EFFECTS OF RATE VARIATION ON THE COUNTING STATISTICS OF DEAD-TIME-MODIFIED POISSON PROCESSES*

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Expressions are obtained for the mean and variance of the number of events in a fixed sampling time for a nonparalyzable dead-time counter. The input process is assumed to be Poisson with a rate that is a known function of time. The mean and variance are shown to depend explicitly on the details of the rate variation during the sampling time; by contrast, in the absence of dead time the mean and variance are uniquely determined by the statistics of the rate integrated over the sampling time (total energy). Experiments performed with triangularly and sinusoidally modulated laser radiation provide results that are in accord with the theory.

1. Introduction

The probability distribution for a dead-time-modified pulse counter [1,2] has been studied by a number of researchers in a broad variety of disciplines such as photon counting [3–5], nuclear counting [6–8], and neural counting [5,9,10]. Many cases have been studied in detail including paralyzable and nonparalyzable counting under blocked, unblocked, and equilibrium conditions. Attention has also been given to the variable dead time case. Müller has recently compiled a comprehensive bibliography on dead time effects [11], and has summarized the results of a number of authors [7,8].

Though most of the work cited above is applicable only when the input to the counter is a Poisson point process with constant rate, a few results are also available for the case where the rate is not constant. Cantor and Teich [4], Teich and McGill [5], and Bédard [3] present expressions for the photon counting distribution when the intensity of the light is a random process, with the sampling interval much smaller than the coherence time of the light.

In this paper we find general expressions for the dead-time-modified mean and variance, for a nonparalyzable counter, when the rate of the input process is an arbitrary function of time, under the constraint that it vary slowly with respect to the duration of the dead time. No constraints on the length of the sampling interval are imposed. Of course, the results also apply to a spatial dead-time-modified process under the appropriate conditions.

2. Dead-time-modified mean

Consider a Poisson counting process whose instantaneous rate is a known function of time that we denote $\lambda(t)$ $[\lambda(t) \geqslant 0]$. The probability of n pulses occurring in the interval (t_1, t_2) is, by definition [12],

$$p_{t_1,t_2}(n) = M^n \,\mathrm{e}^{-M}/n! \tag{1}$$

where $M = \int_{t_1}^{t_2} \lambda(t)\,\mathrm{d}t$ is both the expected value and the variance of n.

Let this process be the input to a nonparalyzable dead-time counter, i.e., a counter that does not record pulses during a time interval of fixed duration τ after recording a given pulse (the pulses that arrive at the input to the counter during this dead time are lost). We consider the case for which the counter is always connected to the input process; this is the "equilib-

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rium counter" as opposed to the "blocked" or "unblocked" counter‡. We are concerned with the statistics of the number of pulses counted in a certain time interval (t_1, t_2).

If the rate $\lambda(t)$ is constant $[\lambda(t) = \lambda_0]$, then the probability distribution for the number of pulses counted in the interval (t_1, t_2) is well known [6–9] as is its expected value $\bar{n}$ [2,8,9],

$$\bar{n} = \frac{\lambda_0}{1 + \tau\lambda_0}(t_2 - t_1) . \tag{2}$$

If $\lambda(t)$ is not constant, on the other hand, we simply divide the interval (t_1, t_2) into many shorter intervals of duration Δt during which $\lambda(t)$ can be considered constant. The expected number of pulses during the entire interval will then be the sum (integral) of the expected values in the short intervals, i.e.,

$$\bar{n} = \int_{t_1}^{t_2} \frac{\lambda(t)}{1 + \tau\lambda(t)} \, \mathrm{d}t . \tag{3}$$

It is clear that eq. (3) is valid only if eq. (2) applies to each of the short intervals. We shall see that this, in turn, is true only if $\lambda(t)$ varies slowly enough to be virtually constant for any time interval with duration of the order of τ. The probability of recording a pulse at a given time depends not only on the value of $\lambda(t)$ at that time, but also on the probability of a dead time being in effect, which in turn, depends on the values of $\lambda(t)$ at previous times in a range of the order of τ. Eq. (2) has been obtained under the assumption that $\lambda(t)$ is constant and equal to λ_0 for a long enough time, previous to t_1, for the counting process to reach equilibrium (cf. refs. [7,8]). Thus it is expected that eq. (3) will be correct if $\lambda(t)$ varies sufficiently slowly, as indicated earlier.

3. Dead-time-modified variance

A similar result can be obtained for the dead-time-modified variance under somewhat different conditions due to the particular characteristics of the variance: Whereas the expected value of a sum of random variables is always equal to the sum of the expected values of the individual random variables, the same additivity law for the variance is not generally true, but is valid in the special case of independent random variables. Thus, in dividing the interval (t_1, t_2) into many short intervals of duration Δt we require Δt to be short enough so that $\lambda(t)$ can be considered constant during Δt. But at the same time, Δt must be long enough so that the number of pulses counted during a given interval be independent of the number counted in another interval. It is not difficult to visualize how the dead time generated by the last pulse recorded in one interval may overflow into the following interval and cause the two random variables to be correlated. It is therefore clear that a necessary condition to achieve additivity for the variance is

$$\Delta t \gg \tau . \tag{4}$$

In this limit, and for $\lambda(t)$ constant and equal to λ_0, we can write the variance $\sigma^2_{t,t+\Delta t}$ of the number of pulses counted during the interval Δt as (see eq. (33) of ref. [7]),

$$\sigma^2_{t,t+\Delta t} = \frac{\lambda_0}{(1+\lambda_0\tau)^3}\Delta t + \frac{1}{6}\frac{(\lambda_0\tau)^2}{(1+\lambda_0\tau)^4}[6 + 4(\lambda_0\tau) + (\lambda_0\tau)^2] . \tag{5}$$

In eq. (5) we recognize two terms: The first is proportional to Δt whereas the second is independent of Δt. If Δt is chosen to satisfy the additivity requirements discussed previously, viz.,

$$\sigma^2_{t,t+2\Delta t} = \sigma^2_{t,t+\Delta t} + \sigma^2_{t+\Delta t,t+2\Delta t} , \tag{6}$$

then the variance must be proportional to Δt. Thus the second term in eq. (5) must be negligible with respect to the first term, or equivalently

$$\frac{1}{6}\frac{\tau}{\Delta t}\frac{\lambda_0\tau}{1+\lambda_0\tau}[6 + 4(\lambda_0\tau) + (\lambda_0\tau)^2] \ll 1 . \tag{7}$$

For $\lambda_0\tau < 1$ or $\lambda_0\tau \approx 1$, it is clear by inspection that eq. (7) is satisfied if eq. (4) is satisfied. For $\lambda_0\tau > 1$, we use the identity

$$\frac{\lambda_0\tau}{1+\lambda_0\tau} = \sum_{j=0}^{\infty}\left(\frac{-1}{\lambda_0\tau}\right)^j , \tag{8}$$

‡ In the limits where our results are applicable, the number of pulses recorded during a sampling interval is $\gg 1$ and therefore the differences among blocked, unblocked, and equilibrium counters become negligible so that our results are equally valid for all three types of counter.

and the left-hand-side of eq. (7) may be rewritten as

$$\frac{\tau}{\Delta t}\,[\tfrac{1}{6}(\lambda_0\tau)^2 + \tfrac{1}{2}(\lambda_0\tau) + \tfrac{1}{2} - \tfrac{1}{2}(\lambda_0\tau)^{-1} + \tfrac{1}{2}(\lambda_0\tau)^{-2} - ...] \,. \tag{9}$$

The first term in square brackets above dominates the sum and thus the condition expressed in eq. (7) may be written equivalently as

$$\frac{\tau}{\Delta t}\,\tfrac{1}{6}(\lambda_0\tau)^2 \ll 1 \tag{10}$$

or

$$\Delta t \gg \tfrac{1}{6}(\lambda_0\tau)^2\tau \,. \tag{11}$$

From the foregoing, we postulate that if Δt satisfies eqs. (4) and (11), the variance will be additive and

$$\sigma^2_{t,t+\Delta t} = \frac{\lambda_0}{(1+\tau\lambda_0)^3}\,\Delta t \,. \tag{12}$$

In the case where $\lambda(t)$ is not constant but varies sufficiently slowly, we can divide the interval (t_1, t_2) into subintervals of appropriate duration Δt during which $\lambda(t)$ is virtually constant, and such that eqs. (4) and (11) are satisfied. Under these conditions, the variance for the interval (t_1, t_2) is the sum of the variances for the subintervals [given by eq. (12) with λ_0 replaced by the appropriate value of $\lambda(t)$]. Since $\lambda(t)$ is virtually constant during each sub-interval of duration Δt, the sum of variance can be expressed as the integral

$$\sigma^2_{t_1,t_2} = \int_{t_1}^{t_2} \frac{\lambda(t)}{[1+\tau\lambda(t)]^3}\,\mathrm{d}t \,. \tag{13}$$

4. Dead-time-modified mean for ramp and for sinusoidal modulation

We explicitly evaluate eq. (3) when $\lambda(t)$ is the ramp:

$$\lambda_r(t) = \lambda_0(1-m) + 2m\lambda_0 t/T, \quad 0 \leq t < T \tag{14}$$

and the sinusoidal time function:

$$\lambda_s(t) = \lambda_0(1 - m\cos 2\pi t/T), \quad 0 \leq t < T \,. \tag{15}$$

In both cases the modulation depth m is [13]

$$m = (\lambda_{max} - \lambda_{min})/(\lambda_{max} + \lambda_{min}) \,, \tag{16}$$

the average value is λ_0, and the parameter M appearing in eq. (1) has the value $M = \lambda_0 T$, independent of m and τ. Evaluating the integral in eq. (3) for the ramp is straightforward. The expected number of counts in the interval $(0, T)$ is

$$\bar{n}_r = \int_0^T \frac{\lambda_r(t)}{1+\tau\lambda_r(t)}\,\mathrm{d}t = \frac{T}{\tau}\left[1 - \frac{1}{2m\lambda_0\tau}\ln\left(1 + \frac{2m\lambda_0\tau}{1+\lambda_0\tau(1-m)}\right)\right] . \tag{17}$$

For the sinusoidal case the result is

$$\bar{n}_s = \int_{t_1}^{t_2} \frac{\lambda_s(t)}{1+\tau\lambda_s(t)}\,\mathrm{d}t = \frac{T}{\tau}\left\{1 - \pi^{-1}[(1+\lambda_0\tau)^2 - (m\lambda_0\tau)^2]^{-1/2} \times \tan^{-1}\left[\left(\frac{1+\lambda_0\tau(1-m)}{1+\lambda_0\tau(1+m)}\right)^{1/2}\tan(\pi t/T)\right]\right\}\Bigg|_{t=t_1}^{t=t_2} . \tag{18}$$

For $t_1 = 0$ and $t_2 = T$ this yields the expected number of counts in the interval $(0, T)$

$$\bar{n}_s = \frac{T}{\tau}\{1 - [1 + 2\lambda_0\tau + (\lambda_0\tau)^2(1-m^2)]^{-1/2}\} . \tag{19}$$

In the limit as $\tau \to 0$ the counting distribution should reduce to eq. (1) indeed $\lim_{\tau\to 0}\bar{n}_r = \lim_{\tau\to 0}\bar{n}_s = \lambda_0 T$. Furthermore for $\lambda_0\tau \gg 1$, $\bar{n}_r \approx \bar{n}_s \approx T/\tau$ which reflects the behaviour of the counter when the rate of arrival of input pulses is very high with respect to τ: The counted pulses are evenly spaced in time, the interval between two subsequent pulses being only slightly larger than τ. Clearly, this asymptotic behavior should be the same for both $\bar{n}_r$ and $\bar{n}_s$ as well as for the case where $\lambda(t)$ is constant. It is also easy to verify that for $m = 0$ the expressions for $\bar{n}_r$ and $\bar{n}_s$ reduce to

$$\bar{n} = \lambda_0 T/(1 + \lambda_0\tau) \,, \tag{20}$$

i.e., the dead-time-modified mean for constant rate $\lambda(t) = \lambda_0$ given in eq. (2).

In fig. 1 we present a plot of the theoretical counting efficiency $(\bar{n}/\lambda_0 T)$ as a function of $\lambda_0\tau$. The solid curve represents $\lambda(t) = \lambda_0$ [see eq. (20)] whereas the dashed and dotted curves represent the ramp and sinusoid respectively [see eqs. (17) and (19)], both with $m = 1$. The dash-dot curve represents the asymptotic behavior $(1/\lambda_0\tau)$.

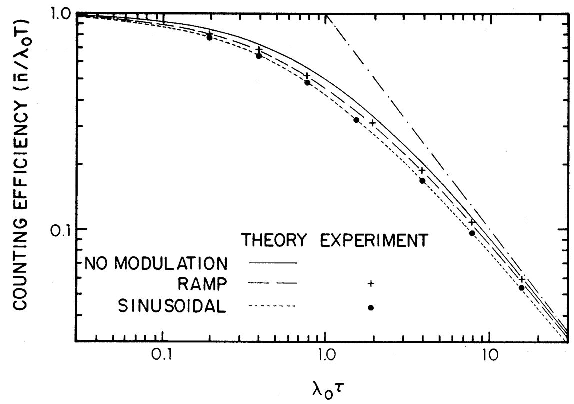

Fig. 1. Counting efficiency ($\bar{n}/\lambda_0 T$) versus $\lambda_0\tau$ where λ_0 is the average input count rate and τ is the dead time. Theoretical curves are for a Poisson process where the rate is constant (solid curve), a ramp (dashed curve), and a sinusoidal function of time (dotted curve). The dash-dot curve represents the asymptotic behavior, which is the same for all three curves. It is clear that the efficiency is significant reduced when the rate is not constant (up to 20% for sinusoidal modulation). Experimental values for the ramp (+) and sinusoid (•) are in good accord with the theoretical curves.

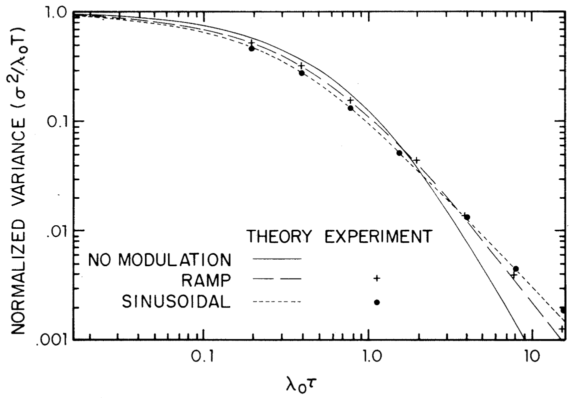

Fig. 2. Normalized dead-time-modified variance ($\sigma^2/\lambda_0 T$) versus $\lambda_0\tau$. Theoretical curves are for a Poisson process where the rate is constant (solid curve), a ramp (dashed curve), and a sinusoidal function of time (dotted curve). Note that the curves as well as the asymptotic behavior are significantly different. Experimental values for the ramp (+) and sinusoid (•) are in good accord with the theoretical curves.

5. Dead-time-modified variance for ramp and for sinusoidal modulation

Using eq. (14) for the ramp, the integral in eq. (13) yields

$$\sigma_r^2 = \int_0^T \frac{\lambda_r(t)}{[1+\tau\lambda_r(t)]^3}\,\mathrm{d}t = \lambda_0 T \frac{1+\lambda_0(1-m^2)}{[1+2\lambda_0\tau+(\lambda_0\tau)^2(1-m^2)]^2} \tag{21}$$

whereas for the sinusoidal case, we use eq. (15) to obtain

$$\sigma_s^2 = \int_0^T \frac{\lambda_s(t)}{[1+\tau\lambda_s(t)]^3}\,\mathrm{d}t = \lambda_0 T \frac{2+(4-3m^2)\lambda_0\tau+2(1-m^2)(\lambda_0\tau)^2}{2[1+2\lambda_0\tau+(1-m^2)(\lambda_0\tau)^2]^{5/2}}\,. \tag{22}$$

It is easy to verify that for $\tau = 0$

$$\sigma_r^2 = \sigma_s^2 = \lambda_0 T \tag{23}$$

and for $m = 0$

$$\sigma_r^2 = \sigma_s^2 = \lambda_0 T/(1+\lambda_0\tau)^3\,, \tag{24}$$

as is expected from eqs. (1) and (12) respectively.

In fig. 2 we present a plot of the normalized dead-time-modified variance ($\sigma^2/\lambda_0 T$) as a function of $\lambda_0\tau$. The solid curve represents $\lambda(t) = \lambda_0$ [see eq. (12)] whereas the dashed and dotted curves represent the ramp and sinusoid respectively [see eqs. (21) and (22)], both with $m = 1$.

6. Upper limit on dead-time-counter efficiency

It appears from fig. 1 that for all values of $\lambda_0\tau$ the counting efficiency $\bar{n}/\lambda_0 T$ is greatest when $\lambda(t)$ is constant. Intuitively we would indeed expect this result since the input pulses tend to bunch when $\lambda(t)$ varies, thereby enhancing the effects of dead time (i.e., reducing the efficiency) in the presence of modulation. In the following we show explicitly that the efficiency is maximum when $\lambda(t)$ is constant. Define

$$f(\lambda) = \lambda/(1+\lambda\tau), \quad \tau \geq 0\,; \tag{25}$$

this is a concave downward function of λ for all $\lambda > 0$. The curve representing $f(\lambda)$ will therefore lie entirely under any straight line tangent to it at any point.

Let $f^*(\lambda)$ represent the equation of the line tangent to the curve at the point $\lambda = \lambda_0$,

$$f^*(\lambda) = f(\lambda_0) + (\lambda - \lambda_0)\,\mathrm{d}f/\mathrm{d}\lambda|_{\lambda=\lambda_0} . \tag{26}$$

Since

$$f^*(\lambda) \geq f(\lambda), \qquad \forall \lambda \tag{27}$$

which implies

$$f^*(\lambda(t)) \geq f(\lambda(t)), \qquad \forall t \tag{28}$$

we obtain (by integrating both sides of eq. (28) with respect to t)

$$\int_0^T f^*(\lambda(t))\,\mathrm{d}t \geq \int_0^T f(\lambda(t))\,\mathrm{d}t . \tag{29}$$

Evaluating the integral on the left-hand-side of eq. (29) we obtain

$$f(\lambda_0)T + \frac{\mathrm{d}f}{\mathrm{d}\lambda}\bigg|_{\lambda=\lambda_0}\left[\int_0^T \lambda(t)\,\mathrm{d}t - \lambda_0 T\right]$$

$$\geq \int_0^T \frac{\lambda(t)}{1+\tau\lambda(t)}\,\mathrm{d}t . \tag{30}$$

If λ_0 is the average value of $\lambda(t)$ in the interval $(0, T)$ then $\lambda_0 T = \int_0^T \lambda(t)\,\mathrm{d}t$ and eq. (30) becomes

$$\int_0^T \frac{\lambda(t)}{1+\tau\lambda(t)}\,\mathrm{d}t \leq \frac{\lambda_0 T}{1+\lambda_0\tau} , \tag{31}$$

where the equality is valid if and only if $\lambda(t)$ is constant and equal to λ_0. Q.E.D. Behavior of this kind is not observed for the variance, as can be seen from fig. 2.

7. Experiment

To verify the validity of our calculations, we performed a series of photocounting experiments. The source was a Spectra-Physics Model 162 Ar^+ ion laser operated at 514.5 nm. The radiation was fed into an acousto-optic modulator that modulated the intensity of the beam with a ramp or a sinusoid. The modulated radiation was attenuated sufficiently for the photon statistics to be observable and was polarized and detected by an RCA 8575 photomultiplier tube. The output pulses from the anode of the photomultiplier tube were counted by a pulse counter with an electronically-generated nonparalyzable dead time whose value could be set arbitrarily.

Data were taken for a ramp and for sinusoidal modulation with the following parameters[‡] (in both cases): $T = 10$ ms, $m = 1$, $\lambda_0 \approx 33$ counts/ms, and the dead time τ was varied from a minimum of 6 μs ($\lambda\tau \approx 0.2$) to a maximum of 1.2 ms ($\lambda\tau \approx 40$). At the low end τ is sufficiently small so that eqs. (4) and (11) can be satisfied and therefore eqs. (3) and (13) are expected to be applicable. For the largest values of τ, however, we would expect that the experimental results should depart somewhat from the predictions of eqs. (3) and (13) since eqs. (4) and (11) cannot be so well satisfied. The experimentally measured values of the mean and variance of the photocounting distributions are presented in figs. 1 and 2, respectively, for both ramp and sinusoidally modulated radiation. Though it is clear that the data are in good accord with the theoretical curves, it is evident from figs, 1 and 2 that, as expected, the experimental points depart somewhat from the theoretical predictions for the largest values of τ.

8. Conclusion

It is evident from the foregoing that when $\lambda(t)$ is not constant, the statistics of the number of pulses counted by a nonparalyzable dead-time-counter in a given time interval depend on more than just the statistics of the total energy (integrated rate) arriving at the counter during that time. Thus these statistics cannot, in general, be deduced from the statistics of the total energy alone as is the case in the absence of dead time. Our results clearly show that the details of the variation of the rate during the sampling interval must be accounted for in order to correctly evaluate the dead-time-modified counting statistics. The particular cases of the ramp and sinusoidal modulation presented

‡ The modulation was generated by driving the acousto-optic modulator with a sinusoidal waveform for sinusoidal modulation, and with a triangular waveform for the ramp. In both cases the period of the wave was equal to the duration of the sampling interval. The sampling intervals were not synchronized with the phase of the wave. Because the integrals in eqs. (3) and (13) are to be evaluated over a full period, it is easy to see that the results of the evaluation are the same as those given in eqs. (17), (19), (21) and (22) for the two types of modulation.

illustrate this point particularly well. This is because the statistics of the integrated rate are the same for both modulation formats and for the no-modulation case, yet the values of the experimental and theoretical dead-time-modified mean and variance are significantly different. These result underscore the fact that existing formulas for the dead-time-modified counting distribution produced by randomly fluctuating light [3–5] are valid only in the limit where the sampling interval is much smaller than the coherence (or modulation) time of the light, and cannot be extended in any simple way to cases outside that limit.

Whereas we have explicitly considered the mean and variance in this paper, we can use the same technique (i.e., dividing the sampling interval into small intervals in which the rate is virtually constant but such that the number of pulses counted in a given interval is independent of other intervals) to arrive at a dead-time-modified expression for any statistical parameter which is additive for independent random variables. For example, the quantity $\alpha(x)$ defined by

$$\alpha(x) = \langle x^3 \rangle - 3\langle x^2 \rangle \langle x \rangle + 2\langle x \rangle^3 , \tag{32}$$

which involves the third-order moment of the random variable x, is additive for independent random variables, i.e., $\alpha(x+y) = \alpha(x) + \alpha(y)$ if x and y are independent random variables.

The technique used in this paper can therefore be used to find a dead-time-modified expression for α, in the case of a Poisson process with slowly varying rate.

The expressions given in eqs. (3) and (13) represent the dead-time-modified mean and variance when the rate is a known function of time. Kikkawa et al. [14] have calculated the efficiency of a dead-time photon counter with Gaussian-Lorentzian light (stochastic rate variation). Their results, however, are only valid in the limit where the correlation time is much smaller than the mean time interval between pulses (see remarks following eq. (10) in ref. [14]). In a forthcoming publication we shall present expressions for the dead-time-modified mean and variance in the general case where the rate is a stochastic process.

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