Reprinted from JOURNAL OF MATHEMATICAL PSYCHOLOGY
All Rights Reserved by Academic Press, New York and London

Vol. 21, No. 2, April 1980
Printed in Belgium

An Increment Threshold Law for Stimuli of Arbitrary Statistics[1]

PAUL R. PRUCNAL AND MALVIN CARL TEICH

Department of Electrical Engineering, Columbia University, New York, New York 10027

We obtain a generalized increment threshold law that describes the relationship between the just detectable average signal increment and the noise when the threshold is stabilized, for the usual yes-no (nonorthogonal) signal format. The results are applicable to audition and vision, for arbitrary (discrete or continuous) signal and noise statistics, and arbitrary detection probabilities. The results are most useful when the noise and signal-plus-noise probability densities have the same form. Previous treatments of this problem have been restricted to Gaussian and Poisson signal and noise densities.

I. INTRODUCTION

The use of counting mechanisms to model both visual and auditory sensory processes is by now well established (Hecht, Shlaer, & Pirenne, 1942; Barlow, 1956, 1977; Treisman, 1963; Siebert, 1965; McGill, 1967, 1971; Penner, 1972; Teich & McGill, 1976; Teich, Prucnal, & Vannucci, 1977; Teich, Matin, & Cantor, 1978; Teich, Prucnal, Vannucci, Breton, & McGill, 1979; Teich & Cantor, 1978; Teich & Lachs, 1979). Many of these counting models specify that a continuous stimulus energy is transformed into a discrete sequence of neural impulses which ultimately constitutes a flow of information in the sensory channel. (The specific details of the energy-to-impulse transformation vary from model to model and are not of concern to us here.) The sequence of impulses is assumed to be monitored at some central counting location and then processed in an optimal fashion. The mathematical formulation of optimal processing utilizing likelihood ratios was developed long ago in the signal detection theory literature (Peterson, Birdsall, & Fox, 1954; Tanner & Swets, 1954a, 1954b; Green & Swets, 1966; VanTrees, 1968; Egan, 1975). More recently, we have shown that under a broad range of conditions, a simple comparison of the number of counts with a single threshold is a mathematically optimal form of processing (Teich *et al.*, 1977; Prucnal & Teich, 1978, 1979a).

The detection of a signal in noise within the framework of these counting models is assumed to be accomplished in the following way. A continuing sequence of noise impulses, corresponding to a background stimulus together with internal spontaneous discharges, is counted during a fixed period of time (the counting interval). An auxiliary stimulus (the signal) is presented to the subject on some subset of the counting intervals, and results in an increased average number of counts during those intervals. If, during a

[1] This work was supported by the National Science Foundation and the National Institutes of Health.

0022-2496/80/020168-10$02.00/0
Copyright © 1980 by Academic Press, Inc.
All rights of reproduction in any form reserved.

given counting interval, the number of impulses counted exceeds a fixed threshold, the signal is declared present, whether or not it actually is. Similarly, if the number of impulses counted is below this threshold, the signal is declared absent.

Using this model, we may formulate the detection problem more carefully as follows. The total number of noise counts during a counting interval is represented by a discrete random variable N' with probability density $p_{N'}(n') = \Pr(N' = n')$. We form the continuous random variable N with probability density defined by $f_N(n) = p_{N'}(n')$, where $n' \leqslant n < n' + 1$, and assume $f_N(n)$ is expressed as a function of n, its mean value $\langle n \rangle$, and other parameters of the distribution $q_1, ..., q_j$. The probability of false alarm is (VanTrees, 1968)

$$P_F(t, \langle n \rangle, q_1, ..., q_j) = \int_t^\infty f_N(n)\, dn, \tag{1}$$

where t is the threshold. The threshold is stabilized, so that the probability of false alarm is maintained at some fixed level $P_F = a$.

The total number of signal-plus-noise counts in the same counting interval is represented by the discrete random variable X' with probability density $p_{X'}(x') = \Pr(X' = x')$. We form the continuous random variable X with probability density defined by $f_X(x) = p_{X'}(x')$, where $x' \leqslant x < x' + 1$, and assume $f_X(x)$ is expressed as a function of x, its mean value $\langle x \rangle$, and the parameters $q_{j+1}, ..., q_k$. The probability of detection is (VanTrees, 1968)

$$P_D(t, \langle x \rangle, q_{j+1}, ..., q_k) = \int_t^\infty f_X(x)\, dx. \tag{2}$$

For particular values of $a, \langle n \rangle, q_1, ..., q_k$, the average signal increment needed to produce a desired probability of detection $P_D = b$ is represented by $\Delta(a, b, \langle n \rangle, q_1, ..., q_k)$. (The "just detectable increment" (McGill, 1971) often corresponds to the particular case of $b = \frac{3}{4}$.)

The graph of $\Delta(a, b, \langle n \rangle, q_1, ..., q_k)$ versus $\langle n \rangle$, where the parameters $a, b, q_1, ..., q_k$ are specified, is called the "increment threshold law" (or sensitivity). The conventional increment threshold law in the literature applies only to the case of known fixed signal in independent Gaussian noise, for which

$$\Delta(a, b, \langle n \rangle, \operatorname{var} n) = d' D(\langle n \rangle), \tag{3}$$

where the detectability parameter d' is a constant depending on a and b, and the variance is some function D^2 of the mean (Peterson *et al.*, 1954; Tanner *et al.*, 1954a, 1954b). It is of interest to note that the slope of the conventional increment threshold law in log–log coordinates is $(\langle n \rangle / D)(dD/d\langle n \rangle)$, and that this slope is a constant m if and only if $D = c\langle n \rangle^m$, where c is an arbitrary constant. Other (non-Gaussian) cases have been treated by assuming that the distributions of noise and signal-plus-noise have identical variances and are approximately Gaussian (Yellott, 1977, 1978) and, in addition, that $b = \frac{1}{2}$ (Peterson *et al.*, 1954; Barlow, 1957; Bouman, Vos, & Walraven, 1963; McGill, 1967, 1971; Cohn, Thibos, & Kleinstein, 1974; Cohn, 1976). An exception is the use of the square-root transformation by McGill (1971, p. 264) to normalize (i.e., to transform to Gaussian) the Poisson distribution, in the visual and auditory cases.

The purpose of this paper is to extend McGill's work by postulating a general normalizing transformation which is utilized to derive a generalized increment threshold law that is valid for arbitrary signal and noise statistics (i.e., not necessarily Gaussian or independent of each other), with arbitrary detection probability $P_D = b$. The result is most useful when the noise and the signal-plus-noise distributions have the same form. The results we present are valid for audition and vision, and for both discrete and continuous random variables (in the continuous case, $f_N(n) = p_{N'}(n')$ for $n = n'$, and $f_X(x) = p_{X'}(x')$ for $x = x'$). These results are also valid when t, $\langle n \rangle$, $q_1, \ldots, q_k$ are time-dependent, in which case the increment threshold law is also time-dependent.

II. Theory

We assume that g and h are known functions that transform n and x into the Gaussian (normal) random variables $\zeta \sim N(\langle \zeta \rangle, \operatorname{var} \zeta)$ and $\xi \sim N(\langle \xi \rangle, \operatorname{var} \xi)$, through the transformations

$$\zeta = g(n) \tag{4}$$

and

$$\xi = h(x). \tag{5}$$

The transformations g and h can be chosen to be monotonic nondecreasing. To see this we observe that as n increases, $\Pr(N \leqslant n)$ is nondecreasing. Then $\Pr(g(N) \leqslant g(n)) \equiv \Pr(N \leqslant n)$ is nondecreasing, which implies that $g(n)$ is also nondecreasing. Using the monotonicity of g and h, Eqs. (4) and (5) transform Eqs. (1) and (2) into

$$\Psi\left[\frac{g(t) - \langle \zeta \rangle}{(\operatorname{var} \zeta)^{1/2}}\right] = 1 - a \tag{6}$$

and

$$\Psi\left[\frac{h(t) - \langle \xi \rangle}{(\operatorname{var} \xi)^{1/2}}\right] = 1 - b, \tag{7}$$

where

$$\Psi[z] \equiv \int_{-\infty}^{z} \frac{\exp(-z^2/2)\, dz}{(2\pi)^{1/2}} \tag{8}$$

and t is the threshold. Inverting Eqs. (6) and (7) and subtracting, we obtain

$$\langle \xi \rangle - \langle \zeta \rangle - h(t) + g(t) = (\operatorname{var} \xi)^{1/2}\, \Psi^{-1}(b) - (\operatorname{var} \zeta)^{1/2}\, \Psi^{-1}(a). \tag{9}$$

The function Ψ^{-1} is the inverse of the cumulative standard Gaussian distribution Ψ defined in Eq. (8) (Burington, 1965). Since ζ and ξ are Gaussian random variables, their means are independent of their variances. The left side of Eq. (9) is therefore independent of the right side, and must be equal to some constant d, where

$$d \equiv (\operatorname{var} \xi)^{1/2}\, \Psi^{-1}(b) - (\operatorname{var} \zeta)^{1/2}\, \Psi^{-1}(a), \tag{10}$$

so that the generalized increment threshold law is specified implicitly by

$$\langle \xi \rangle - \langle \zeta \rangle - h(t) + g(t) = d. \tag{11}$$

Equation (11) is valid for arbitrary signal and noise statistics, and arbitrary values of P_F and P_D, and is a function of the threshold t. For situations where t is not known, a family of generalized increment threshold laws may be plotted parametrically in t. The threshold t is eliminated when $p_{N'}(n')$ and $p_{X'}(x')$ are of the same form, so that the functions g and h are identical, and Eq. (11) reduces to the simpler form

$$\langle \xi \rangle - \langle \zeta \rangle = d. \tag{12}$$

We shall see in the following examples that Eq. (12) yields a generalized increment threshold law in terms of the quantities $\langle x \rangle$ and $\langle n \rangle$. For counting models in which the stimulus is proportional to the number of counts, we can compare $\langle x \rangle$ and $\langle n \rangle$ with the known stimulus values to obtain the proportionality (efficiency) factor. For other counting models, we must use the assumed stimulus-to-count transformation to relate empirical quantities to Eq. (12). Finally, we note that Eq. (9) provides an expression for the receiver operating characteristic (ROC), in which $P_D = b$ is plotted as a function of $P_F = a$ for given values of $\langle n \rangle$, $\langle x \rangle$, $q_1, \ldots, q_k$.

III. Examples

The usefulness of the generalized increment threshold law will be illustrated by several examples where $p_{N'}(n')$ and $p_{X'}(x')$ are of the same form.

Let n and x be Gaussian random variables, with $n \sim N(\langle n \rangle, \operatorname{var} n)$ and $x \sim N(\langle x \rangle, \operatorname{var} x)$. In this case, Eqs. (4) and (5) are the identity transformation, and Eq. (12) yields

$$\Delta(a, b, \langle n \rangle, \operatorname{var} n, \operatorname{var} x) = (\operatorname{var} x)^{1/2}\, \Psi^{-1}(b) - (\operatorname{var} n)^{1/2}\, \Psi^{-1}(a). \tag{13}$$

In the special case where the signal and noise are additive, independent Gaussian random variables with equal variance, Eq. (13) reduces to the conventional result expressed in Eq. (3).

As a second example, let n and x be lognormal random variables with coefficients of variation c_{vn} and c_{vx} (Prucnal *et al.*, 1979b). By definition, the logarithm of a lognormal random variable is normally distributed (Woodroofe, 1975, p. 199). In this case Eqs. (4) and (5) are logarithmic transformations and Eq. (12) yields

$$\begin{aligned} &\Delta(a, b, \langle n \rangle, c_{vn}, c_{vx}) \\ &\quad = -\langle n \rangle + \langle n \rangle\, C_x^{1/2} C_n^{-1/2} \exp\{[\ln C_x]^{1/2}\, \Psi^{-1}(b) - [\ln C_n]^{+1/2}\, \Psi^{-1}(a)\}, \end{aligned} \tag{14}$$

where $C_z \equiv c_{vz}^2 + 1$. Due to the permanence of the lognormal density, in the special case where the signal and noise are additive, independent lognormal random variables, the signal-plus-noise is also approximately lognormal (Mitchell, 1968).

IV. Approximations

For many probability densities, the exact transformations to the Gaussian density expressed in Eqs. (4) and (5) are not readily available or easily derived. A number of approximate forms of these transformations have been developed for specific probability densities, such as the Poisson, binomial, and negative-binomial (Bartlett, 1936; Cochran, 1940; Curtiss, 1943; Anscombe, 1948; Freeman & Tukey, 1950). In these cases, we utilize the mean and variance of the approximately transformed random variable in Eqs. (12) and (10).

We consider in some detail the case of Poisson random variables. In this case, transformations in Eqs. (4) and (5) are

$$g(z) \simeq (z + 3/8)^{1/2}, \tag{15}$$

which are distributed, approximately, according to

$$g(z) \sim N\left(g(\langle z\rangle) - \frac{1}{8}\langle z\rangle^{-1/2} + \frac{1}{64}\langle z\rangle^{-3/2}, \frac{1}{4} + \frac{1}{64}\langle z\rangle^{-2}\right), \tag{16}$$

where $z = x$ or $z = n$ (Anscombe, 1948). Equation (12) reduces to

$$g(\langle x\rangle) - \frac{1}{8}\langle x\rangle^{-1/2} + \frac{1}{64}\langle x\rangle^{-3/2} - g(\langle n\rangle) + \frac{1}{8}\langle n\rangle^{-1/2} - \frac{1}{64}\langle n\rangle^{-3/2} \simeq d, \tag{17}$$

where

$$d \simeq \left(\frac{1}{4} + \frac{1}{64}\langle x\rangle^{-2}\right)^{1/2} \Psi^{-1}(b) - \left(\frac{1}{4} + \frac{1}{64}\langle n\rangle^{-2}\right)^{1/2} \Psi^{-1}(a), \tag{18}$$

and where $g(z)$ is given in Eq. (15). The increment threshold law for Poisson random variables is then a graph of

$$\Delta(a, b, \langle n\rangle) = \langle x\rangle - \langle n\rangle \tag{19}$$

versus $\langle n\rangle$, where the relationship between $\langle x\rangle$ and $\langle n\rangle$ is specified implicitly by Eqs. (17) and (18). A number of other approximate transformations from the Poisson to the Gaussian density have been developed to achieve variance stabilization, such as $\zeta \simeq (n)^{1/2} + (n + 1)^{1/2}$ (Freeman *et al.*, 1950), $\zeta \simeq (n + \frac{1}{2})^{1/2}$ (Bartlett, 1936), and the simple square-root transformation $\zeta \simeq (n)^{1/2}$ (Mattick, McClemont, & Irwin, 1935). The square-root transformation, although somewhat less accurate than Eq. (15) (Freeman *et al.*, 1950), was used by McGill (1971, p. 264), together with a Taylor series approximation, to provide an independent derivation of the deVries–Rose law for visual intensity discrimination (deVries, 1943; Rose, 1948). For a comparison of the accuracy of the above-described approximate normalizing transformations, a complete discussion of their variance-stabilizing capabilities is presented by Freeman and Tukey (1950).

In certain situations an approximate transformation of a probability density to the Gaussian density may not be available. The transformation

$$g(n) \propto \int \frac{d\langle n\rangle}{D(\langle n\rangle)}, \tag{20}$$

where the integral is evaluated at $\langle n\rangle = n$, may be used to transform densities for which the variance can be expressed as a function D^2 of the mean

$$\operatorname{var} n = D^2(\langle n\rangle) \tag{21}$$

(Kendall & Stuart, 1977, p. 88). However, the correctness of the normalizing transformation given in Eq. (20) is somewhat controversial (Curtiss, 1943).

V. Comparison of Generalized and Conventional Increment Threshold Laws

In this section we compare the conventional increment threshold law, which is used in the literature, with the generalized increment threshold law derived in this paper. The Gaussian, lognormal, and Poisson cases will be discussed and plotted in log–log coordinates. We assume throughout that the noise and signal-plus-noise densities have the same form. We choose the parameters $a = 0.25$ and $b = 0.75$.

The generalized increment threshold law is specified implicitly by

$$\langle\xi\rangle - \langle\zeta\rangle = d, \tag{22}$$

where

$$d \equiv (\operatorname{var} \xi)^{1/2}\, \Psi^{-1}(b) - (\operatorname{var} \zeta)^{1/2}\, \Psi^{-1}(a). \tag{23}$$

(Eqs. (12) and (10) have been repeated for convenience.) This generalized increment threshold law applies to arbitrary probability densities of the same form, with arbitrary values of P_D and P_F, and is represented explicitly by a plot of

$$\Delta(a, b, \langle n\rangle, q_1, \ldots, q_k) = \langle x\rangle - \langle n\rangle \tag{24}$$

versus $\langle n\rangle$.

By assumption, $a = 0.25$ and $b = 0.75$, so Eq. (23) reduces to

$$d = 0.675[(\operatorname{var} \xi)^{1/2} + (\operatorname{var} \zeta)^{1/2}]. \tag{25}$$

In the case of Gaussian random variables of unequal variance, Eq. (24) reduces to Eq. (13). The independent parameters $\operatorname{var} x$ and $\operatorname{var} n$ may be specified arbitrarily, and we choose $\operatorname{var} x = \langle x\rangle^2$ and $\operatorname{var} n = \langle n\rangle^2$ (i.e., $c_{vn} = c_{vx} = 1$) for the purpose of illustration. In practice, the variances will be determined by the particular physical situation, and might have no simple analytic relation to the mean. The choice of unity coefficient of variation is arbitrary, but will be used consistently in all examples presented here. With these assumptions, Eq. (13) reduces to

$$\Delta(0.25, 0.75, \langle n \rangle, \mathrm{var}\, n, \mathrm{var}\, x) = 4.154\langle n \rangle, \tag{26}$$

which is represented by solid line A in Fig. 1.

For the case of lognormal random variables Eq. (24) reduces to Eq. (14). The independent parameters c_{vx} and c_{vn} may be specified arbitrarily, but we choose $c_{vn} = c_{vx} = 1$ for comparison with the above Gaussian case. With this assumption, Eq. (14) reduces to

$$\Delta(0.25, 0.75, \langle n \rangle, c_{vn}, c_{vx}) = 2.077\langle n \rangle, \tag{27}$$

which is represented by solid line B in Fig. 1.

For the case of Poisson random variables, Eqs. (22) and (23) reduce to Eqs. (17) and (18). There are no independent parameters since the Poisson is completely specified by its mean value. With $a = 0.25$ and $b = 0.75$,

$$\Delta(0.25, 0.75, \langle n \rangle) = \langle x \rangle - \langle n \rangle, \tag{28}$$

where $\langle x \rangle$ and $\langle n \rangle$ are related by Eq. (17) with $d \simeq 1$. Eq. (28) is represented by solid curve C in Fig. 1.

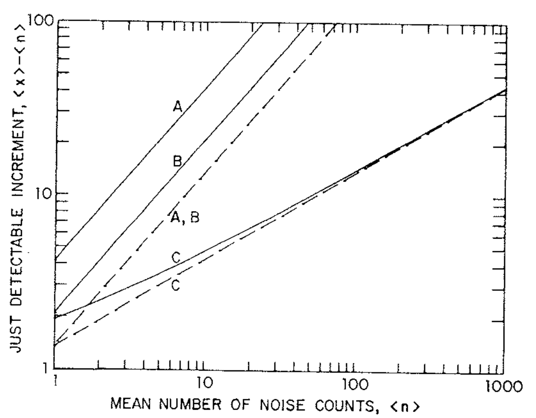

FIG. 1. Gaussian (A), lognormal (B), and Poisson (C) increment threshold laws are shown on a graph of $\log(\langle x \rangle - \langle n \rangle)$ versus $\log(\langle n \rangle)$, for the case where the noise and signal-plus-noise densities have the same form. In all cases $P_F = 25\%$ and $P_D = 75\%$. The solid lines represent the generalized increment threshold law derived in this paper. Solid lines A [Eq. (26)] and B [Eq. (27)] are exact with $c_{vn} = c_{vx} = 1$, whereas solid line C [Eqs. (28) and (17) with $d \simeq 1$] is approximate. The dashed lines A [Eq. (30)], B, and C [Eq. (31)] represent the conventional increment threshold law corresponding to the above cases. In each case, the conventional increment threshold law requires the approximation $\mathrm{var}\, x \approx \mathrm{var}\, n$.

For comparison, we now apply the conventional increment threshold law (see Eq. (13))

$$\Delta(a, b, \langle n \rangle, \operatorname{var} n) = [\Psi^{-1}(b) - \Psi^{-1}(a)](\operatorname{var} n)^{1/2}, \tag{29}$$

to the above-described Gaussian, lognormal, and Poisson cases. We conform to the same choice of parameters in each case as in the above-described cases (i.e., $\operatorname{var} n = \langle n \rangle^2$, and $c_{vn} = 1$). We note that Eq. (29) is exact only for the restricted case of Gaussian probability densities with equal variance, which contain a total of three independent parameters, $\langle x \rangle$, $\langle n \rangle$, and $\operatorname{var} n$. To use the conventional increment threshold law approximately for probability densities with an arbitrary number of parameters $\langle x \rangle$, $\langle n \rangle$, $q_1, \ldots, q_k$, the set of parameters must artificially be reduced to $\langle x \rangle$, $\langle n \rangle$, and $\operatorname{var} n$ only. This "reduction" may be neither convenient nor accurate, as is illustrated in the following examples.

To apply the conventional increment threshold law to the above case (A) of Gaussian random variables of unequal variance, we are required to make the possibly unreasonable assumption that $\operatorname{var} x \approx \operatorname{var} n$. With this assumption, Eq. (29) reduces to

$$\Delta(0.25, 0.75, \langle n \rangle, \operatorname{var} n) = 1.35\langle n \rangle, \tag{30}$$

which is represented by dashed line A in Fig. 1.

To apply the conventional increment threshold law to the above case (B) of lognormal random variables with $c_{vn} = c_{vx} = 1$, and the above case (C) of Poisson random variables, we are required to assume that $\operatorname{var} x \approx \operatorname{var} n$, which implies that $\langle x \rangle \approx \langle n \rangle$. The implication that $\langle x \rangle \approx \langle n \rangle$ may be unreasonable. Nevertheless, with these assumptions in the lognormal case, Eq. (29) reduces to Eq. (30), which is represented by dashed line B in Fig. 1 (the same as dashed line A). With these assumptions in the Poisson case, Eq. (29) reduces to

$$\Delta(0.25, 0.75, \langle n \rangle, \operatorname{var} n) = 1.35\langle n \rangle^{1/2}, \tag{31}$$

which is represented by dashed line C in Fig. 1.

Using the conventional increment threshold law in the above three cases has required that we approximate the noise and signal-plus-noise densities by Gaussians of equal variance. This forces us to ignore the parameters $q_2, \ldots, q_k$, which specify the detailed shape of these densities.

The first consequence of ignoring $q_2, \ldots, q_k$ is the assumption that $\operatorname{var} x \simeq \operatorname{var} n$ in situations where we would expect $\operatorname{var} x > \operatorname{var} n$. Underestimating $\operatorname{var} x$ causes us to overestimate the "precision" with which we can predict x, and therefore underestimate the just detectable increment. This error is exhibited in Fig. 1, where the dashed lines are everywhere below the solid lines.

The second consequence of ignoring the detailed shape of the densities is that different densities can lead to indistinguishable increment threshold laws, as occurs with dashed lines A and B. In contrast, the generalized increment threshold law for different densities may be distinguishable, as occurs with solid lines A and B.

The third consequence is that the asymmetry of distributions, such as the Poisson exhibits at low means, is ignored by the conventional increment threshold law (dashed curve C). At high means, however, where the Poisson approaches the Gaussian, solid curve C approaches dashed curve C asymptotically.

In summary, the generalized increment threshold law includes all parameters of the densities, whereas the conventional increment threshold law ignores many parameters and tends to underestimate the just detectable increment. It is possible that the generalized increment threshold law may prove useful in distinguishing between different kinds of subject noise through specially designed experiments in which the statistical characteristics of the stimulus are varied.

References

Abromowitz, M. & Stegun, I. A. *Handbook of mathematical functions with formulas, graphs, and mathematical tables.* Washington, D. C.: U. S. Government Printing Office, 1972.

Anscombe, F. J. The transformation of Poisson, binomial and negative-binomial data. *Biometrika,* 1948, **35**, 246–254.

Barlow, H. B. Retinal noise and absolute threshold. *Journal of the Optical Society of America,* 1956, **46**, 634–639.

Barlow, H. B. Increment thresholds at low intensities considered as signal/noise discriminations. *Journal of Physiology,* 1957, **136**, 469–488.

Barlow, H. B. A method of determining the overall quantum efficiency of visual discriminations. *Journal of Physiology,* 1962, **160**, 155–168.

Barlow, H. B. Retinal and central factors in human vision limited by noise. In H. B. Barlow & P. Fatt (Eds.). *Vertibrate Photoreception* London: Academic Press, 1977. Pp. 337–358.

Bartlett, M. S. The square root transformation in analysis of variance. *Journal of the Royal Statistical Society Supplement,* 1936, **3**, 68–78.

Bouman, M. A., Vos, J., & Walraven, P. Fluctuation theory of luminance and chromaticity discrimination. *Journal of the Optical Society of America,* 1963, **53**, 121–128.

Burington, R. S. *Handbook of mathematical tables and formulas.* New York: McGraw–Hill, 1965.

Cochran, W. G. The analysis of variance when experimental errors follow the Poisson or binomial laws. *Annals of Mathematical Statistics,* 1940, **9**, 335–347.

Cohn, T. E., Thibos, L. N., & Kleinstein, R. N. Detectibility of a luminance increment. *Journal of the Optical Society of America,* 1974, **64**, 1321–1327.

Cohn, T. E. Detectability of luminance increment: effect of superimposed random luminance fluctuations. *Journal of the Optical Society of America,* 1976, **66**, 1426–1428.

Curtiss, J. H. On transformations used in the analysis of variance. *Annals of Mathematical Statistics,* 1943, **14**, 107–122.

Egan, J. P. *Signal detection theory and ROC analysis.* New York: Academic Press, 1975.

Freeman, M. F. & Tukey, J. W. Transformations related to the angular and square root. *Annals of Mathematical Statistics,* 1950, **21**, 607–611.

Green, D. M. & Swets, J. A. *Signal detection theory and psychophysics.* New York: Wiley, 1966.

Hecht, S., Shlaer, S., & Pirenne, M. H. Energy, quanta and vision. *Journal of General Physiology,* 1942, **25**, 819–840.

Kendall, M. G. & Stuart, A. *The advanced theory of statistics, Vol.3.* New York: Hafner, 1966. Pp. 88–94.

Mattick, A. T. R., McClemont, J., & Irwin, J. O. The plate count of milk. *Dairy Research,* 1935, **6**, 130–147.

McGill, W. J. Neural counting mechanisms and energy detection in audition. *Journal of Mathematical Psychology,* 1967, **4**, 351–376.

McGill, W. J. & Goldberg, J. P. A study of the near-miss involving Weber's law and pure-tone intensity discrimination. *Perception and Psychophysics,* 1968, **4**, 105–109.

McGill, W. J. Poisson counting and detection in sensory systems. In E. F. Beckenbach & C. B.

Tompkins (Eds.), *Concepts of Communication: Interpersonal, Intrapersonal, and Mathematical.* New York: Wiley, 1971. Pp. 257–281.

MITCHELL, R. L. Permanence of the log-normal distribution. *Journal of the Optical Society of America*, 1968, **58**, 1267–1272.

PAPOULIS, A. *Probability, random variables, and stochastic processes.* New York: McGraw–Hill, 1965.

PENNER, M. Neural or energy summation in a Poisson counting model. *Journal of Mathematical Psychology*, 1972, **9**, 286–293.

PETERSON, W., BIRDSALL, T., & FOX, W. The theory of signal detectability. *IRE Transactions, Professional Group on Information Theory*, 1954, **PGIT-4**, 171–212.

PRUCNAL, P. R. & TEICH, M. C. Single-threshold detection of a random signal in noise with multiple independent observations 1: discrete case with application to optical communications. *Applied Optics*, 1978, **17**, 3576–3583.

PRUCNAL, P. R. & TEICH, M. C. Single-threshold detection of a random signal in noise with multiple independent observations 2: continuous case. *IEEE Transactions on Information Theory*, 1979a, **IT-25**, 213–218.

PRUCNAL, P. R. & TEICH, M. C. Statistical properties of counting distributions for intensity-modulated sources. *Journal of the Optical Society of America*, 1979b, **69**, 539–544.

ROSE, A. The sensitivity performance of the human eye on an absolute scale. *Journal of the Optical Society of America*, 1948, **38**, 196–208.

SIEBERT, W. M. Some implications of the stochastic behavior of primary auditory neurons. *Kybernetik*, 1965, **2**, 206–215.

TANNER, W. P. & SWETS, J. A. A decision-making theory of visual detection. *Psychological Review*, 1954a, **61**, 401–409.

TANNER, W. P. & SWETS, J. A. The human use of information—1. Signal detection for the case of the signal known exactly. *IRE Transactions, Professional Group on Information Theory*, 1954b, **PGIT-4**, 171–212.

TEICH, M. C., PRUCNAL, P. R., & VANNUCCI, G. Optimum photon detection with a simple counting processor. *Optics Letters*, 1977, **1**, 208–210.

TEICH, M. C. & CANTOR, B. I. Information, error and imaging in dead-time-perturbed doubly stochastic Poisson counting systems. *IEEE Journal on Quantum Electronics*, 1978, **QE-14**, 993–1003.

TEICH, M. C., MATIN, L. & CANTOR, B. I. Refractoriness in the maintained discharge of the cat's retinal ganglion cell. *Journal of the Optical Society of America*, 1978, **68**, 386–402.

TEICH, M. C., PRUCNAL, P. R., VANNUCCI, G., BRETON, M. E., & MCGILL, W. J. Role of quantum fluctuations and the Neyman Type-A distribution in human vision. *Journal of the Optical Society of America*, 1979, **69**, 1469.

TEICH, M. C. & LACHS, G. A neural counting model incorporating refractoriness and spread-of-excitation. 1. Application to intensity discrimination. *Journal of the Acoustical Society of America*, 1979, **66**, 1738–1749.

TEICH, M. C. & MCGILL, W. J. Neural counting and photon counting in the presence of dead time. *Physical Review Letters*, 1976, **36**, 754–758, 1473.

TREISMAN, M. Noise and Weber's law. *Psychological Review*, 1963, **71**, 314–330.

VANTREES, H. L. *Detection, estimation and modulation theory, Part 1.* New York: Wiley, 1968.

deVRIES, H. L. The quantum character of light and its bearing upon the threshold of vision, the differential sensitivity and acuity of the eye. *Physica*, 1943, **10**, 553–564.

WOODROOFE, M. *Probability with applications.* New York: McGraw–Hill, 1975.

YELLOTT, J. I. The relationship between Luce's choice axiom, Thurstone's theory of comparative judgement, and the double exponential distribution. *Journal of Mathematical Psychology*, 1977, **15**, 109–144.

YELLOTT, J. I. On a functional equation related to Thurstone models. *Journal of Mathematical Psychology*, 1978, **17**, 266–270.

RECEIVED: July 30, 1979

Printed by the St. Catherine Press Ltd., Tempelhof 37, Bruges, Belgium