N-Fold Joint Photocounting Distribution for Modulated Laser Radiation: Transmission Through the Turbulent Atmosphere*

M. C. TEICH, S. ROSENBERG
Department of Electrical Engineering and Computer Science, Columbia University, New York, New York 10027, USA

Received 26 October 1970; in revised form 12 March 1971

In this paper a generalised result for the N-fold joint photoelectron counting distribution for independently modulated radiation is given. We extend the recent results of Diament and Teich, for the one-fold photoelectron counting distribution for light propagated through an atmosphere characterised by log-normal irradiance fluctuations, to the N-fold joint photoelectron counting distribution. An approximate solution for this N-fold distribution is obtained, for detection intervals $\{T_i\} \ll \tau_a$ where τ_a is the characteristic time of the atmospheric turbulence. We present specifically the two-fold joint photocounting distribution for amplitude-stabilised laser radiation passing through such an atmosphere for several levels of turbulence and degrees of correlation. Cases including additive, independent, non-interfering Poisson noise are considered. Computer generated plots of the photocounting distribution are presented. For noise-free detection, the otherwise narrow-peaked photocounting distribution is seen to broaden markedly and shift its peak to lower counts as the turbulence level increases. Furthermore, a non-singular counting distribution is obtained for fully correlated detection. In the presence of additive noise and varying only the signal-to-noise ratio γ, the probability surface is intermediate between that of the Poisson and that of the noise-free log-normal fading counting distribution. The peak, however, is observed to decrease and then again increase in magnitude as $\gamma \to \infty$, for correlated detection only. These results are expected to be of use in the study of atmospheric turbulence, as well as in the evaluation of certain stochastic functionals that occur in optical communication theory for the turbulent atmospheric channel.

1. Introduction

Optical communication through the turbulent atmosphere has recently received considerable attention in the literature. However, most of the investigations to date have considered detector configurations which yield a continuous current as the observable [1-7]. In much of the work, the statistics of the atmospheric turbulence are assumed to be log-normal since the predominance of recent experimental work supports this model. That is to say, the logarithm of both the field amplitude and intensity are normally distributed [4-7]. Diament and Teich [8-10] have recently considered the photodetection of laser radiation passing through the turbulent

*This work was supported in part by the US National Science Foundation under Grant Number NSF-GK-16649.

atmosphere at low intensity levels and for short detection time intervals, for which case the detector output consists of discrete random pulses rather than a continuous current. They evaluated the one-fold photocounting distribution for signal noise limited detection of coherent as well as coherent plus chaotic radiation for several levels of turbulence. In that work, the effect of the turbulence was interpreted as equivalent to modulation of the mean of the undisturbed counting distribution. The detection regime considered there, i.e. low radiation levels and short intervals, is of great importance in the evaluation of high data rate optical communications and radar systems. In this regime, then, over a detection interval $(t, t+T)$, n photoelectrons are counted. The distribution of these counts translate into statistics which can reveal properties of both the source and of the intervening medium.

For a single detector, the photocounting distribution $p(n, t, T)$, is defined as the probability of counting n photoelectrons in the time interval $(t, t+T)$. If we consider N-time interval detection at a single detector, the N-fold joint-time photocounting distribution, $p(n_1, t_1, T_1; n_2, t_2, T_2; \ldots; n_N, t_N, T_N)$, is defined as the probability of observing n_1 counts over the time interval (t_1, t_1+T_1), and n_2 counts over $(t_2, t_2+T_2), \ldots,$ and n_N counts over (t_N, t_N+T_N). Similarly for N simultaneously illuminated detectors the N-fold space-time joint photocounting distribution, $p(n_1, \mathbf{r}_1, T_1; n_2, \mathbf{r}_2, T_2; \ldots; n_N, \mathbf{r}_N, T_N)$, is defined as the probability of observing n_1 counts at a detector at the space point $\mathbf{r}_1$ over the time interval $(t, t+T_1)$, and n_2 counts at $\mathbf{r}_2$ over $(t, t+T_2), \ldots,$ and n_N counts at $\mathbf{r}_N$ over $(t, t+T_N)$. These joint photocounting distributions have been evaluated for various mixtures of coherent and chaotic radiation [11-14]; however, none have presented the effects of modulation (or of the atmosphere).

In this paper we consider the N-fold joint photocounting distribution for optical radiation passing through a turbulent atmosphere inducing log-normal scintillation, and detected either with or without additive, independent, non-interfering noise. We present an approximate general solution of the N-fold joint photocounting distribution for light of arbitrary statistics provided only that weak conditions are obeyed by the undisturbed counting distribution. In addition, we investigate specifically the two-fold joint distribution for laser radiation when detection time intervals are short compared to both the fluctuation time of the atmosphere τ_a, usually of the order of 1 msec, and the coherence time of the radiation source τ_c. These distributions are obtained for a receiver consisting of an array of detectors, for the general case of correlated detection. This is in contrast to the single detector receiver considered previously [8, 9]. Computer generated plots of the two-fold counting distribution for an amplitude-stabilised source are given for several levels of turbulence, degrees of correlation of the detected field, and for several signal-to-noise ratios.

2. Theory

If a quasi-monochromatic, linearly polarised beam of light falls perpendicularly on a single photodetector, then the photocount distribution is given by the Mandel formula [15, 16],

$$p(n, t, T) = \left\langle \frac{W^n \, \mathrm{e}^{-W}}{n!} \right\rangle_W , \tag{1}$$

where the brackets $< >$ indicate an ensemble average over the statistics of W, which is the integrated intensity and is defined as

$$W \equiv \int_t^{t+T} \alpha I(t') \mathrm{d}t' . \tag{2}$$

Here, the quantum efficiency α includes the detector area. The instantaneous intensity (or irradiance), $I(t')$, may be expressed as the product $V^*(\mathbf{r}, t)\, V(\mathbf{r}, t)$, where $V(\mathbf{r}, t)$ is the complex analytic field [16]. In quantum-mechanical terms, $I(t') = |\mathcal{E}(\mathbf{r}, t)|^2$ where $\mathcal{E}(\mathbf{r}, t)$ is the eigenvalue of the photon annihilation operator of the quantised field [17]. Since the source of radiation is fluctuating, $I(t')$ will be stochastic with a density function $p(I)$. Moreover, W is a time integral of $I(t')$, and one cannot generally relate the statistics of W to those of $I(t')$. However, for short counting intervals, i.e. when the counting time T is much smaller than the coherence time of the light, τ_c, then

$$W = \alpha I(t)T \tag{3}$$

and the statistics of W are directly related to those of $I(t)$. Thus, assuming $I(t)$ is stationary, and that $T \ll \tau_c$, the one-fold counting distribution may be written as

$$p_0(n, N_0) = \int_0^\infty \frac{W^n \, e^{-W}}{n!} p(I) dI = \int_0^\infty \frac{(\alpha IT)^n \, e^{-\alpha IT}}{n!} p(I) dI \ , \tag{4}$$

where $N_0 = \alpha \langle I \rangle T$.

When the radiation is itself modulated either deterministically or stochastically, the integrated intensity W is reinterpreted as reflecting the variation of both the source fluctuation and the modulation. In many cases both effects can be accounted for while leaving equation 2 intact, if I is interpreted as the source intensity and α is taken to include not merely the quantum efficiency of the detector but also the overall transfer function from source to photosensitive surface. This can include the effects of various optical components, as well as the characteristics of the media through which the light is passed. The factor α can hence be time varying and must remain under the integral in equation 2. This model is valid when the modulation medium between the source and the detector acts as a linear irradiance filter.

We now consider both α and I to be randomly fluctuating, independently, with both processes stationary, i.e. independent of t in equation 2. We emphasise that the effects of the modulation are attributed to α, rather than I. For detection intervals short compared to the characteristics times of both the modulation, τ_m, and the source, τ_c, such variations imposed on the instantaneous irradiance are clearly equivalent to modulation of its mean [10].

Thus, the ensemble average in equations 1 and 4 is seen to be over the statistics of both the source and the modulation. Since α and I are independent, the counting distribution is then

$$p(n, N_0) = \int_\alpha p_0(n, N_0) p(\alpha) d\alpha \tag{5}$$

where the counting distribution in the absence of modulation is given by equation 4.

The N-fold counting distribution in the presence of modulation is similarly obtained. For every detector in a multidetector array, the Mandel formula applies independently. That is, given the integrated intensity W_i for the ith detector, we may write

$$p(n_1, N_1; n_2, N_2; \ldots ; n_N, N_N \,|W_1, W_2, \ldots , W_N) = \prod_{i=1}^{N} \frac{W_i^{n_i} \, e^{-W_i}}{n_i!} \ . \tag{6}$$

It then follows that the joint photoelectron counting distribution is [11]

$$p(n_1, N_1; n_2, N_2; \ldots ; n_N, N_N) = \left\langle \prod_{i=1}^{N} \frac{W_i^{n_i} \, e^{-W_i}}{n_i!} \right\rangle_{\{W_i\}} , \tag{7}$$

where the ensemble average is over the joint statistics of $\{W_i\}$. For counting intervals $\{T_i\} \ll \tau_c$, τ_m, the integrated intensity is given by $W_i = \alpha_i I_i T_i$.

If we express the undisturbed joint photocounting distribution in the absence of independent modulation as

$$p_0(n_1, N_1; n_2, N_2; \ldots ; n_N, N_N) = \left\langle \prod_{i=1}^{N} \frac{W_i^{n_i} \, e^{-W_i}}{n_i!} \right\rangle_{\{I_i\}} , \tag{8}$$

then the doubly fluctuating counting distribution, in the presence of independent modulation, is found from the averaging integral

$$p([n]; [N]) = \int \ldots \int_\alpha p_0([n]; [N]) \, p([\alpha]) d\alpha_1 \ldots d\alpha_N \ , \tag{9}$$

where $[x]$ is defined as the ordered set $\{x_1, x_2, \ldots , x_N\}$.

As in the single dimensional case, we interpret the effect of the modulation, for the short counting intervals, as equivalent to modulating the mean of each of the W_i. The N-fold counting distribution can then be written as

$$p([n];[N];A) = \int_0^\infty \ldots \int_0^\infty p_0([n];[K])\, p([K];[N];A)\mathrm{d}K_1 \ldots \mathrm{d}K_N\ , \tag{10}$$

where each K_i is the randomly distributed mean count of $p_0([n];[K])$, the undisturbed counting distribution; $\{N_i\}$ are the overall mean counts for the individual photodetectors contained in $p([n];[N];A)$, and A is a parameter containing the modulation level and irradiance correlations. This result applies to an arbitrary form of modulation.

3. Statistics of the Turbulent Atmosphere

In this section we consider, in particular, modulation induced by a turbulent atmosphere characterised by log-normal statistics.

The basic phenomenon responsible for atmospherically induced irradiance scintillation is the random fluctuation of the refractive index produced by turbulent mixing of globules of air at different temperatures. Tatarski [1] and others [18-20] have evaluated the statistical properties of radiation propagating through a turbulent atmosphere with the result that the logarithm of the field $V(\mathbf{r}, t)$ is normally distributed. That is, $\ln V(\mathbf{r}, t) = \ln A|z(\mathbf{r}, t)| + j\theta(\mathbf{r}, t)$ is a complex Gaussian process, completely specified by the mean and covariance functions of its real and imaginary parts. The received field is assumed to be spatially homogeneous and isotropic, when the field in the absence of turbulence is a plane-wave. Letting $V(\mathbf{r}, t) = A\, z(\mathbf{r}, t) = A|z|\mathrm{e}^{j\theta}$, where A is a constant amplitude and $z(\mathbf{r}, t)$ is the varying part, the probability density of z is given by

$$p(|z|, \theta) = \frac{1}{2\pi}\,\frac{1}{\sqrt{2\pi\sigma'^2}\,|z|}\exp\left[-\frac{(\ln|z|-m)^2}{2\sigma'^2}\right]\ ,\quad |z|>0\ . \tag{11}$$

The variance of the phase angle θ is observed to be quite large and θ is thus taken to be uniformly distributed [3]. The quantity σ'^2 is the log-amplitude variance and for the plane wave case considered here, $m = -\sigma'^2$ [18]. Although we have specifically considered N-space points over the intervals $\{t, t+T_i\}$, and thus $z(\mathbf{r}_i, t) \rightarrow z(\mathbf{r}_i)$ over these intervals, the results are valid for the more general N-space-time points with time intervals $\{t_i, t_i+T_i\}$. In the latter case, of course, the appropriate space-time covariance function is required.

Of the several models for the atmosphere, none has proven to be entirely satisfactory. However, as long as log-normal statistics are assumed, the counting distributions given here are valid regardless of the detailed form of the log-amplitude covariance function $C_l(\rho)$, where $\rho = |\mathbf{r}_1 - \mathbf{r}_2|$. This covariance function is dependent on several variables such as path length, the scale of turbulence, altitude, and the type of wave propagated. As an example, Ochs, Bergman, and Snyder [5] give experimental normalised covariance functions for laser radiation propagated over path lengths from 5.5 to 145 km. For $\lambda = 6328$ Å and a path length of 5.5 km, the normalised covariance falls to its 3 dB point for a detector spacing of 2 cm, and falls off to zero for $\rho \geqslant 9.5$ cm. For a path length of 15 km, the 3 dB and zero points correspond to $\rho = 3$ cm and $\rho \geqslant 16$ cm respectively.

Likewise, the log-intensity is also normal [18, 20] since $I/\langle I\rangle = |z|^2$, and thus $\log(I/\langle I\rangle) = 2\log|z|$. Carrying this over to the present case where α is considered as the parameter that contains the effect of the atmosphere, then $\alpha/\langle\alpha\rangle \leftrightarrow |z|^2$, and thus

$$p(\alpha) = \frac{1}{\sqrt{2\pi\sigma^2}\,\alpha}\exp\left[\frac{-\left(\ln\dfrac{\alpha}{\langle\alpha\rangle}+\dfrac{\sigma^2}{2}\right)^2}{2\sigma^2}\right],\quad \alpha>0\ . \tag{12}$$

Here, σ^2 is the log-intensity variance and is given by $\sigma^2 = 4\sigma'^2$, where σ'^2 has been defined as the log-amplitude variance, and is given by $C_l(0)$. Similarly the log-intensity covariance is given by $C_{II}(\rho) = 4C_l(\rho)$ [21, 22].

As already noted in section 2, this model for attributing the modulation to a multiplicative process α is valid only when the medium between the source and the detector acts as a linear irradiance filter or, as in the present case, when an effective multiplicative function, which may result from interference of the field due to the turbulence, can be found. The joint log-normal density for $(\alpha_1, \alpha_2, \ldots \alpha_N)$ is then written [23]

$$p([\alpha]; [\langle\alpha\rangle]; \Lambda) = \frac{1}{(2\pi)^{N/2}\,|\Lambda|^{1/2}\,\alpha_1\alpha_2\ldots\alpha_N} \exp\left(-\tfrac{1}{2}\mathbf{X}'^{\dagger}[\Lambda]^{-1}\mathbf{X}'\right), \quad \alpha_i > 0 \tag{13}$$

where Λ is the log-intensity covariance matrix

$$\Lambda = \langle\{\log\boldsymbol{\alpha} - \langle\log\boldsymbol{\alpha}\rangle\}\{\log\boldsymbol{\alpha} - \langle\log\boldsymbol{\alpha}\rangle\}^{\dagger}\rangle \equiv \begin{bmatrix} C_{II}(\rho_{11}) & C_{II}(\rho_{12})\cdots & C_{II}(\rho_{1N}) \\ C_{II}(\rho_{21}) & C_{II}(\rho_{22})\cdots & \vdots \\ \vdots & \vdots & \vdots \\ C_{II}(\rho_{N1})\cdots & \cdots & C_{II}(\rho_{NN}) \end{bmatrix} \tag{14}$$

and $\rho_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$. Thus, for $i = j$, $\rho_{ii} = 0$, and $C_{II}(\rho_{ii}) = \sigma_i^2$. Letting $R_{ij} = \frac{C_{II}(\rho_{ij})}{\sigma_i\sigma_j}$ be the correlation coefficient, then

$$\Lambda = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2R_{12}\cdots & \sigma_1\sigma_NR_{1N} \\ \sigma_2\sigma_1R_{21} & \sigma_2^2 \quad \cdots & \vdots \\ \sigma_N\sigma_1R_{N1} & \vdots\cdots\cdots & \sigma_N^2 \end{bmatrix}. \tag{15}$$

The column vector $\mathbf{X}'$ is defined as

$$\mathbf{X}' \equiv \begin{bmatrix} \ln(\alpha_1/\langle\alpha_1\rangle) + \frac{\sigma_1^2}{2} \\ \ln(\alpha_2/\langle\alpha_2\rangle) + \frac{\sigma_2^2}{2} \\ \vdots \\ \ln(\alpha_N/\langle\alpha_N\rangle) + \frac{\sigma_N^2}{2} \end{bmatrix}. \tag{16}$$

Recalling the interpretation that the atmospheric turbulence modulates the mean irradiance, and thus the corresponding mean count, we define $\{K_i\}$ as the mean counts of the undisturbed counting distribution. These $\{K_i\}$ are now log-normally distributed with covariance Λ and expected values $\{N_i\}$. The distribution of the mean counts is then, following equation 13,

$$p([K]; [N]; \Lambda) = \frac{1}{(2\pi)^{N/2}|\Lambda|^{1/2}K_1K_2\ldots K_N} \exp\left(-\tfrac{1}{2}\mathbf{X}^{\dagger}[\Lambda]^{-1}\mathbf{X}\right) \tag{17}$$

where

$$K_i = \alpha_i\langle I_i\rangle T_i \;;\; N_i = \langle\alpha_i\rangle\langle I_i\rangle T_i \;,$$

and

$$X_i = \ln\frac{K_i}{N_i} + \frac{\sigma_i^2}{2} \;. \tag{18}$$

The overall counting distribution from equation 10 is then given by

$$p([n];[N];\Lambda) = \int_0^\infty \ldots \int_0^\infty p_0([n];[K]) \frac{\exp\{-\frac{1}{2}\mathbf{X}^\dagger[\Lambda]^{-1}\mathbf{X}\}}{(2\pi)^{N/2}|\Lambda|^{1/2}K_1K_2\ldots K_N} \mathrm{d}K_1\ldots\mathrm{d}K_N \, . \quad (19)$$

The above general integral is valid for short detection intervals and for any undisturbed counting distribution $p_0([n];[K])$.

4. Approximate Solution of the N-Fold Integral

In the previous section the N-fold counting distribution for the atmospherically modulated irradiance was shown to take the general form given in equation 19.

Following the method used by Diament and Teich [8, 9] we express the undisturbed counting distribution $p_0([n];[K])$ as an exponential of its own logarithm and we make a change of variable from K to $\ln K$. The integral then becomes

$$\int_{-\infty}^\infty \ldots \int_{-\infty}^\infty \frac{\exp\{\ln p_0([n];[K])\}\exp\{-\frac{1}{2}\mathbf{X}^\dagger[\Lambda]^{-1}\mathbf{X}\}}{(2\pi)^{N/2}|\Lambda|^{1/2}} \mathrm{d}(\ln K_1)\ldots\mathrm{d}(\ln K_N) \, . \quad (20)$$

We proceed to obtain the point where this quantity is a maximum, i.e. the stationary point, and obtain a Taylor series expansion of the exponent about this point.

Defining the combined exponent as $H([K])$, that is $H([K]) \equiv \{\ln p_0([n];[K])\} - \frac{1}{2}\mathbf{X}^\dagger[\Lambda]^{-1}\mathbf{X}$, and defining $Y_i \equiv \ln K_i$, we obtain

$$H([Y]) = H([Y_0]) + \left.\frac{\partial H}{\partial \mathbf{Y}}\right|^\dagger_{\mathbf{Y}=\mathbf{Y}_0}(\mathbf{Y}-\mathbf{Y}_0) + \tfrac{1}{2}(\mathbf{Y}-\mathbf{Y}_0)^\dagger\left(\frac{\partial}{\partial \mathbf{Y}}\left\{\frac{\partial H}{\partial \mathbf{Y}}\right\}^\dagger\right)(\mathbf{Y}-\mathbf{Y}_0) \quad (21)$$

plus terms of the order of $\|\mathbf{Y}-\mathbf{Y}_0\|^3$.

The stationary point $\mathbf{Y}_0$ is then obtained implicitly from the stationary condition for the combined exponent,

$$\frac{\partial}{\partial \mathbf{Y}}\{\ln p_0([n];[K]) - \tfrac{1}{2}\mathbf{X}^\dagger[\Lambda]^{-1}\mathbf{X}\} = 0 \, , \quad (22)$$

which takes the form

$$\begin{bmatrix} \dfrac{\partial \ln p_0}{\partial(\ln K_1)} \\ \dfrac{\partial \ln p_0}{\partial(\ln K_2)} \\ \vdots \\ \dfrac{\partial \ln p_0}{\partial(\ln K_N)} \end{bmatrix} - [\Lambda]^{-1}\mathbf{X} = 0 \, . \quad (23)$$

Defining the quantities $Q_i^{(m)}([n];[K])$ as

$$Q_i^{(m)}([n];[K]) \equiv \frac{\partial^m\{\ln p_0([n];[K])\}}{\partial(\ln K_i)^m} \, , \quad (24)$$

equation 23 can be expressed compactly as

$$\mathbf{Q}^{(1)}([n];[K]) - [\Lambda]^{-1}\mathbf{X} = 0 \, . \quad (25)$$

The solution to these N simultaneous equations is the stationary point $[K_0] \equiv \{K_{01}, K_{02}, \ldots K_{0N}\}$; similarly $[Y_0] \equiv [\ln K_0]$. Defining the matrix $\mathbf{B}$ as

$$\mathbf{B} \equiv \left.\frac{\partial}{\partial \mathbf{Y}}\left\{\frac{\partial H}{\partial \mathbf{Y}}\right\}^\dagger\right|_{\mathbf{Y}_0} \quad (26)$$

and with

$$Q_{ij}^{(2)}([n];[K]) \equiv \frac{\partial^2\{\ln p_0([n];[K])\}}{\partial(\ln K_i)\,\partial(\ln K_j)} \quad (27)$$

it can be shown that

$$\mathbf{B} = \begin{bmatrix} Q_{11}^{(2)} - \dfrac{\sigma_1^2}{|\Lambda|} & Q_{12}^{(2)} + \dfrac{\sigma_1\sigma_2 R_{12}}{|\Lambda|} \dots & Q_{1N}^{(2)} + \dfrac{\sigma_1\sigma_N R_{1N}}{|\Lambda|} \\ Q_{21}^{(2)} + \dfrac{\sigma_2\sigma_1 R_{21}}{|\Lambda|} & Q_{22}^{(2)} - \dfrac{\sigma_2^2}{|\Lambda|} \quad \dots & \vdots \\ \vdots & \vdots & \vdots \\ Q_{N1}^{(2)} + \dfrac{\sigma_N\sigma_1 R_{N1}}{|\Lambda|} & \dots\dots\dots\dots & Q_{NN}^{(2)} - \dfrac{\sigma_N^2}{|\Lambda|} \end{bmatrix}, \tag{28}$$

where $|\Lambda|$ is the determinant of the covariance matrix Λ.

Recalling that $X_i = \ln\left(\dfrac{K_i}{N_i}\right) + \dfrac{\sigma_i^2}{2}$, we define $X_{0i} = \ln\left(\dfrac{K_{0i}}{N_i}\right) + \dfrac{\sigma_i^2}{2}$ at the stationary point K_{0i} and thus

$$H([Y]) = \ln\{p_0([n];[K_0])\} - \tfrac{1}{2}\mathbf{X}_0{}^{\dagger}[\Lambda]^{-1}\mathbf{X}_0 + \tfrac{1}{2}(\mathbf{Y}-\mathbf{Y}_0)^{\dagger}\mathbf{B}(\mathbf{Y}-\mathbf{Y}_0) + R_N\ , \tag{29}$$

where R_N contains terms of the order $\|\mathbf{Y}-\mathbf{Y}_0\|^3$.
Since, at the stationary point $[K_0]$, $\mathbf{Q}^{(1)}([n];[K_0]) = [\Lambda]^{-1}\mathbf{X}_0$, then

$$H([Y]) = \ln\{p_0([n];[K_0])\} - \tfrac{1}{2}\mathbf{X}_0{}^{\dagger}\mathbf{Q}^{(1)}([n];[K_0]) + \tfrac{1}{2}(\mathbf{Y}-\mathbf{Y}_0)^{\dagger}\mathbf{B}(\mathbf{Y}-\mathbf{Y}_0) + R_N\ . \tag{30}$$

If we make the approximation of neglecting all terms higher than second order, the averaging integral becomes

$$\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\frac{p_0([n];[K_0])}{(2\pi)^{N/2}|\Lambda|^{1/2}}\exp\{-\tfrac{1}{2}\mathbf{X}_0{}^{\dagger}\mathbf{Q}^{(1)}([n];[K_0])\}\exp\{\tfrac{1}{2}(\mathbf{Y}-\mathbf{Y}_0)^{\dagger}\mathbf{B}(\mathbf{Y}-\mathbf{Y}_0)\}\,\mathrm{d}Y_1\dots\mathrm{d}Y_N \tag{31}$$

which gives rise to the counting distribution

$$p([n];[N];\Lambda) \simeq \frac{p_0([n];[K_0])}{|\Lambda|^{1/2}|\mathbf{B}|^{1/2}}\exp\{-\tfrac{1}{2}\mathbf{X}_0{}^{\dagger}\mathbf{Q}^{(1)}([n];[K_0])\}\ . \tag{32}$$

This expression is valid provided that the matrix **B** is negative definite. For the amplitude-stabilised radiation source considered in section 5, this condition is satisfied.

Recall again that $[K_0]$ is a function of $[n]$ and $[N]$, and thus equation 32 must be evaluated for every set $[n]$. It should be noted that equation 32 is a general approximate solution valid for arbitrary light statistics and for detection intervals $\{T_i\} \ll \tau_c, \tau_a$. If, on the other hand, $\tau_c \ll \{T_i\} \ll \tau_a$ then the fluctuations of the source are averaged out and the undisturbed counting distribution $p_0([n];[N])$ reverts to the N-fold Poisson distribution. However, for $\tau_a \ll \{T_i\} \ll \tau_c$ the atmospheric modulation need not be considered, as only fluctuations due to the source will be observed. The counting distribution in that case is merely the undisturbed counting distribution $p_0([n];[N])$. Similarly for $\{T_i\} \gg \tau_c, \tau_a$ both fluctuations are averaged out and an N-fold Poisson distribution is observed.

5. Two-Fold Counting Distribution for a Stable Source

In this section we present the two-fold photocounting distribution for a single-mode amplitude stabilised laser source. We consider the case where the log-intensity variances are identically distributed, i.e. $\sigma_1 = \sigma_2$ and for detector separations corresponding to several degrees of correlation of the log-normally distributed mean counts.

For an ideal amplitude-stabilised laser, the undisturbed joint counting distribution for $N = 2$ is

$$p_0(n_1, n_2; K_1, K_2) = \frac{K_1^{n_1}\,\mathrm{e}^{-K_1}}{n_1!}\,\frac{K_2^{n_2}\,\mathrm{e}^{-K_2}}{n_2!}\ . \tag{33}$$

In the absence of turbulence, and for identical detectors, counting intervals, and a uniform illumination, K_1 would equal K_2. However, in the presence of turbulence, K_1 and K_2 are stochastic. The stationary condition for this case is

$$\begin{aligned} Q_1^{(1)}(n_1, n_2; K_1, K_2) - \frac{[X_1 - RX_2]}{\sigma^2[1-R^2]} = 0 \\ Q_2^{(1)}(n_1, n_2; K_1, K_2) - \frac{[X_2 - RX_1]}{\sigma^2[1-R^2]} = 0 \end{aligned} \tag{34}$$

where we have taken $R \equiv R_{12} = R_{21}$ and where

$$Q_1^{(1)}(n_1, n_2; K_1, K_2) = n_1 - K_1; \quad Q_2^{(1)}(n_1, n_2; K_1, K_2) = n_2 - K_2 \ .$$

These equations have the general form $\mathbf{G}(\mathbf{Y}) = 0$, where

$$\mathbf{G}(\mathbf{Y}) = \mathbf{Q}^{(1)}(\mathbf{Y}) - [\Lambda]^{-1}\, \mathbf{X} = 0 \ . \tag{35}$$

The solution to this N-dimensional implicit equation may be solved by several methods. For the two-dimensional case considered here, the Newton-Raphson [24] algorithm was utilised. This is an iteration method that converges rapidly near the true solution, and is given by

$$\mathbf{Y}(j+1) = \mathbf{Y}(j) - \left[\frac{\partial \mathbf{G}}{\partial \mathbf{Y}}\right]^{-1}\Bigg|_{\mathbf{Y}(j)} \mathbf{G}(\mathbf{Y}(j)) \ . \tag{36}$$

Since $\mathbf{G}(\mathbf{Y}) = 0$ is a function of n_1, n_2, the solution requires solving such an iteration formula for every value of n_1, n_2 desired. The solution thus obtained, the remaining quantities in the counting distribution can be evaluated:

$$\begin{aligned} Q_{11}^{(2)}(n_1, n_2; K_{01}, K_{02}) = -K_{01}; \quad Q_{22}^{(2)}(n_1, n_2; K_{01}, K_{02}) = -K_{02} \\ Q_{12}^{(2)}(n_1, n_2; K_{01}, K_{02}) = Q_{21}^{(2)}(n_1, n_2; K_{01}, K_{02}) = 0 \end{aligned} \tag{37}$$

and

$$|\Lambda| = \sigma^4[1-R^2] \ .$$

Additionally,

$$\mathbf{B} = \begin{bmatrix} -K_{01} - \dfrac{1}{\sigma^2[1-R^2]} & 0 + \dfrac{R}{\sigma^2[1-R^2]} \\ 0 + \dfrac{R}{\sigma^2[1-R^2]} & -K_{02} - \dfrac{1}{\sigma^2[1-R^2]} \end{bmatrix}, \tag{38}$$

and therefore

$$|\mathbf{B}|^{\frac{1}{2}} = \frac{1}{\sigma^2[1-R^2]}\left[\{\sigma^2[1-R^2]K_{01}+1\}\,\{\sigma^2[1-R^2]K_{02}+1\} - R^2\right]^{\frac{1}{2}} \ . \tag{39}$$

The counting distribution is then

$$p(n_1, n_2; N_1, N_2; \Lambda) = \frac{p_0(n_1, n_2; K_{01}, K_{02}) \exp -\frac{1}{2}\{X_{01}Q_1^{(1)}(K_{01}) + X_{02}Q_2^{(1)}(K_{02})\}}{|\Lambda|^{\frac{1}{2}}|\mathbf{B}|^{\frac{1}{2}}} \ . \tag{40}$$

For the particular case of amplitude-stabilised radiation,

$$p(n_1, n_2; N_1, N_2; \sigma; R) = \frac{K_{01}^{n_1}\, e^{-K_{01}}}{n_1!} \frac{K_{02}^{n_2}\, e^{-K_{02}}}{n_2!}$$

$$\times \frac{[1-R^2]^{\frac{1}{2}} \exp -\dfrac{1}{2}\left\{\left(\ln K_{01} - \ln N_1 + \dfrac{\sigma^2}{2}\right)\left(n_1 - K_{01}\right) + \left(\ln K_{02} - \ln N_2 + \dfrac{\sigma^2}{2}\right)\left(n_2 - K_{02}\right)\right\}}{[\{\sigma^2(1-R^2)K_{01}+1\}\,\{\sigma^2(1-R^2)K_{02}+1\} - R^2]^{\frac{1}{2}}} \ . \tag{41}$$

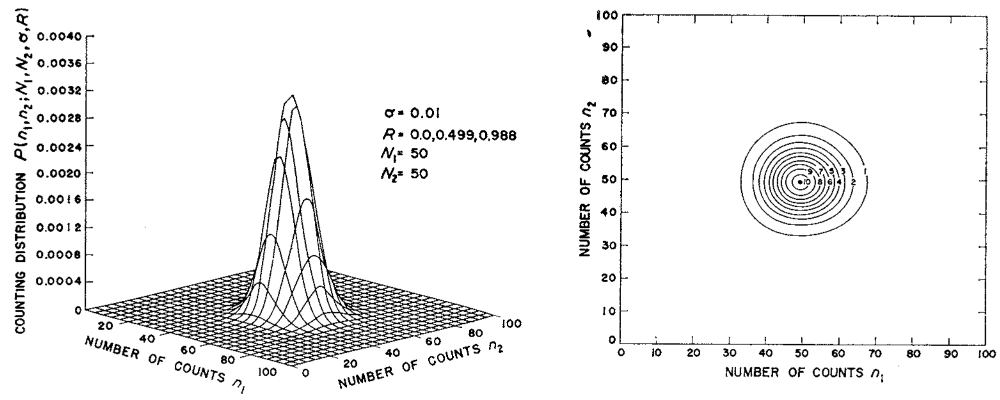

Figure 1 Two-fold joint photocounting distribution for a turbulence level corresponding to $\sigma = 0.01$, and correlation coefficients $R = 0.0, 0.499$ and 0.998. Slight asymmetry in the level contours is due to the plotting field of the computer plotter. $N_1 = N_2 = 50$. Peak occurs at $n_1 = 50$, $n_2 = 50$ and $p(50, 50) = 3.15 \times 10^{-3}$. $\Delta p = 3.15 \times 10^{-4}$, for adjacent contour levels 1 through 10.

6. Discussion of the Two-Fold Photocounting Distribution

In fig. 1, we present the counting distribution for a negligibly low level of turbulence, corresponding to $\sigma = 0.01$, and for correlation coefficient values $R = 0.0, 0.499$, and 0.998. The value $R = 0$ corresponds to completely uncorrelated detection (large distance or time separations), while $R = 1$ corresponds to completely correlated detection (small distances or time separations). As can be seen from the surface plot and the level contour plot, the distribution is singly peaked about the mean count $N_1 = N_2 = 50$, and falls off rapidly in all directions from this point. The distribution is independent of the correlation coefficient, as expected in the absence of turbulence, and reduces essentially to a two-fold Poisson distribution. The flattening evident in the contour plots is due to the particular characteristics of the plotting field of the computer plotting program. (If both the n_1 and n_2 axes were of equal length, the contours would not appear flattened.) The accuracy of the method is found to be better than 1 % for the case considered here, as discussed in the appendix.

For a moderately turbulent atmosphere ($\sigma = 0.5$), the counting distribution for $R = 0.499$ is presented graphically in fig. 2. By comparison with fig. 1, the peak is seen to occur at lower count numbers, (31, 31), and a marked broadening of the counting distribution occurs.

Figs 3, 4, 5 show the counting distribution for $\sigma = 1.0$ and correlation coefficients of $R = 0.0, 0.499$, and 0.998. As in the previous case, the peak is seen to shift to even lower count

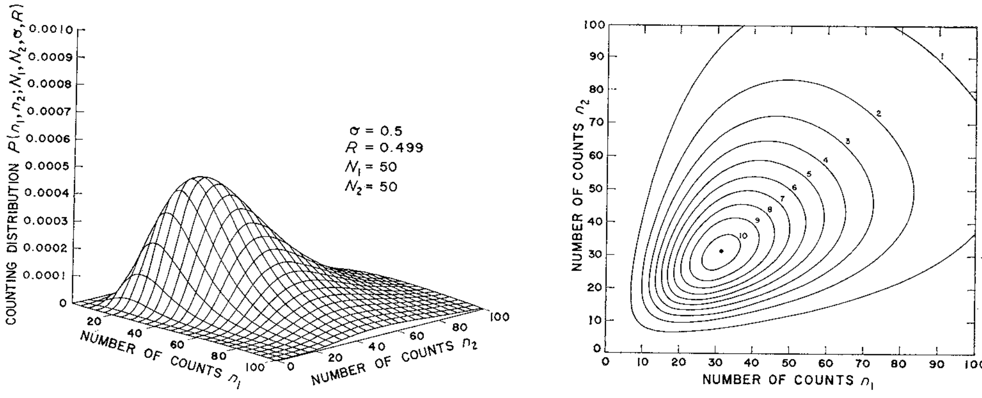

Figure 2 Photocounting distribution for $\sigma = 0.5$ and $R = 0.499$. $N_1 = N_2 = 50$. Peak is at $n_1 = 31$, $n_2 = 31$ and $p(31, 31) = 4.67 \times 10^{-4}$. $\Delta p = 4.67 \times 10^{-5}$, for adjacent contour levels 1 through 10.

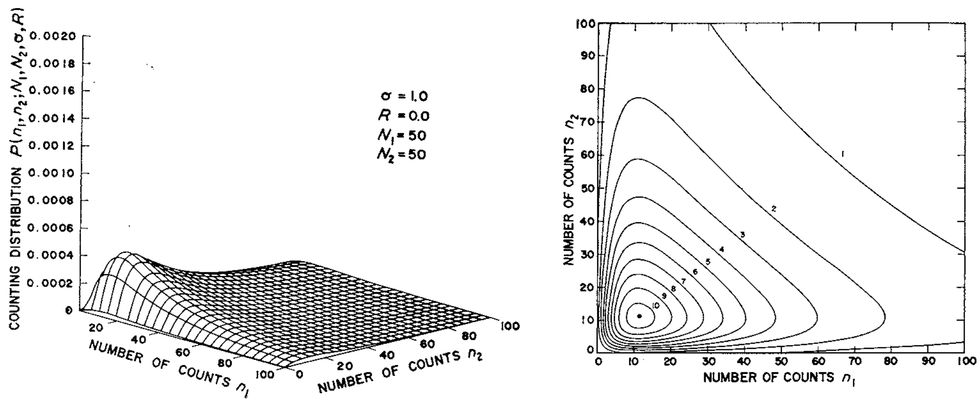

Figure 3 Photocounting distribution for $\sigma = 1.0$ and $R = 0.0$. $N_1 = N_2 = 50$. Peak is at $n_1 = 11$, $n_2 = 11$ and $p(11, 11) = 4.28 \times 10^{-4}$. $\Delta p = 4.28 \times 10^{-5}$, for adjacent contour levels 1 through 10.

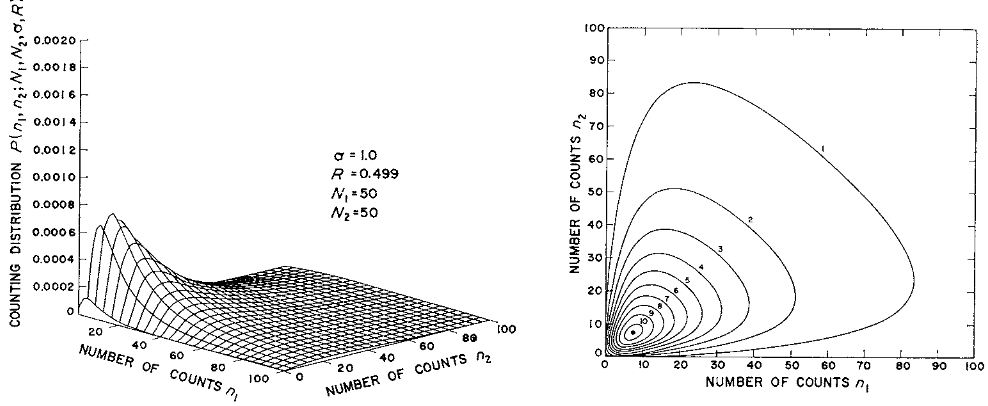

Figure 4 Photocounting distribution for $\sigma = 1.0$ and $R = 0.499$. $N_1 = N_2 = 50$. Peak is at $n_1 = 7$, $n_2 = 7$ and $p(7, 7) = 7.48 \times 10^{-4}$. $\Delta p = 7.48 \times 10^{-5}$, for adjacent contour levels 1 through 10.

numbers, now occurring at $(n_1, n_2) = (11, 11)$, $(7, 7)$, and $(6, 6)$ for the three values of correlation indicated, while the peak value increases as $R \to 1$. For R identically equal to unity, the photocounting distribution concentrates almost entirely on the $n_1 = n_2$ line. This is to be expected since the mean counts K_1 and K_2 are then equal with probability one. However, the counting distribution remains nonsingular because two physical detectors are involved. That is, two independent Poisson transformations take place at detection and thus the distribution has width about the $n_1 = n_2$ line. The occurrence of a nonsingular counting distribution is also evident from the stationary condition for the case $R = 1.0$:

$$n_1 + n_2 - 2K - \left[\ln K - \ln N_1 + \frac{\sigma^2}{2}\right] \Big/ \sigma^2 = 0 \ . \tag{42}$$

For a given value of K, and $n_1 + n_2 =$ constant, there are many values of n_1 and n_2 that satisfy $n_1 + n_2 =$ constant. For the equal mean case considered here (i.e. $N_1 = N_2$) and $R = 1.0$, the maximum occurs along the line $n_1 = n_2$. With this particular value of N_1 and σ, the counting distribution for $R = 0.998$ peaks to a value more than twice that for $R = 0.499$, and almost five times the value for $R = 0.0$.

As is evident from the photocounting distributions presented, the overall effect of the atmospheric turbulence is to markedly broaden the probability surface and shift the peak toward lower counts. The probability of observing small count numbers becomes larger as the

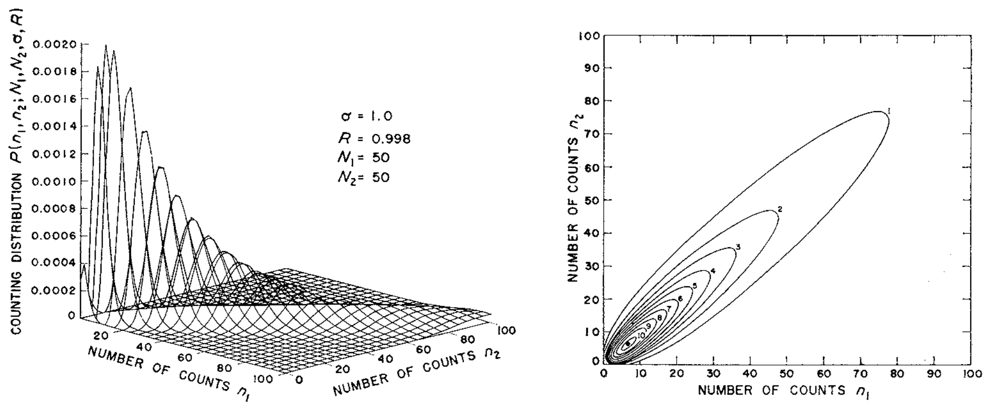

Figure 5 Photocounting distribution for $\sigma = 1.0$ and $R = 0.998$. $N_1 = N_2 = 50$. Peak is at $n_1 = 6$, $n_2 = 6$ and $p(6, 6) = 1.99 \times 10^{-3}$. $\Delta p = 1.99 \times 10^{-4}$, for adjacent contour levels 1 through 10.

turbulence increases, while the probabilities of observing extremely high counts, that were previously unlikely to occur, is also increased. The otherwise narrow-peaked two-fold Poisson counting distribution is thus altered considerably, as it is in the single dimensional case [8, 9]. The effect of the atmosphere, then, is to cluster the photons so that the most likely observed count numbers are reduced while clusters of many photons arrive with greater probability than would be the case for a quiescent atmosphere.

7. Detection with Additive Noise

In this section we develop the photoelectron counting distributions for radiation similarly modulated, but in the presence of independent, additive, Poisson distributed noise counts. This arises for relatively large detector bandwidths, as is the case when optical interference filters are utilised ($\gtrsim$ 1 Å), and when signal and noise radiation interference is not detected. The characteristic time of the background radiation τ_n is then $\lesssim 10^{-12}$ sec. For detection intervals $\{T_i\} \gg \tau_n$, which is the region of present experimental capability, the noise fluctuations are manifested in additive, independent, Poisson distributed counts. This has been shown to be the limiting case for additive narrow-band Gaussian noise [8, 25]. In addition, detector dark current also produces Poisson counts and may be included in this independent background.

If $[n]$ is the detected count in the presence of noise, $[s]$ the signal count, and $[z]$ the Poisson distributed noise count, then

$$[n] = [s]+[z] \ . \tag{43}$$

Since $[s]$ and $[z]$ are independent, the overall counting distribution is given by [26]

$$p([n]; [N]; \Lambda) = p([s]; [N_s]; \Lambda)*p([z]; [N_z]) \ , \tag{44}$$

where

$$N_{si} = \frac{N_i \gamma_i}{1+\gamma_i}$$

$$N_{zi} = \frac{N_i}{1+\gamma_i} \qquad i = 1, 2, \ldots M \ , \tag{45}$$

and

$$N_i = N_{si}+N_{zi} \ , \tag{46}$$

and where $*$ indicates convolution summation.

The ratio of signal-to-noise mean counts at the ith detector, γ_i, is defined by

$$\gamma_i = \frac{N_{si}}{N_{zi}} \tag{47}$$

The overall counting distribution is then

$$p([n];[N];\Lambda) = \sum_{m_1=0}^{n_1} \sum_{m_2=0}^{n_2} \cdots \sum_{m_M=0}^{n_M} \frac{p_0([m];[K_0]) \exp\{-\frac{1}{2}\mathbf{X}_0^{\dagger}\mathbf{Q}^{(1)}([m];[K_0])\}}{|\Lambda|^{\frac{1}{2}}\,|\mathbf{B}|^{\frac{1}{2}}} \times \prod_{i=1}^{M} \frac{(N_{zi})^{n_i-m_i}\,e^{-N_{zi}}}{(n_i-m_i)\,!} \,. \qquad (48)$$

The counting distribution given in equation 48 was evaluated numerically, and graphically displayed by digital computer for the case of an ideal amplitude-stabilised source, for $M = 2$, $\sigma_1 = \sigma_2 = 1.0$, $N_1 = N_2 = 12$, $R = 0.0, 0.499, 0.998$, and several values of γ. The distribution for $R = 0$ is analogous to the one-fold distribution given in [8]. For partially correlated detection, $R = 0.499$, the effect of increasing signal-to-noise ratio is to markedly broaden the probability surface from the sharply peaked Poisson distribution ($\gamma = 0$), to the limiting case of pure log-normal fading ($\gamma = \infty$). As discussed in section 6, the most prominent effect is the shifting of the peak to lower count numbers as well as the broadening of the probability surface. However, unlike the one-fold distribution, and the case $R = 0.0$, the peak first decreases and then again increases as γ increases. Even at moderate values of γ, e.g. $\gamma = 4$, the surface is not too different from that in the noise-free case. As $R \to 1$, the surface behaves in a similar manner except that the magnitude of the peak does not decrease measurably while shifting to lower count numbers. Exemplary plots of the counting distributions are given in figs 6 and 7. It was observed that the effect of decreasing γ is similar to decreasing σ in the noise-free case.

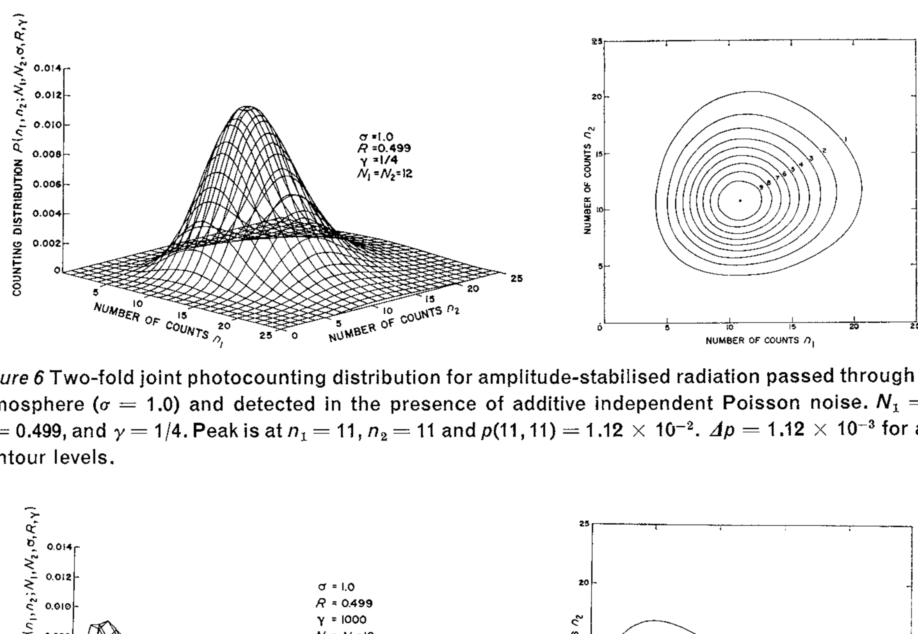

Figure 6 Two-fold joint photocounting distribution for amplitude-stabilised radiation passed through a turbulent atmosphere ($\sigma = 1.0$) and detected in the presence of additive independent Poisson noise. $N_1 = N_2 = 12$, $R = 0.499$, and $\gamma = 1/4$. Peak is at $n_1 = 11$, $n_2 = 11$ and $p(11, 11) = 1.12 \times 10^{-2}$. $\Delta p = 1.12 \times 10^{-3}$ for all adjacent contour levels.

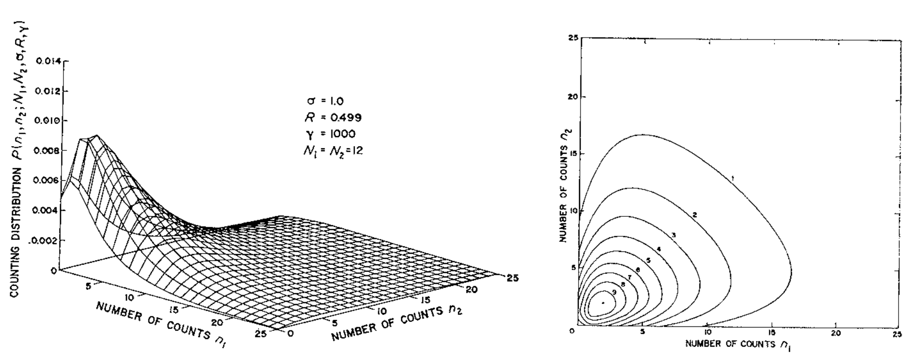

Figure 7 Same distribution as in fig. 6 except that $\gamma = 1000$. Peak is at $n_1 = 2$, $n_2 = 2$ and $p(2, 2) = 0.911 \times 10^{-2}$. $\Delta p = 0.911 \times 10^{-3}$ for all adjacent contour levels.

As in the case of the one-fold distribution, this similarity may explain the observed decrease in σ below its saturation value for long path lengths, and a departure from log-normal statistics [4, 5, 8].

8. Conclusion

We have presented an approximate method of solving the N-fold joint photocounting averaging integral for the case of optical radiation of arbitrary statistics passing through a turbulent atmosphere characterised by log-normal statistics. Several detection time regimes have been considered. The effect of the atmosphere on the irradiance may be interpreted as a modulation of the mean count of the undisturbed joint photocounting distribution, under the assumption that the source and the atmosphere are independent random processes. In evaluating these distributions, it has been assumed that the log-normal model is a sufficiently complete description of the statistics of the turbulent atmosphere.

In obtaining the solution to the averaging integral, an N-dimensional Taylor series involving logarithmic derivatives and the method of steepest descent were used. The error in the amplitude-stabilised case was found to be at worst $\sim 3\%$, and depends on the nature of the undisturbed counting distribution. It is quite apparent that great care must be exercised in using this method if the undisturbed counting distribution is not singly peaked. This has been shown to be the case when the radiation source is itself irradiance modulated [10].

We have specifically evaluated the two-fold photocounting distribution for ideal laser light for several levels of turbulence and degrees of correlation, with and without additive independent non-interfering Poisson noise. The most prominent effects on the undisturbed counting distribution are the marked broadening of the probability surface, the shifting of the peak towards the origin, and the non-singular counting distribution for fully correlated detection ($R = 1$). For constant turbulence level and correlation, increasing the signal-to-noise ratio broadens the surface in the same way as does increasing the level of turbulence. For partially correlated detection, however, the peak decreases and then again increases as $\gamma \rightarrow \infty$. For an amplitude-stabilised source, the results presented are valid for all values of the detection to coherence time ratio T/τ_c.

The interaction of source, channel, and detector array, the three essential elements of an optical information system, have been considered. Since the receiver structure is not specified in this work, these results provide maximum system information and are expected to be useful in studies of communication through the atmosphere as well as in studies of the atmosphere itself. Specific characterisation such as likelihood detection, as recently considered in great depth by Hoversten, Harger, and Halme [27], and channel capacity, should therefore benefit from this work. *Exact* expressions for the first-, second-, and third-order cumulants have been obtained, and will be published elsewhere [28].

Appendix

In the course of this work, we have made an approximation to the averaging integral (equation 19) by the method of steepest descent, arriving at the approximate integral (equation 31).

To obtain equation 31, we had to assume that the matrix norm $\|\mathbf{B}\|$ was much larger than unity. This ensured that the integrand in equation 31 was a sharply peaked Gaussian surface, peaked at the stationary point $\mathbf{Y}_0$. For the two-fold case, we can write the norm as

$$\|\mathbf{B}\| = \begin{cases} \max\limits_{Y_{0i}} \left\{ e^{Y_{0i}} + \dfrac{1}{\sigma^2[1-R^2]} \right\} & 0 \leqslant R < 1 \\ & i = 1, 2 \\ 2e^{Y_0} + \dfrac{1}{\sigma^2} & R = 1 \end{cases} \tag{49}$$

for the partially and fully correlated cases, respectively. For $N = 50$ and $\sigma = 1$, min $\|B\| = e^{0.91} + 1 \approx 3.5$ for $R = 0$. The minimum value of $\|B\|$ increases as n_1, n_2 increase and thus the integrand becomes more sharply peaked. In addition the integrand is broadest for $R = 0$, uncorrelated detection, and thus the errors for this case are expected to be largest, which was verified numerically. This is only true, however, when the undisturbed counting distribution is independent of R, as it is in the amplitude stabilised case. For other probability surfaces, appropriate care must be exercised in estimating the largest error. For $N = 50$, the worst relative was less than 1%, while it was less than 1.5% for $N = 12$. The error is a complicated function of N_i, σ, n_i, and R, and is greatest for small overall mean count numbers. For $N_i \geqslant 1$ and $\sigma \leqslant 1$, the greatest error is less than 3%, and within this range the method is an excellent approximation. It should be noted that it is precisely over this range of parameters that most of the data on log-intensity variance and covariance functions has been obtained.

References

1. R. I. TATARSKI, "Wave Propagation in a Turbulent Atmosphere" (McGraw-Hill, New York, 1961), chapters 3 and 7.
2. D. L. FRIED and R. A. SCHMELTZER, *Appl. Optics* **6** (1967) 1729-1737.
3. R. S. KENNEDY and E. V. HOVERSTEN, *IEEE Trans. Information Theory* **IT-14** (1968) 716-724.
4. G. R. OCHS and R. S. LAWRENCE, *J. Opt. Soc. Am.* **59** (1969) 226-227.
5. G. R. OCHS, R. R. BERGMAN, and J. R. SNYDER, *ibid* **59** (1969) 231-234.
6. P. H. DEITZ and N. J. WRIGHT, *ibid* **59** (1969) 527-535.
7. G. E. MEVERS, M. P. KEISTER, JR., and D. L. FRIED, *ibid* **59** (1969) 491A.
8. P. DIAMENT and M. C. TEICH, *ibid* **60** (1970) 1489-1494.
9. P. DIAMENT and M. C. TEICH, *Applied Optics,* **10** (1971) 7.
10. *Idem, J. Opt. Soc. Am.* **60** (1970) 682-689.
11. G. BÉDARD, *Phys. Rev.* **161** (1967) 1304-1308.
12. D. DIALETIS, *J. Physics* **A2,** Ser. 2 (1969) 229-235.
13. F. T. ARECCHI, A. BERNÉ, and A. SONA, *Phys. Rev. Letts.* **17** (1966) 260-263.
14. G. L. FILLMORE, *Phys. Rev.* **182** (1969) 1384-1386.
15. L. MANDEL, *Proc. Phys. Soc. (London)* **72** (1958) 1037-1048.
16. L. MANDEL and E. WOLF, *Rev. Mod. Phys.* **37** (1965) 231-287.
17. R. J. GLAUBER, in "Quantum Optics and Electronics", C. de Witt, A. Blandin, and C. Cohen-Tannoudji, Eds. (Gordon and Breach, New York, 1965), p. 65.
18. D. L. FRIED, *J. Opt. Soc. Am.* **57** (1967) 169-175.
19. J. W. STROHBEHN, *Proc. IEEE* **56** (1968) 1301-1318.
20. D. L. FRIED, G. E. MEVERS, and M. P. KEISTER, JR., *J. Opt. Soc. Am.* **57** (1967) 787-797.
21. D. L. FRIED, *ibid* **57** (1967) 175-180.
22. D. L. FRIED and S. B. SEIDMAN, *ibid* **57** (1967) 181-185.
23. J. AITCHINSON and T. BROWN, "The Log-Normal Distribution" (Cambridge University Press, London, 1957).
24. T. E. STERN, "Theory of Non-Linear Networks and Systems" (Addison-Wesley, Reading, Mass., 1965), pp. 44-51.
25. S. KARP and J. R. CLARK, *IEEE Trans. Information Theory* **IT-16** (1970) 672-680.
26. A. PAPOULIS, "Probability, Random Variables, and Stochastic Processes" (McGraw-Hill, New York, 1965).
27. E. V. HOVERSTEN, R. O. HARGER, and S. J. HALME, *Proc. IEEE* **58** (1970) 1626-1650.
28. S. ROSENBERG and M. C. TEICH, to be published.

ERRATUM

M. C. Teich and S. Rosenberg
Opto-electronics 3 (1971) 63-76

"N-Fold Joint Photocounting Distribution for Modulated Laser Radiation: Transmission Through the Turbulent Atmosphere"

---

page 69. Equation (28) should read:

$$\underset{\sim}{B} = \begin{bmatrix} \underline{Q}_{11}^{(2)} & \underline{Q}_{12}^{(2)} & \cdots & \underline{Q}_{1N}^{(2)} \\ \underline{Q}_{21}^{(2)} & \underline{Q}_{22}^{(2)} & \cdots & \vdots \\ \vdots & \vdots & & \vdots \\ \underline{Q}_{N1}^{(2)} & \cdots & \cdots & \underline{Q}_{NN}^{(2)} \end{bmatrix} - \underset{\sim}{\Lambda}^{-1}$$

page 71. Lettering on Figure 1 should read: R = 0.0, 0.499, 0.998

page 76. Reference 9 should read: Applied Optics 10 (1971) 1664.

page 76. Reference 28 should read: J. Appl. Phys., to be published.