1] = \eta[\langle(\Delta\hat{n}_{Vt})^2\rangle/\langle\hat{n}_{Vt}\rangle - 1], \tag{9d}$$

as is obtained classically [4]. Conversely, if eq. (7) is obeyed, then the relation for independent photon deletions (eq. (6)) follows.

From (7), we can also obtain the corresponding relations between the photon-number probability distributions $p(m_{Vt})$ and $P(n_{Vt})$, and between the moments $\langle\hat{m}^k_{Vt}\rangle$ and $\langle\hat{n}^j_{Vt}\rangle$, since

$$p(m_{Vt}) = \frac{1}{m_{Vt}!}\left.\frac{\partial^{m_{Vt}}\langle\mathrm{e}^{-\lambda\hat{m}_{Vt}}\rangle}{\partial(-\lambda)^{m_{Vt}}}\right|_{\lambda=1}, \tag{10a}$$

and

$$\langle\hat{m}^k_{Vt}\rangle = \left.\frac{\partial^k}{\partial(-\lambda)^k}\langle\mathrm{e}^{-\lambda\hat{m}_{Vt}}\rangle\right|_{\lambda=0}. \tag{10b}$$

In this way we show that the m photon-number distribution is a binomially deleted version of the n photon-number distribution:

$$p(m_{Vt}) = \sum_{n_{Vt}=m_{Vt}}^{\infty}\binom{n_{Vt}}{m_{Vt}}\eta^{m_{Vt}}(1-\eta)^{n_{Vt}-m_{Vt}}P(n_{Vt}). \tag{11}$$

Furthermore,

$$\langle\hat{m}^k_{Vt}\rangle = \sum_{j=1}^{k} a_{k-j}\eta^j\langle\hat{n}_{Vt}(\hat{n}_{Vt}-1)\ldots(\hat{n}_{Vt}-j+1)\rangle, \tag{12a}$$

where

$$a_{k-j} = (-1)^j\sum_{s=1}^{j}(-1)^s\frac{s^k}{s!(j-s)!}. \tag{12b}$$

Of course, for $\eta = 1$, we recover $\langle\hat{m}^k_{Vt}\rangle = \langle\hat{n}^k_{Vt}\rangle$.

Note that (6) is typical for bilinear interactions of boson quantum systems in the rotating-wave approximation, which lead to Heisenberg-Langevin equations involving only annihilation operators. Such interactions leave the initial statistics of the system unchanged (in particular, a coherent initial state remains coherent). The results derived here are also applicable for interactions in which photons interact with electrons and atoms whose fermion properties play no role. Of course, if the photon deletions are not independent (as in a multiphoton interaction, for example), the results presented here do not apply.

More generally, an additional noise field $\hat{C}(x, t)$ may also be present, in which case

$$\hat{A}(x, t) = \alpha\hat{B}(x, t) + \hat{C}(x, t), \qquad (13)$$

where α is an amplification or attenuation factor. We then obtain

$$\langle(\Delta\hat{m}_{Vt})^2\rangle = \eta(1-\eta)\langle\hat{n}_{Vt}\rangle + \eta^2\langle(\Delta\hat{n}_{Vt})^2\rangle + \langle\hat{r}\rangle, \qquad (14)$$

with $\eta = |\alpha|^2$, under the assumption that the number of degrees of freedom is large (so that Var $\int_V \hat{C}^\dagger \cdot \hat{C} \times d^3x \approx \langle\int_V \hat{C}^\dagger \cdot \hat{C}\, d^3x\rangle = \langle\hat{r}\rangle$, and the interference component proportional to $\int_V \langle\hat{B}^\dagger \cdot \hat{B}\rangle\langle\hat{C}^\dagger \cdot \hat{C}\rangle d^3x$ is negligible). Eq. (14) is the quantum analog of the generalized classical Burgess theorem, in which independent additive Poisson noise counts have been included (see [3], eq. (25)).

The calculations carried out here demonstrate explicitly that the conclusions reached by classical arguments [3] hold for boson quantum fields. In particular, the super- or sub-poissonian nature of a light beam is conserved under the influence of Bernoulli deletion and/or additive independent Poisson noise.

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