verse relations. Thus, the foregoing criteria permit estimates of the maximum value of the source coherence length from estimates of the minimum intensity far-zone range.

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Observation of dead-time-modified photocounting distributions for modulated laser radiation

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We report a series of optical experiments that verify the theoretical photocounting distributions obtained by Diament and Teich for triangularly and sinusoidally modulated laser radiation. Another series of experiments validates the (nonparalyzable) dead-time-modified versions of these formulas obtained by Cantor and Teich. A new expression is obtained for the counting distribution in the presence of modulation and paralyzable dead time. The results have application in photon counting, nuclear counting, and neural counting.

I. INTRODUCTION

The photocounting statistics for intensity-modulated radiation have been studied by a number of researchers.[1-9] In particular, theoretical results for various periodic modulation formats (square-wave, triangular, and sinusoidal) have been obtained by Diament and Teich[4,5] for arbitrary modulation depth and short sampling times. There have been few experimental measurements of these counting statistics, however, though some time ago Teich and Diament observed the flat γ-ray counting distribution resulting from a linearly swept mean.[4,10]

For an unmodulated (Poisson) source, the effect of detector dead time on the counting statistics has also been studied extensively, primarily in the context of nuclear particle counting and neural pulse counting.[11-14] In the presence of intensity modulation, the theoretical dead-time-modified counting statistics have been obtained only under the special conditions of a nonparalyzable counter with a sampling time short in comparison with the fluctuation time of the source.[15-17] A particularly useful form for this distribution has been presented by Cantor and Teich.[16] (A comprehensive bibliography on dead-time effects was compiled by Müller in 1975.[18])

In this paper we report a series of experiments that verify the theoretical photocounting distributions for triangular and sinusoidal modulation given by Diament and Teich,[5] and the nonparalyzable-dead-time-modified versions of these formulas obtained using the method suggested by Cantor and Teich.[16] We also present a new result for the paralyzable-dead-time-modified counting distribution for intensity-modulated radiation and short sampling times. In a related

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paper,[19] we use the formulas discussed here to investigate likelihood-ratio detection, channel capacity, and maximum-likelihood image estimation in doubly stochastic Poisson counting systems subject to nonparalyzable dead-time effects.

II. THEORY

We first consider a detector irradiated by an amplitude-stabilized spatially coherent polarized source of intensity $I(t)$. In this case, the photocounting distribution $p_0(n|W)$, representing the probability of registering n photocounts in the fixed sampling time $(t, t+T)$, is given by

$$p_0(n|W) = W^n e^{-W}/n!, \tag{1}$$

where the integrated intensity W is

$$W = \alpha \int_t^{t+T} I(t')\, dt' \tag{2}$$

and α is the quantum efficiency of the detector.

If W is a random variable [by virtue of t being a random variable or $I(t)$ being a random process, or both] the photocounting distribution is given by Mandel's formula[20]

$$p(n,\langle n\rangle) = \langle p_0(n|W)\rangle_W = \int_0^\infty p_0(n|W)P(W)\, dW, \tag{3}$$

where $\langle n\rangle$ is the average count number ($\langle n\rangle = \langle W\rangle$).

For the special case where $I(t)$ is a triangular waveform with (uniformly distributed) random phase, period $T_M \gg T$ and modulation depth m, Eq. (3) yields[4,5,10]

$$p(n,m,\langle n\rangle) = \frac{\exp[-\langle n\rangle(1-m)]}{2m\langle n\rangle}\sum_{k=0}^{n}\frac{[\langle n\rangle(1-m)]^k}{k!} - \frac{\exp[-\langle n\rangle(1+m)]}{2m\langle n\rangle}\sum_{k=0}^{n}\frac{[\langle n\rangle(1+m)]^k}{k!}. \tag{4}$$

Again, $\langle n\rangle$ is the mean number of counts. Equation (4) represents the family of flat counting distributions studied by Teich and Diament.[4,5,10] For sinusoidal modulation of depth m, with random phase and period $T_M \gg T$, the counting distribution is the finite sum[5]

$$p(n,m,\langle n\rangle) = \frac{\langle n\rangle^n e^{-\langle n\rangle}}{n!} \times \sum_{l=0}^{n}\binom{n}{l}\left(\frac{-m}{2}\right)^l \sum_{k=0}^{l}\binom{l}{k} \mathrm{I}_{|l-2k|}(m\langle n\rangle), \tag{5}$$

where $\mathrm{I}_q(\cdot)$ is the modified Bessel function of order q.

Note that for cases in which the intensity fluctuates very rapidly and the degeneracy parameter is much less than unity,[20] $p(n,\langle n\rangle)$ approaches the Poisson distribution.[20,21]

By definition, a nonparalyzable-dead-time counter cannot record counts (i.e., it is dead) for a time interval of fixed duration τ immediately following the registration of a count. We consider in detail the nonparalyzable(or nonextended)–dead-time counter unblocked at the beginning of the counting interval (see Refs. 13 and 22 for a more detailed description of the various kinds of dead-time counters). For an unmodulated amplitude-stabilized source, $I(t) = I_0$ (constant) and $W = \alpha I_0 T$. In this case the exact dead-time-modified counting distribution $p(n|W,\tau/T)$ is[12,14,16]

$$p(n|W,\tau/T) = \begin{cases} \sum_{k=0}^{n} p_0(k|W[1-n\tau/T]) - \sum_{k=0}^{n-1} p_0(k|W[1-(n-1)\tau/T]), & n < T/\tau \\ 1 - \sum_{k=0}^{n-1} p_0(k|W[1-(n-1)\tau/T]), & T/\tau \le n < T/\tau + 1 \\ 0, & n \ge T/\tau + 1. \end{cases} \tag{6}$$

The exact nonparalyzable-dead-time-modified counting distributions for the blocked and equilibrium counters have also been obtained (see Ref. 14 for the appropriate formulas). In the usual situation the mean count is much greater than 1, in which case the differences among the blocked, unblocked, and equilibrium counters are not substantial.[22] In that event, Eq. (6) may be approximated by

$$p(n|W,\tau/T) \approx p_0(n|W[1-n\tau/T]) \tag{6a}$$

for all three types of counter when $n \le T/\tau$. (Simple asymptotic expressions for the dead-time-modified count mean and variance are also available for all of these cases.[14])

When W is random, Cantor and Teich[16] have indicated that the nonparalyzable-dead-time-modified counting distribution is obtained by averaging Eq. (6) over the statistics of W, i.e.,

$$p(n,\langle n\rangle,\tau/T) = \langle p(n|W,\tau/T)\rangle_W = \begin{cases} \sum_{k=0}^{n} \langle p_0(k|W[1-n\tau/T])\rangle_W - \sum_{k=0}^{n-1} \langle p_0(k|W[1-(n-1)\tau/T])\rangle_W, & n < T/\tau \\ 1 - \sum_{k=0}^{n-1} \langle p_0(k|W[1-(n-1)\tau/T])\rangle_W, & T/\tau \le n < T/\tau + 1 \\ 0, & n \ge T/\tau + 1. \end{cases} \tag{7}$$

The averaging operation is valid, however, only if $I(t)$ is virtually constant during the sampling time (i.e., if $T \ll T_M$ or, when $I(t)$ is a random process, if $T \ll \tau_c$ where τ_c is the coherence time of the source[20]). The need for this condition stems from the fact that Eq. (6) is applicable only when the intensity is constant $[I(t) = I_0]$. Equation (7) is therefore not valid when appreciable intensity fluctuations occur during the sampling time, as clearly demonstrated by Vannucci and Teich.[23]

We can now combine Eqs. (3) and (7) to obtain the nonparalyzable-dead-time-modified counting statistics for an arbitrary modulation format. For triangular and sinusoidal modulation we combine Eq. (7) with Eqs. (4) and (5), respectively. Again, for a mean count much greater than 1, the distinction among blocked, unblocked, and equilibrium counters is often not significant in which case Eq. (7) may be approximated by

$$p(n, \langle n \rangle, \tau/T) = \langle p(n|W, \tau/T) \rangle_W \approx \langle p_0(n|W[1 - n\tau/T]) \rangle_W \quad \text{(7a)}$$

for all three types of counter when $n \leq T/\tau$. Equation (7a) is obtained by averaging Eq. (6a) over the statistics of W. (Closed-form expressions for the dead-time-modified count mean and variance in this case are mathematically identical to those obtained by Vannucci and Teich for a related problem.[23])

III. EXPERIMENT

A series of experiments was performed to verify the theoretical photocounting distributions for triangular and sinusoidal modulation in the absence of dead time, as well as in the presence of nonparalyzable dead time. The source was a Spectra-Physics Model 162 Ar^+ ion laser[24] operated at 514.5 nm. The radiation was fed into an acousto-optic modulator that modulated the intensity of the beam with a triangular or a sinusoidal wave. The modulated radiation was attenuated sufficiently for the photocounting statistics to be observable and was polarized and detected by an RCA Type 8575 photomultiplier tube. The output pulses from the anode of the photomultiplier tube were counted by an (unblocked) pulse counter with an electronically generated nonparalyzable dead time whose value could be set arbitrarily.

Data were taken for triangular and sinusoidal modulation for various combinations of values of the modulation depth m and the dead-time ratio τ/T. Other experimental parameters were the period of the wave $T_M = 1$ s, the sampling interval $T = 1$ ms, and the number of observation samples $N = 10^5$. These parameters were the same for all sets of data, except where explicitly indicated in the figure captions.

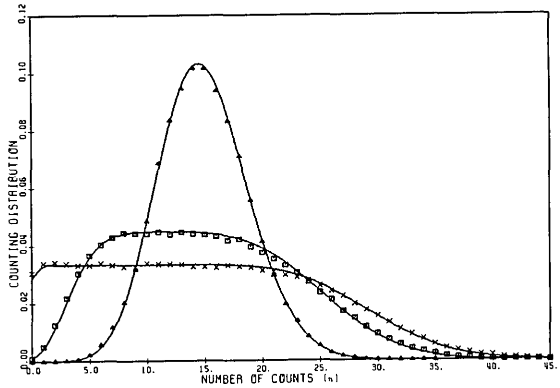

FIG. 2. Theoretical counting distributions (solid curves) and experimental data for triangularly modulated radiation in the absence of dead time ($\tau/T = 0$). The modulation depth m takes on three values: $m = 0$ (Δ), $m = 0.74$ ($\square$, $N = 200\,000$), and $m = 0.99$ (X, $N = 200\,000$). The mean count is approximately the same for all three distributions ($\langle n \rangle \approx 15$). Note the flat counting distribution [see Refs. 4, 5, and 10] obtained when $m \approx 1$.

The experimental data (data points indicated by $\square$, Δ, X, $\uparrow$) as well as the theoretical counting distributions for the same parameters (solid curves) are presented in Figs. 1–6. Figures 1 and 2 display results for sinusoidal and triangular modulation, respectively, in the absence of dead time [see Eqs. (5) and (4)]. The modulation depth is varied parametrically. In Fig. 3 the light is unmodulated ($m = 0$) so that the results correspond to the nonparalyzable-dead-time-modified Poisson distribution [see Eq. (6)]. This case has been studied in considerable detail in the context of nuclear counting,[11,14,22] neural counting,[12,25] and photon counting.[15,16] The more general cases of triangular and sinusoidal modulation in the presence of nonparalyzable dead time are presented in Figs. 4–6 [see Eqs. (7), (4), and (5)].

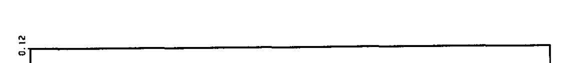

FIG. 1. Theoretical counting distributions (solid curves) and experimental data for sinusoidally modulated radiation in the absence of dead time ($\tau/T = 0$). The modulation depth m takes on three values: $m = 0$ (X), $m = 0.75$ (Δ, $T_M = 5$ s, $T = 10$ ms, $N = 50\,000$), and $m = 1.0$ ($\square$, $T_M = 5$ s, $T = 10$ ms, $N = 50\,000$) [see Ref. 29]. The mean count is approximately the same for all three distributions ($\langle n \rangle \approx 17$).

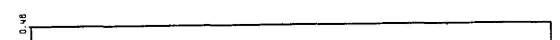

FIG. 3. Theoretical nonparalyzable-dead-time-modified counting distributions (solid curves) and experimental data in the absence of modulation ($m = 0$). The dead-time ratio takes on four values: $\tau/T = 0$ ($\uparrow$, Poisson distribution, $T = 10$ ms, $N = 50\,000$), $\tau/T = 0.02$ (X), $\tau/T = 0.05$ (Δ), and $\tau/T = 0.1$ ($\square$). The unmodified mean count (i.e., the mean count before the dead-time reduction) is approximately the same for all four distributions ($\langle n \rangle \approx 15$).

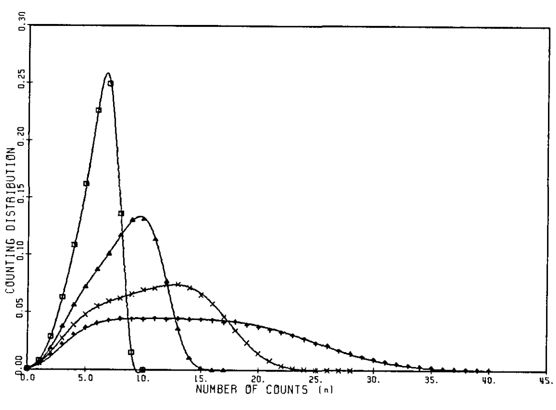

FIG 4. Theoretical nonparalyzable-dead-time-modified counting distributions (solid curves) and experimental data for triangularly modulated radiation ($m \approx 0.75$). The dead-time ratio takes on four values: $\tau/T = 0$ (↑, $N = 200\,000$), $\tau/T = 0.02$ (X, $N = 200\,000$), $\tau/T = 0.05$ (Δ), and $\tau/T = 0.1$ (□). The unmodified mean count is approximately the same for all four distributions ($\langle n\rangle \approx 15$).

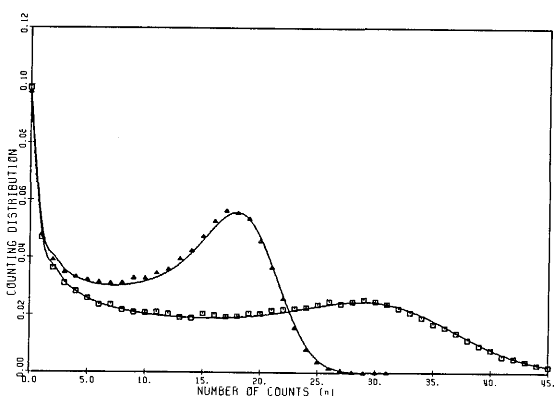

FIG. 6. Theoretical nonparalyzable-dead-time-modified counting distributions (solid curves) and experimental data for sinusoidally modulated radiation (m ≈ 1) [see Ref. 29]. The dead-time ratio takes on two values: $\tau/T = 0$ (□) and $\tau/T = 0.02$ (Δ). The unmodified mean count is approximately the same for both distributions ($\langle n\rangle \approx 17$).

In examining the figures, it is clear that the theory is in excellent agreement with all of the experimental data. It is also apparent that modulation broadens the counting distributions (see Figs. 1 and 2); this is interpretable as accentuated photon bunching. Dead time, on the other hand, decreases both the mean and variance of the counting distribution as well as the variance-to-mean ratio (see Figs. 3–6), corresponding to a loss of counts and to count antibunching. The dead-time-modified counting distributions converge to the unmodified distributions for low count numbers n where dead-time effects are least important.

IV. PARALYZABLE DEAD-TIME COUNTER

All of the results discussed to this point are specifically for the nonparalyzable (or nonextended) dead-time counter. When W is a random variable (and $T \ll T_M, \tau_c$) and the counter is of the extended-dead-time type, the paralyzable-dead-time-modified counting distribution $\pi(n, \langle n\rangle, \tau/T)$ is obtained by using Libert's results[26] for the constant-intensity paralyzable counting distribution $\pi(n\,|\,W, \tau/T)$ in conjunction with the technique of evaluating the ensemble average $\langle \pi(n\,|\,W, \tau/T)\rangle_W$ used earlier. Thus for the blocked paralyzable dead-time counter,[26,27] for example,

$$\pi(n\,|\,W, \tau/T) = \sum_{k=n}^{\lfloor T/\tau \rfloor} \frac{(-1)^{k-n}}{n!(k-n)!} e^{-kW\tau/T} \left[W\left(1 - k\frac{\tau}{T}\right)\right]^k, \tag{8}$$

so that

$$\begin{aligned}\pi(n, \langle n\rangle, \tau/T) &= \langle \pi(n\,|\,W, \tau/T)\rangle_W \\ &= \sum_{k=n}^{\lfloor T/\tau \rfloor} \frac{(-1)^{k-n}}{n!(k-n)!} \left\langle e^{-kW\tau/T} \left[W\left(1 - k\frac{\tau}{T}\right)\right]^k \right\rangle_W. \end{aligned} \tag{9}$$

Here $\lfloor T/\tau \rfloor$ denotes the largest integer smaller than T/τ. Results for the unblocked and equilibrium counters are similar in form. The same technique can be used with the type-p counter[28] which reduces to paralyzable and nonparalyzable behavior as special cases.

In practice, it is often the discriminator following the photodetector that provides the major contribution to dead time so that the detailed structure of the system is important in determining which formulas should be used.

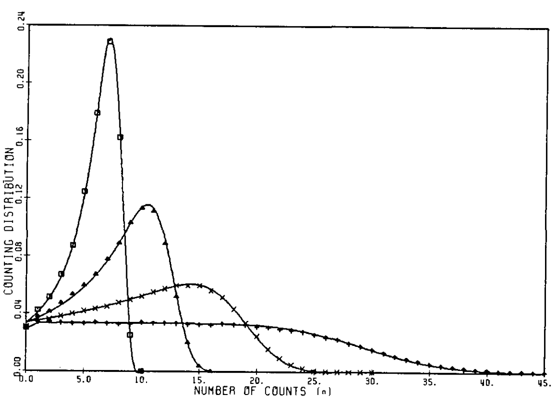

FIG. 5. Theoretical nonparalyzable-dead-time-modified counting distributions (solid curves) and experimental data for triangularly modulated radiation ($m \approx 1$). The dead-time ratio takes on four values: $\tau/T = 0$ (↑, $N = 200\,000$), $\tau/T = 0.02$ (X, $N = 200\,000$), $\tau/T = 0.05$ (Δ), and $\tau/T = 0.1$ (□). The unmodified mean count is approximately the same for all four distributions ($\langle n\rangle \approx 15$).

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