Dead-time-corrected photocounting distributions for laser radiation*

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Exact forward photocounting distributions for a source of optical radiation with arbitrary statistics are obtained in the presence of photodetector dead time. In particular, we examine the counting and pulse-interval distributions that arise from amplitude-stabilized radiation, from chaotic radiation, and from a Van der Pol laser with arbitrary excitation. The exact dead-time-corrected Poisson distribution is graphically compared with a previous approximate result and with the uncorrected Poisson. Plots of the Bose–Einstein distribution clearly indicate the dramatic anti-bunching effects of the dead time in overcoming the inherent bunching of this distribution. A simplified approximate solution is also found for the Van der Pol laser above threshold; this result is similar to light from an amplitude-stabilized source incident on a photodetector with a gaussian-distributed dead time. Information about photodetector dead-time variation can therefore be obtained either by using an amplitude-stabilized source or by properly choosing system parameters such that irradiance fluctuations are averaged out.

Index Headings: Source; Lasers; Detection.

During the past decade, there has been a great deal of study of the forward-photocounting distributions[1–7] for a variety of radiation sources, modulation schemes, and stochastic channels. Laboratory measurements have shown the need to consider the effects of the photodetector dead time on the form of these distributions.[4] The dead time τ is a fixed period of time, after the registration of a photoelectron, during which the photodetector cannot emit another electron. As optical communications systems operate at increasing rates of information transmission, sampling times become smaller and the effects of dead time more pronounced.

The effects of dead time on counting distributions that arise in nuclear-counting problems have been studied by De Lotto *et al.*,[8] who present a number of results in the Laplace domain for both paralyzable and nonparalyzable systems. Johnson *et al.*[4] applied this work to both amplitude-stabilized and chaotic sources for sampling times $T \gg \tau$; their calculation of the counting distribution breaks down, however, for count numbers n such that $n\tau$ approaches the value of the counting interval T. Under such conditions, it is necessary to make use of the exact solution for the distribution, including dead time. Although some progress toward this end has been made by Bédard,[9] it is of interest to obtain dead-time-corrected counting distributions for arbitrary sources of radiation, such as the Van der Pol laser. Such results can also provide information about the distribution of dead times manifested by a detector (where τ is not fixed) for a source whose radiation statistics are known.

I. DEAD-TIME-CORRECTED POISSON COUNTING DISTRIBUTION

In the absence of dead time, the probability of recording n counts in a time interval T from a detector illuminated by an amplitude-stabilized source is[1]

$$p(n, \lambda) = (\lambda T)^n \exp(-\lambda T)/n! \ , \tag{1}$$

with

$$\lambda = \alpha I \ .$$

This is the well-known Poisson distribution, with α the quantum efficiency of the detector and I the (constant) irradiance at the detector. Poisson counting is also observed for a source of arbitrary statistics, provided that $T \gg \tau_c$, where τ_c is the coherence time of the source.[5–7] In the presence of a dead time τ, the results of De Lotto *et al.*[8] for a nonparalyzable system are used (see Appendix) to show that the probability of registering n counts in a time interval T is given exactly by

$$p_0(n, \lambda, \tau) = \sum_{k=0}^{n} \{\lambda^k (T - n\tau)^k / k!\} \exp\{-\lambda(T - n\tau)\} - \sum_{k=0}^{n-1} \{\lambda^k [T - (n-1)\tau]^k / k!\} \exp\{-\lambda[T - (n-1)\tau]\} \ . \tag{2}$$

This expression is valid for counts n such that $n\tau < T$, and assumes that the counter is unblocked at the beginning of the sampling interval. By the definition of dead time, count numbers greater than T/τ are forbidden. The mean and variance of this distribution can be written as

$$\bar{n} = \lambda T(1 + \lambda\tau)^{-1} + \tfrac{1}{2}(\lambda\tau)^2(1 + \lambda\tau)^{-2} \tag{3a}$$

and

$$\sigma^2 = \lambda T(1 + \lambda\tau)^{-3} \ , \tag{3b}$$

respectively.[8,10] Equation (2) can also be obtained from Eq. (23) in Ref. 9, by use of the properties of the incomplete gamma function.

Figure 1 shows the deviation from the original Poisson distribution when the effects of dead time are included and also provides a comparison with a dead-time-corrected distribution whose mean is approximately equal to that of the original Poisson distribution. The ratio of dead time to sampling time τ/T is 0.025 for all curves. The mean of the original Poisson distribution (solid curve) along with its variance is 25.0; Fig. 1 shows that the mean of the dead-time-corrected distribution (dotted curve) is reduced to 16.3, whereas its variance is 5.6. The dashed curve was obtained for an irradiance that was increased such that $\lambda T = 66.5$ to compensate approximately for dead-time losses in the mean. This clearly demonstrates the reduction of variance to

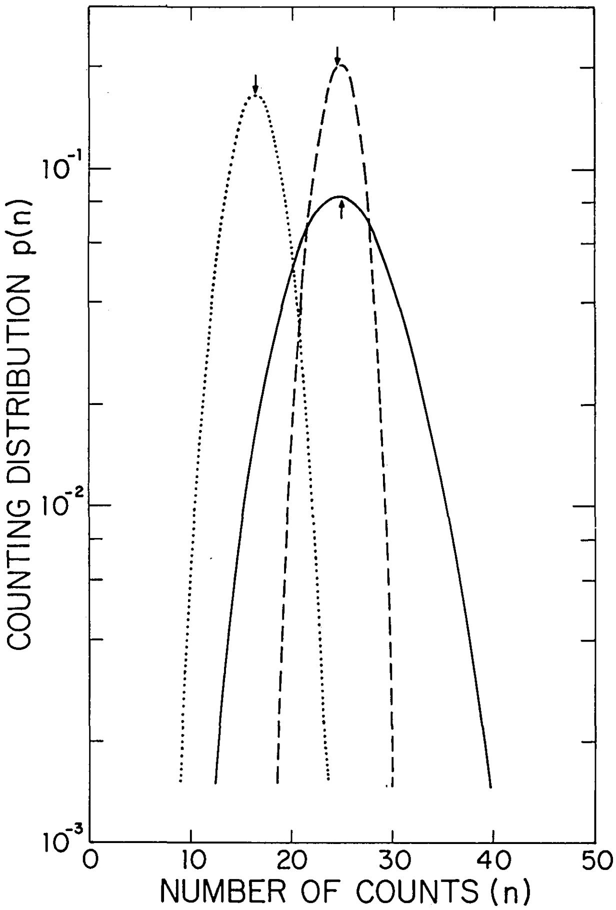

FIG. 1. The Poisson distribution with no dead time is represented by the solid curve (mean = variance = 25.0). The effects of dead time produce the distribution shown by the dotted curve (mean = 16.3, variance = 5.6). Note the reduction of the mean (shown by arrows) due to the elimination of output pulses. The dead-time-corrected distribution normalized to approximately the same mean as the uncorrected Poisson distribution is shown by the dashed curve (mean = 24.6, variance = 3.6); this clearly demonstrates the decrease of variance brought about by dead-time effects. The ratio of dead time to sampling time τ/T was 0.025 for all curves.

3.6 by the dead-time effects. Figure 2 presents a comparison of the exact result (solid curve) obtained above with the approximate result (dashed curve) obtained by Johnson *et al.*[4] for the same irradiance, with $\tau/T = 0.005$. The two distributions are sufficiently different so that the exact results should be used, particularly for count numbers greater than the mean. These distributions correspond to a value of $\bar{n}_0 = 29$ (Johnson *et al.*[4]) and $\lambda = 2.9$, with $T = 10$ in both cases. For $\tau/T = 0.001$, the exact and approximate results cannot be distinguished graphically. The effects of the detector dead time are therefore to shift the mean to a lower value, and to reduce the variance of the distribution.

II. DEAD-TIME-CORRECTED ARBITRARY COUNTING DISTRIBUTION

If, for convenience, we define a function $p_k(n, \lambda)$ as

$$p_k(n, \lambda) = \{\lambda^k (T - n\tau)^k / k!\} \exp\{-\lambda(T - n\tau)\} , \qquad (4)$$

the dead-time-corrected statistics for an amplitude-stabilized source can be written as [see Eq. (2)]

$$p_0(n, \lambda, \tau) = \sum_{k=0}^{n} p_k(n, \lambda) - \sum_{k=0}^{n-1} p_k(n-1, \lambda) . \qquad (5)$$

When illuminated by a source whose irradiance fluctuates, then, the photoelectron-counting distribution will be given by

$$p(n, \lambda, \tau) = \langle p_0(n, \lambda, \tau) \rangle_M , \qquad (6a)$$

with

$$M = \alpha \int_t^{t+T} I(t')\, dt' . \qquad (6b)$$

This represents an ensemble average of the underlying dead-time-corrected Poisson distribution over the statistics of the integrated irradiance M. Using Eq. (5), we see that Eq. (6a) becomes

$$p(n, \lambda, \tau) = \sum_{k=0}^{n} \langle p_k(n, \lambda) \rangle_M - \sum_{k=0}^{n-1} \langle p_k(n-1, \lambda) \rangle_M . \qquad (7)$$

Each of the $p_k(n, \lambda)$ is simply the probability of registering k counts in a time interval $(T - n\tau)$ for a process that obeys a Poisson probability law with rate λ. Thus, each of the ensemble averages in Eq. (7) represents an appli-

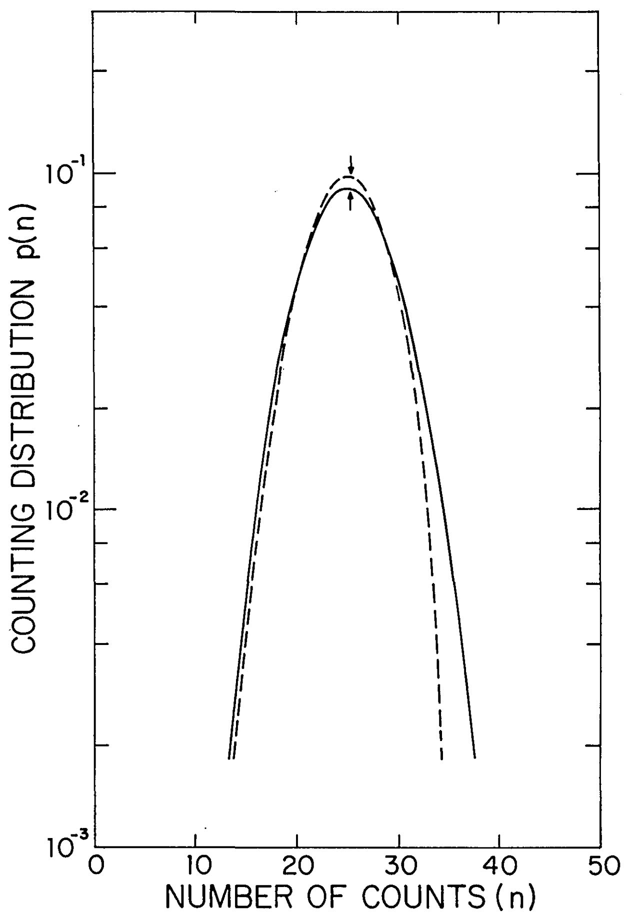

FIG. 2. Photocounting distributions corresponding to exact (solid curve) and approximate (dashed curve) corrections for dead time. The mean value for both curves (shown by arrows) is 25.3. These curves are normalized to the same irradiance and correspond to a value $\tau/T = 0.005$.

cation of Mandel's formula[1] that uses the statistics of the integrated irradiance M. We have therefore obtained a simple expression for the exact dead-time-corrected distribution for any source whose statistics are known. In the following, we explicitly calculate a number of dead-time-corrected distributions.

III. DEAD-TIME-CORRECTED COUNTING DISTRIBUTION FOR A VAN DER POL LASER

Armstrong and Smith[5] and Chang *et al.*[11] have shown by photocounting experiments that the behavior of a single-mode laser is well described throughout the range from well-below to near to well-above threshold by Risken's irradiance distribution[12,13]

$$P(I) = 2\pi^{-1/2}[I_1(1+\text{erf}w)]^{-1}\exp\{-[(I/I_1)-w]^2\} . \quad (8)$$

Because $I \geq 0$, this is a truncated gaussian. The parameter w describes the state of excitation of the laser; it is negative below threshold, zero at threshold, and positive above threshold.[5,6,12,13]

For sampling times $T \ll \tau_c$, Eq. (6b) shows that the statistics of M are identical with the statistics of I. In this case, then, the ensemble average in Eq. (7) yields[6]

$$\langle p_k(n,\lambda)\rangle_M = [\exp(w^2)\,\text{erfc}(-w)]^{-1} \times\{(\nu t)^k \exp[(\nu t/2)-w]^2\, i^k \text{erfc}[(\nu t/2)-w]\} , \quad (9)$$

with $t=(T-n\tau)$ and $\nu=\alpha I_1$. The function $i^k\text{erfc}$ represents the kth-iterated integral of the complementary error function.[6,14] The average laser output $\langle I\rangle$ can be related to the parameters w and I_1 by the expression[6]

$$\langle I\rangle = I_1[w+\pi^{-1/2}(1+\text{erf}w)^{-1}\exp(-w^2)] . \quad (10)$$

The results represented by Eqs. (7) and (9) above are not only exact, but also provide dead-time-corrected counting distributions for a source that has a broad range of statistical behavior. Thus, for large negative w ($w<-3$), we obtain the dead-time-corrected distribution for a source that is essentially chaotic.

For arbitrary T/τ_c, the dead-time-corrected counting distribution for the rotating-wave Van der Pol laser can be obtained in a similar way by use of Eq. (7) and the results obtained by Lax and Zwanziger.[15] This exact distribution would then permit a highly accurate comparison with experimentally observed distributions,[16] where dead-time losses must be considered.

We can, furthermore, obtain the exact pulse-interval distribution $f(t)$. For fixed dead time, this is given by the expression $-\partial p(0,T)/\partial T$. Using Eq. (9) and a form of Eq. (7) corresponding to a counter blocked at the beginning of the sampling interval,[8] we find

$$f(t)=\begin{cases}\nu\pi^{-1/2}[\exp(w^2)\,\text{erfc}(-w)]^{-1}\{1-\pi^{1/2}[(\nu/2)(t-\tau)-w]\exp[(\nu/2)(t-\tau)-w]^2\,\text{erfc}[(\nu/2)(t-\tau)-w]\} , & t\geq\tau \\ 0 , \quad t<\tau .\end{cases} \quad (11)$$

In the limit of large w, this properly reduces to the pulse-interval distribution corresponding to a dead-time-corrected Poisson distribution with rate $w\nu$, i.e., $w\nu\exp[-w\nu(t-\tau)]$, for $t\geq\tau$.

IV. APPROXIMATE SOLUTION FOR A VAN DER POL LASER WELL ABOVE THRESHOLD

Far above threshold, for large positive w, the irradiance distribution in Eq. (8) is gaussian, with mean wI_1 and standard deviation $I_1/(2)^{1/2}$. We now consider this special case for sampling times that are short in comparison with the irradiance fluctuations. Thus, the parameter λ of the counting distribution is related to the instantaneous value of the irradiance by

$$\lambda = \alpha I(t) , \quad (12)$$

and the distribution of λ will be of a form identical to that of the irradiance I. The counting distribution given in Eq. (2) can then be directly averaged over the fluctuations in λ to obtain the dead-time-corrected counting distribution for a photodetector illuminated by a Van der Pol laser above the threshold of oscillation.

Specifically, for $w>5$, the distribution of the rate parameter λ may be written as

$$P(\lambda) = (2\pi)^{-1/2}\sigma^{-1}\exp\{-\tfrac{1}{2}[(\lambda-\bar\lambda)/\sigma]^2\} , \quad (13)$$

with

$$\bar\lambda = \alpha I_1 w$$

and

$$\sigma = \alpha I_1/(2^{1/2}) .$$

The desired counting distribution will then be given by

$$p(n,\sigma,\bar\lambda) = \int_0^\infty p_0(n,\lambda,\tau)P(\lambda)\,d\lambda , \quad (14)$$

where $p_0(n,\lambda,\tau)$ is the distribution with dead time in the absence of source variations, given in Eq. (2). Thus,

$$p(n,\sigma,\bar\lambda) = \int_0^\infty\Big(\sum_{k=0}^{n}\{\lambda^k(T-n\tau)^k/k!\}\exp\{-\lambda(T-n\tau)\} - \sum_{k=0}^{n-1}\{\lambda^k[T-(n-1)\tau]^k/k!\}\exp\{-\lambda[T-(n-1)\tau]\}\Big) \times P(\lambda)\,d\lambda . \quad (15)$$

Since the sums involved are finite, the order of integration and summation may be interchanged. The solution therefore involves the evaluation of integrals of the form

$$p_{k,n} = \int_0^\infty\{\lambda^k(T-n\tau)^k/k!\}\exp\{-\lambda(T-n\tau)\}P(\lambda)\,d\lambda . \quad (16)$$

An approximate evaluation of the integral in Eq. (16)

may be carried out by the method of steepest descent.[6,7] For convenience, we use Eq. (4) and express $p_k(n, \lambda)$ as an exponential of its own logarithm. The point of stationarity for the combined exponential is then sought, and a Taylor-series expansion about that point readily yields an approximate expression for the desired integral. Because all integrals occurring in the sums of Eq. (15) are of the same form, the desired counting distribution can be calculated.

Formally, we let the combined exponent be represented as

$$f(\lambda) = \ln[p_k(n, \lambda)] - (\lambda - \bar{\lambda})^2/2\sigma^2 \quad (17)$$

and define

$$q_m(n, k, \lambda) = \partial^m \ln p_k(n, \lambda)/\partial\lambda^m . \quad (18)$$

We then determine λ_0 implicitly for each n and k from the stationarity condition

$$f'(\lambda_0) = 0 \quad (19)$$

or, specifically,

$$\lambda_0 = \bar{\lambda} + \sigma^2 q_1(n, k, \lambda_0) . \quad (20)$$

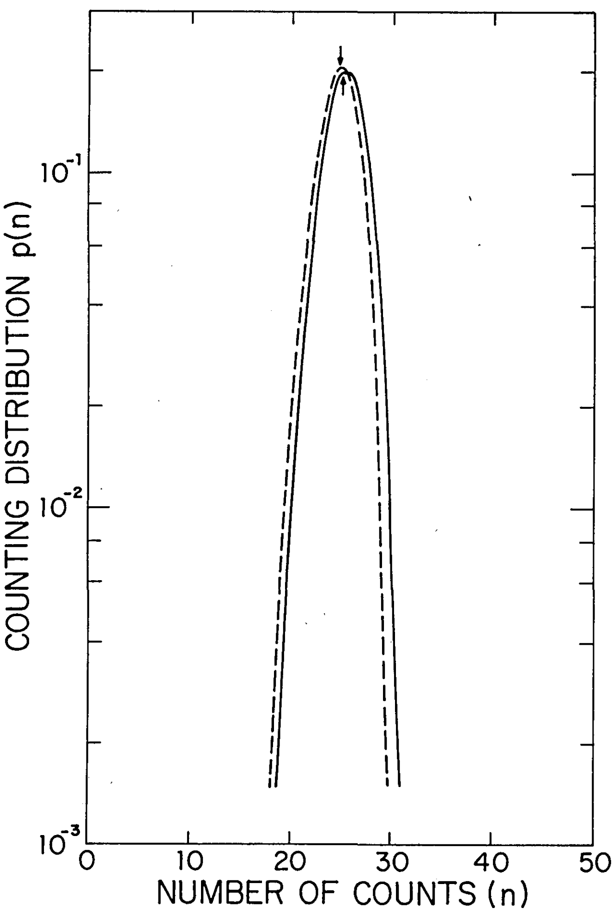

FIG. 3. Dead-time-corrected photocounting distributions for amplitude-stabilized (dashed curve) and Van der Pol laser (solid curve) radiation. The mean irradiance of the Van der Pol laser is equal to the irradiance of the amplitude-stabilized source to first order. The excitation parameter w of the laser is taken to be 12.5. As in Fig. 1, $\tau/T = 0.025$.

A Taylor-series expansion of $f(\lambda)$ in the variable $x = (\lambda - \lambda_0)$ yields

$$f(\lambda) = f(\lambda_0) + f'(\lambda_0)x + f''(\lambda_0)x^2/2! + R(x) \quad (21)$$

in which the remainder term $R(x)$ is of order $f'''(\lambda_0)x^3$. Provided that this can be neglected in comparison with $f''(\lambda_0)x^2$, the integral in Eq. (16) yields

$$p_{k,n} \simeq [1 - \sigma^2 q_2(n, k, \lambda_0)]^{-1/2}\{p_k(n, \lambda_0)\exp[-\tfrac{1}{2}(\lambda_0 - \bar{\lambda})/\sigma^2]\} . \quad (22)$$

The function $p_k(n, \lambda)$ given in Eq. (4) yields $q_m(n, k, \lambda)$ given by

$$q_1(n, k, \lambda) = k/\lambda - (T - n\tau) , \quad (23a)$$

$$q_2(n, k, \lambda) = -k/\lambda^2 , \quad (23b)$$

$$q_3(n, k, \lambda) = 2k/\lambda^3 , \quad (23c)$$

so that the procedure requires the solution of

$$\lambda_0 = \bar{\lambda} + \sigma^2 k/\lambda_0 - \sigma^2(T - n\tau) \quad (24)$$

for $\lambda_0(n, k)$, with $\bar{\lambda}$ and σ as parameters. Thus, the integral in Eq. (16) yields

$$p_{k,n} \simeq (1 + \sigma^2 k/\lambda_0^2)^{-1/2}\{p_k(n, \lambda_0)\exp[-\tfrac{1}{2}(\lambda_0 - \bar{\lambda})/\sigma^2]\} ; \quad (25)$$

the desired photoelectron-counting distribution is therefore

$$p(n, \sigma, \bar{\lambda}) = \sum_{k=0}^{n} p_{k,n} - \sum_{k=0}^{n-1} p_{k,n-1} . \quad (26)$$

Figure 3 shows a comparison of the distributions arising from an amplitude-stabilized source (dashed curve) and a Van der Pol laser (solid curve) for a photodetector with $\tau/T = 0.025$. The mean irradiance from the Van der Pol laser is taken to be equal (to first order) to the irradiance from the stable source, and the variance corresponds to a laser-excitation value $w = 12.5$. The dashed curve is identical to the dashed curve in Fig. 1 (mean = 24.6, variance = 3.6), whereas the solid curve has a mean of 25.2 and a variance of 3.9.

To obtain the approximate counting distributions given above, the averaging integrals in Eq. (16) were evaluated by the saddle-point method. This yields reliable results when the irradiance distribution represented in Eq. (8) has a well-defined peak and falls off rapidly on either side of that peak. This requires $w \gtrsim 5$ and therefore corresponds to the Van der Pol laser somewhat above threshold. When the method of steepest descent cannot be used, a Gauss–Laguerre integration might be applied to provide accurate numerical results.

V. DEAD-TIME-CORRECTED BOSE–EINSTEIN COUNTING DISTRIBUTION

For a chaotic radiation source, the ensemble averages in Eq. (7) are given by

$$\langle p_k(n, \lambda)\rangle_M = \{\lambda[T - n\tau]\}^k/\{1 + \lambda[T - n\tau]\}^{k+1} , \quad (27)$$

yielding a dead-time-corrected Bose–Einstein counting distribution that can be written

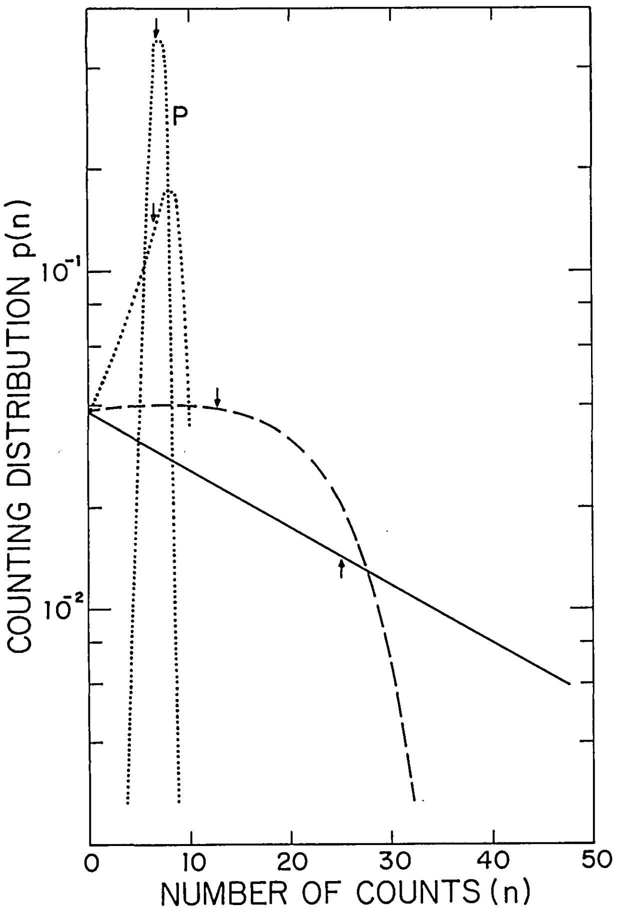

FIG. 4. The Bose–Einstein distribution with no dead time is represented by the solid curve (mean = 25.0, variance = 1275.0). The effects of dead time produce the distributions shown by the dashed curve ($\tau/T = 0.025$, mean = 12.7, variance = 54.5) and by the dotted curve ($\tau/T = 0.1$, mean = 6.5, variance = 4.7). Note the reduction of the mean (shown by arrows) due to the elimination of output pulses. The dotted curve labeled P represents the dead-time-corrected Poisson distribution ($\tau/T = 0.1$, mean = 6.7, variance = 0.7), clearly demonstrating the similarity of diverse counting distributions in the dead-time-limited domain.

$$p(n, \lambda, \tau) = \sum_{k=0}^{n} \{\lambda[T - n\tau]\}^k / \{1 + \lambda[T - n\tau]\}^{k+1} - \sum_{k=0}^{n-1} \{\lambda[T - (n-1)\tau]\}^k / \{1 + \lambda[T - (n-1)\tau]\}^{k+1} . \quad (28)$$

This counting distribution is valid for counts such that $n\tau < T$; count numbers greater than T/τ are forbidden. In Fig. 4, we compare the dead-time-corrected Bose–Einstein distribution with the uncorrected distribution, for several ratios of dead time to sampling time, τ/T. The solid curve represents the original uncorrected Bose–Einstein (mean = 25.0, variance = 1275.0), the dashed curve represents the distribution for $\tau/T = 0.025$, whereas the dotted curve represents the distribution for $\tau/T = 0.1$. For comparison, the dotted curve labeled P represents a dead-time-corrected Poisson distribution with $\tau/T = 0.1$. The original mean was taken to be $\lambda T = 25.0$ for all curves. The dead-time-corrected means and variances are reduced to 6.7 and 0.7 (Poisson, $\tau/T = 0.1$), 6.5 and 4.7 (Bose–Einstein, $\tau/T = 0.1$), and 12.7 and 54.5 (Bose–Einstein, $\tau/T = 0.025$). Note that all Bose–Einstein distributions have the same probability at $n = 0$; this is a result of assuming that the counter is unblocked at $t = 0$. For a blocked counter, the probability at $n = 0$ depends on the dead time τ. The antibunching effects of the dead time rapidly overcome the inherent bunching of the Bose–Einstein distribution. In the highly dead-time-limited case ($\tau/T = 0.1$), the Poisson and the Bose–Einstein distributions appear quite similar, in marked contrast with their behavior in the absence of dead time. Note the sharp cutoff at $n = T/\tau = 10$ counts. The effects of the detector dead time are, once again, to shift the mean to a lower value, and to reduce the variance of the distribution.

VI. COUNTING DISTRIBUTION FOR VARIABLE DEAD TIME

In much the same way that Eq. (6) or Eq. (14) represents an average over the statistics of the irradiance, they could be modified to provide an average over the stochastic variation of dead time for a source of constant intensity. If the distribution of dead time is gaussian with mean $\bar{\tau}$ and variance σ_τ^2, the method of solution will be similar to that presented previously in Eqs. (16)–(22), and an approximate solution can be obtained for $\bar{\tau} \gg \sigma_\tau^2$. Thus, a measurement of the photocounting distribution for a source of constant intensity will provide information about the statistical distribution of the photodetector dead time in those cases in which it is not fixed. Inasmuch as the light from a tungsten lamp has a very small coherence time τ_c, simply choosing $T \gg \tau_c$ will provide a convenient laboratory source suitable for this purpose.[4–7] Applications of the variable-dead-time model to problems in vision research will be discussed in a future publication.

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APPENDIX

The exact dead-time-corrected Poisson counting distribution given in Eq. (2) can be derived using the results of De Lotto *et al.*[8] For simplicity, we use the same notation as these authors. In general, we may write

$$P_n^*(0, t) = \int_0^t [p_n^*(0, t') - p_{n+1}^*(0, t')] dt' , \quad (A1)$$

where $P_n(0, t)$ is the probability of recording exactly n counts in the interval $(0, t)$ and $p_n(0, t')dt'$ is the probability that the nth count occurs between t' and $t' + dt'$ in an observation interval beginning at $t = 0$. The asterisk indicates that the counter is unblocked at $t = 0$, whereas functions without an asterisk represent a counter blocked at $t=0$. In the Laplace domain, Eq. (A1) becomes

$$P_n^*(s)=s^{-1}[p_n^*(s)-p_{n+1}^*(s)]\,. \tag{A2}$$

For a nonparalyzable counter,

$$p_n^*(s)=p_1^*(s)p_{n-1}(s)=p_1^*(s)[p_1(s)]^{n-1}\,, \tag{A3}$$

and Eq. (A2) can be written

$$P_n^*(s)=s^{-1}p_1^*(s)[1-p_1(s)][p_1(s)]^{n-1}\,, \tag{A4}$$

for $n\geq 1$.

For a Poisson input distribution with mean rate λ, $p_1^*(s)$ and $p_1(s)$ are given by[8]

$$p_1^*(s)=p_1(s)=\lambda(s+\lambda)^{-1}\,. \tag{A5}$$

At the output of the counter, taking into account the fixed dead time τ,

$$p_{1,u}^*(s)=\lambda(s+\lambda)^{-1}$$

and (A6)

$$p_{1,u}(s)=\lambda(s+\lambda)^{-1}\exp(-s\tau)\,,$$

where u signifies that Eq. (A6) represents a dead-time-corrected process. Therefore,

$$P_{n,u}^*(s)=s^{-1}\lambda(s+\lambda)^{-1}[1-\lambda(s+\lambda)^{-1}\exp(-s\tau)]\times[\lambda(s+\lambda)^{-1}\exp(-s\tau)]^{n-1}\,, \tag{A7}$$

which reduces to

$$P_{n,u}^*(s)=s^{-1}\lambda^n(s+\lambda)^{-n}\exp[-s\tau(n-1)]-s^{-1}\lambda^{n+1}(s+\lambda)^{-(n+1)}\exp[-(ns\tau)]\,. \tag{A8}$$

A partial-fraction expansion for the term $s^{-1}(s+\lambda)^{-n}$ takes the form

$$s^{-1}(s+\lambda)^{-n}=K_1s^{-1}+K_{2,n}(s+\lambda)^{-n}+K_{2,n-1}(s+\lambda)^{-(n-1)}+\cdots+K_{2,2}(s+\lambda)^{-2}+K_{2,1}(s+\lambda)^{-1}\,. \tag{A9}$$

The coefficients are easily found to be

$$K_1=\lambda^{-n}$$

and (A10)

$$K_{2,n-m}=-\lambda^{-(m+1)}\,,\quad m=0,1,2,\ldots,n-1\,.$$

The inverse transform of this term is therefore given by

$$\mathcal{L}^{-1}\{s^{-1}(s+\lambda)^{-n}\}=u(t)\lambda^{-n}-\sum_{m=0}^{n-1}\lambda^{-(m+1)}\mathcal{L}^{-1}[(s+\lambda)^{-(n-m)}]=u(t)\lambda^{-n}-\sum_{m=0}^{n-1}\lambda^{-(m+1)}[t^{(n-m-1)}e^{-\lambda t}/(n-m-1)!]\,, \tag{A11}$$

where $u(t)$ is the unit step function.

Thus, for the first term in Eq. (A8),

$$\mathcal{L}^{-1}\{\lambda^n\exp[-s\tau(n-1)]s^{-1}(s+\lambda)^{-n}\}=u[t-(n-1)\tau]-\sum_{m=0}^{n-1}\lambda^{n-(m+1)}\{[t-(n-1)\tau]^{(n-m-1)}/(n-m-1)!\}\times\exp\{-\lambda[t-(n-1)\tau]\}\,. \tag{A12}$$

In a similar manner, the second term in Eq. (A8) yields

$$\mathcal{L}^{-1}\{\lambda^{n+1}\exp[-(ns\tau)]s^{-1}(s+\lambda)^{-(n+1)}\}=u(t-n\tau)-\sum_{m=0}^{n}\lambda^{n-m}\{[t-n\tau]^{n-m}/(n-m)!\}\exp[-\lambda(t-n\tau)]\,. \tag{A13}$$

If $k=n-m-1$ in Eq. (A12) and $k=n-m$ in Eq. (A13), the desired counting distribution is

$$p_0(n,\lambda,\tau)=P_{n,u}^*(0,T)=\sum_{k=0}^{n}\{\lambda^k(T-n\tau)^k/k!\}\exp\{-\lambda(T-n\tau)\}-\sum_{k=0}^{n-1}\{\lambda^k[T-(n-1)\tau]^k/k!\}\exp\{-\lambda[T-(n-1)\tau]\}\,, \tag{A14}$$

valid for $n\tau<T$.

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