Poisson branching point processes

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We investigate the statistical properties of a special branching point process. The initial process is assumed to be a homogeneous Poisson point process (HPP). The initiating events at each branching stage are carried forward to the following stage. In addition, each initiating event independently contributes a nonstationary Poisson point process (whose rate is a specified function) located at that point. The additional contributions from all points of a given stage constitute a doubly stochastic Poisson point process (DSPP) whose rate is a filtered version of the initiating point process at that stage. The process studied is a generalization of a Poisson branching process in which random time delays are permitted in the generation of events. Particular attention is given to the limit in which the number of branching stages is infinite while the average number of added events per event of the previous stage is infinitesimal. In the special case when the branching is instantaneous this limit of continuous branching corresponds to the well-known Yule–Furry process with an initial Poisson population. The Poisson branching point process provides a useful description for many problems in various scientific disciplines, such as the behavior of electron multipliers, neutron chain reactions, and cosmic ray showers.

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I. INTRODUCTION

The theory of branching processes provides an important set of mathematical tools which may be applied to many problems in modern physics.[1,2] These range from multiple atomic transitions to extensive air showers produced by cosmic rays. In many of the existing mathematical treatments of these problems, the branching is treated as an instantaneous effect. However, in most physical systems, a random time delay (or spatial dispersion) is inherent in the multiplication process.

In a recent set of papers, we examined a special generalized branching process in which the multiplication of each event is Poisson and a random time delay is introduced at every stage. The first model that we analyzed[3–5] is the two-stage cascaded Poisson, in which each event of a primary Poisson point process produces a virtual inhomogeneous rate function which, in turn, generates a secondary Poisson point process. These secondary point processes are superimposed to form the final point process. In that model, primary events themselves are excluded from the final point process.[3–5] The description turns out to be that of a doubly stochastic Poisson point process (DSPP), which we refer to as the shot-noise-driven process (SNDP).[3] The SNDP is also a special case of the Neyman–Scott cluster process.[3,5] Because of the great body of theoretical results available for the DSPP, our calculations for the statistical properties of the process turned out to be relatively straightforward. In another version of this two-stage model, primary events are carried forward to the final process.[6]

The second system which we analyzed[7] is an m-stage cascade of Poisson processes buffered by linear filters. Each filtered point process forms the input to the following stage, acting as a rate for a DSPP. This is equivalent to a cascaded SNDP. We obtained the counting and time statistics, as well as the autocovariance function. The results of that study are likely to find use in problems where a series of multiplicative effects occur. Examples are the behavior of photon and charged-particle detectors, the production of cosmic rays, and the transfer of neural information.

In this paper, we consider a cascade model in which primary events are carried forward together with secondary events, to form the point process at the input to each successive stage. Since the primary and secondary events comprising the union process at each stage are not independent,[6] the solution is somewhat more difficult than for the cascaded Poisson case considered previously.[7] The initial point process is assumed to be a homogeneous Poisson process (HPP). The final process is itself homogeneous (stationary). This treatment should allow us to model a wide variety of physical phenomena in which particles produce more particles, and so on, with the original particles remaining. Our process may also be regarded as a special generalized branching process,[1] in which each event of the HPP produces an age-dependent point process. However, our interest is in the union of the branching point processes rather than in the statistics of the number of events at a certain time (or place), as is the customary quantity of interest in age-dependent branching processes.

Branching processes with properties such as age dependence, random walk, and diffusion have been studied extensively from a general theoretical point of view.[1] Few of the statistical properties are obtained in a form amenable to numerical solution, however. The present work examines a relatively simple process that describes branching with time delay. Thanks to the simplicity offered by the Poisson assumption, we can obtain explicit formulas for useful statistical properties that may be experimentally measured. Examples are the counting distribution, moments, and power spectral density, as we demonstrate.

In Sec. II, we review the properties of a Poisson branching process in which the branching is instantaneous. This establishes the properties of the limiting situation, to which our process must converge when time delay is negligible. We also consider the limiting case when the number of branching stages approaches infinity while the average number of secondary events per primary event approaches zero. In the instantaneous multiplication case, this results in the Yule–Furry process,[2] driven by HPP initial events.

In Sec. III, we introduce time delay at each stage of branching and define the process formally. We provide expressions for the moment generating functional of the process, from which we compute the moments, counting probability distribution, and autocorrelation function (or power spectral density). In Sec. IV, we discuss the important limit of the continuous branching point process with time delay, showing how it differs from the instantaneous continuous branching case.

II. INSTANTANEOUS POISSON BRANCHING PROCESS

This section is divided into three subsections. In Subsec. A, we briefly discuss the well-known general Galton–Watson (GW) branching process.[1] In Subsec. B, a special Galton–Watson branching process, in which the multiplication is Poisson, is examined. The properties of a Poisson Galton–Watson process, in which the initial number of events is itself Poisson, are examined in detail in Subsec. C.

A. Galton–Watson branching process

Let $N_0,N_1,N_2,\ldots$ be nonnegative integers denoting the successive random variables of a Markov chain, where N_m denotes the size of the population of the mth generation of the branching process. The population N_{m+1} at the $(m+1)$st generation is determined by the sum

$$N_{m+1} = \sum_{k=1}^{N_m} Z_k^m \tag{1}$$

of N_m independent, identically distributed (iid) random variables $Z_1^m, Z_2^m, \ldots, Z_N^m$, each with probability distribution

$$\operatorname{Prob}(Z^m = k) = p_k^m, \qquad k = 0,1,2,\ldots . \tag{2}$$

This determines the transition matrix of the Markov chain. It is assumed that $N_0 = 1$. The chain is known as a Galton–Watson (GW) process.

The basic assumption is that each of the members of a generation branches independently and identically to generate the population of the following generation. The statistical properties of the random number N_m may be determined from its probability generating function

$$G_m(z) = \langle z^{N_m} \rangle , \tag{3}$$

which may be calculated by use of recursive equations. These are easily determined by using the iid assumption:

$$\begin{aligned} &G_0(z) = z , \\ &G_{m+1}(z) = G_m[O_m(z)] , \quad m = 0,1,\ldots , \end{aligned} \tag{4}$$

where

$$O_m(z) = \sum_{k=0}^{\infty} p_k^m z^k \tag{5}$$

is the probability generating function of the random variable Z^m.

B. Poisson Galton–Watson process

We now consider a special case of the GW process by taking

$$p_k^m = \begin{cases} 0, & k = 0, \\ \alpha_m^{k-1} e^{-\alpha_m}/(k-1)!, & k = 1,2,\ldots, \end{cases} \tag{6}$$

i.e., Z_k^m obeys a shifted version of the Poisson distribution[8] of mean α_m. This signifies that each member of the mth generation survives and remains in the $(m+1)$st generation, adding a cluster of offspring which is Poisson distributed with mean α_m. We shall call this special GW process the Poisson GW process (PGW).

By substituting (6) in (5), we obtain

$$O_m(z) = z e^{\alpha_m(z-1)} , \quad m = 0,1,2,\ldots . \tag{7}$$

Therefore, from (4), the probability generating function is

$$\begin{aligned} &G_0(z) = z , \\ &G_{m+1}(z) = G_m[z e^{\alpha_m(z-1)}] , \quad m = 0,1,\ldots . \end{aligned} \tag{8}$$

C. Poisson Galton–Watson process with an initial Poisson population

In this subsection, we define a process in which members of an initial population of random size N_0 each independently generate identical PGW processes. The final process is the sum of these processes. Furthermore, we assume that N_0 is Poisson with mean a.

The properties of this process may be obtained by regarding it as a shifted version of a special GW process in which $N_0 = 1$, and the p_k^m are given by

$$\begin{aligned} &p_k^1 = a^k e^{-a}/k! , \quad k = 0,1,\ldots , \\ &p_k^m = \left.\begin{cases} 0, & k = 0 \\ \alpha_m^{k-1} e^{-\alpha_m}/(k-1)!, & k = 1,2,\ldots \end{cases}\right\} , \quad m = 2,3,\ldots . \end{aligned} \tag{9}$$

Thus $N_1 = Z^1$ is Poisson with mean a, and the branching to generations $m = 2,3,\ldots$ occurs in accordance with a shifted Poisson law (in which no deaths occur) with parameters $\alpha_2,\alpha_3,\ldots$. This allows us to write the probability generating function for this special process as

$$\begin{aligned} &G_0(z) = z , \\ &G_1(z) = e^{a(z-1)} , \\ &G_{m+1}(z) = G_m[z e^{\alpha_m(z-1)}] , \quad m = 1,2,\ldots . \end{aligned} \tag{10}$$

Because (10) forms the limiting case for the process we shall define in Sec. III, some of its important statistical properties will be provided in the following. All of these properties may be determined by using (10).

1. Moment generating function

The moment generating function (mgf) $Q_m(s) = \langle \exp(-sN_m)\rangle$ may be obtained from the probability generating function $G_m(z)$ by the use of [7]

$$Q_m(s) = G_m(e^{-s}). \tag{11}$$

With the help of (10) we can show that for a Poisson branching process, with homogeneous branching (i.e., $\alpha_j = \alpha$), and with a Poisson initial population,

$$Q_m(s) = \exp\{a[D_m(s) - 1]\}, \quad m \geqslant 1, \tag{12}$$

where

$$D_m(s) = D_1(s) \exp\left\{\alpha \sum_{j=1}^{m-1} [D_j(s) - 1]\right\},$$

$$D_1(s) = e^{-s}.$$

For $m = 1$ and $m = 2$, we recover the mgf's for the Poisson and Thomas counting distributions, respectively.[6,8,9]

2. Moments

The moments of the count number N_m may be obtained from (11). The mean and variance are[7]

$$\langle N_m\rangle = a \prod_{k=1}^{m-1} (1 + \alpha_k), \quad m \geqslant 2 \tag{13}$$

and

$$\mathrm{Var}[N_m] = a \sum_{k=0}^{m-1} C_k \prod_{r=k+1}^{m-1} (1 + \alpha_r)^2, \tag{14}$$

where

$$C_0 = 1,$$

$$C_1 = \alpha_1,$$

$$C_k = \alpha_k \sum_{r=1}^{k-1} (1 + \alpha_r), \quad k \geqslant 2.$$

The count variance-to-mean ratio (Fano factor F) provides a suitable index for the degree of deviation from a Poisson counting process for which $F = 1$.[9] We form this ratio with the help of (13) and (14):

$$\begin{aligned} F_m &= \frac{\mathrm{Var}[N_m]}{\langle N_m\rangle} \\ &= \sum_{k=0}^{m-1} \left\{\alpha_k \left[\prod_{r=1}^{k-1} (1+\alpha_r)\right] \left[\prod_{r=k+1}^{m-1} (1+\alpha_r)^2\right]\right\} \\ &\quad \times \left(\prod_{k=1}^{m-1} (1 + \alpha_k)\right)^{-1}, \end{aligned} \tag{15}$$

where

$$\alpha_0 = 1,$$

$$\prod_{r=t}^{s} (\cdot) = 1 \quad \text{for} \quad s < t.$$

For homogeneous branching

$$\langle N_m\rangle = a(1 + \alpha)^{m-1}, \quad m \geqslant 1, \tag{16}$$

$$\mathrm{Var}[N_m] = a(1+\alpha)^{m-2}[(2+\alpha)(1+\alpha)^{m-1} - 1], \quad m \geqslant 1, \tag{17}$$

$$F_m = [1/(1+\alpha)][(2+\alpha)(1+\alpha)^{m-1} - 1], \quad m \geqslant 1. \tag{18}$$

The results for the one- and two-stage cases are clearly identical to those for the Poisson and Thomas distributions, respectively.[6,9]

When the branching is homogeneous, the nth moment of N_m may be determined from the mgf provided in (12). The result is the recurrence relation

$$\langle N_m^{n+1}\rangle = \langle N_m\rangle \sum_{k=0}^{n} \binom{n}{k} \langle N_m^{n-k}\rangle I_m^{(k+1)}, \quad m \geqslant 2, \tag{19}$$

where

$$I_m^{(1)} = 1,$$

$$I_m^{(k+1)} = \frac{(-1)^k}{(1+\alpha)^{m-1}} D_m^{k+1},$$

$$D_m^{(k+1)} = D_m^k + \alpha \sum_{l=0}^{k} \binom{k}{l} D_m^{(k-l)} \sum_{j=1}^{m-1} D_j^{(l+1)},$$

$$D_m^{(0)} = 1,$$

$$D_1^{(k)} = 1, \quad k \geqslant 1,$$

$$\langle N_m\rangle = a(1+\alpha)^{m-1}.$$

3. Counting probability distribution

The probability distribution $p_m(n)$ of N_m may be obtained by differentiating the probability generating function $G_m(z)$,[6]

$$p_m(n) = \frac{1}{n!} \frac{\partial^n}{\partial z^n} G_m(z)\bigg|_{z=0}. \tag{20}$$

Using (10) and (20), we obtain the recurrence relation for the homogeneous case,

$$p_m(0) = e^{-a}, \tag{21}$$

$$(n+1) p_m(n+1) = \langle N_m\rangle \sum_{k=0}^{n} p_m(n-k) J_m^{(k+1)},$$

where

$$J_m^{(k+1)} = \frac{(-1)^{k+1}}{(1+\alpha)^{m-1} k!} E_m^{(k+1)},$$

$$E_m^{(k+1)} = Y_m^{(k)} + \alpha \sum_{l=0}^{k} \binom{k}{l} E_m^{(k-l)} \sum_{j=1}^{m-1} E_j^{(l+1)},$$

$$Y_m^{(k+1)} = \alpha \sum_{l=0}^{k} \binom{k}{l} Y_m^{(k-l)} \sum_{j=1}^{m-1} E_j^{(l+1)},$$

$$Y_m^{(0)} = \exp\left\{\alpha \sum_{j=1}^{m-1} [E_j^{(0)} - 1]\right\},$$

$$E_m^{(k)} = 0 \quad \text{for all } m \geqslant 1, \text{ all } k \geqslant 0, \text{ except } (m,k) = (1,1),$$

$$E_1^{(1)} = -1.$$

In Fig. 1(a), we present a graphical representation of the counting distribution $p_m(n)$ versus the count number n for $m = 2,3,4$, and 10, with $\alpha = 0.5$ and $\langle N_m\rangle = 10$. It is seen that the distribution for $m = 10$ approaches a δ-function at the origin plus a relatively flat component, indicating very strong pulse clustering. In Fig. 1(b), the case for $\alpha = 2.0$ is shown. For both cases, it is clear that the variance of the counting distributions increases as the number of stages increases. It is also apparent that the variance increases with increasing α, when m and $\langle N_m\rangle$ are fixed. The results for $m = 2$ are identical to those for the instantaneous Thomas process.[6,9]

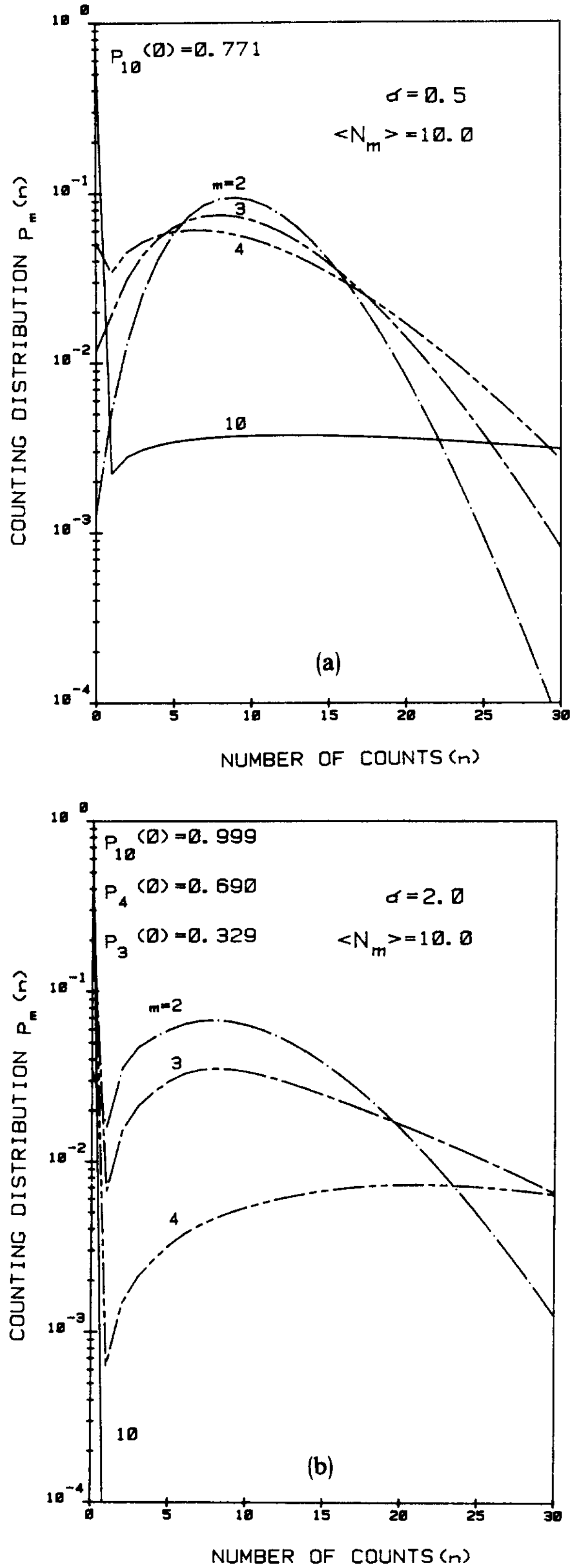

FIG. 1. Counting distribution $p_m(n)$ vs count number n with m as a parameter. The mean count $\langle N_m \rangle = 10.0$ for all cases. (a) $\alpha = 0.5$; (b) $\alpha = 2.0$.

4. Limit of continuous branching

An important special case is one in which the number of branching stages approaches infinity, while the branching at each stage becomes infinitesimal. Let

$$m \to \infty , \tag{22a}$$

$$\alpha \to 0 , \tag{22b}$$

with the product

$$m\alpha = x \tag{22c}$$

remaining finite. In this limit, we denote N_m and $Q_m(s)$ as N_x and $Q_x(s)$, respectively. The limit of (12) yields

$$Q_x(s) = \exp\{a[D_x(s) - 1]\} , \tag{23a}$$

where $D_x(s)$ satisfies the differential equation

$$\frac{\partial}{\partial x} D_x(s) = D_x(s)\,[D_x(s) - 1] , \tag{23b}$$

and the initial condition is

$$D_0(s) = e^{-s} . \tag{23c}$$

Equation (23) has the solution

$$Q_x(s) = \exp\left\{ -a \frac{1 - e^{-s}}{1 - (1 - e^{-x})e^{-s}} \right\} , \tag{24}$$

which is recognized as the moment generating function for the linear birth (Yule–Furry) process with a Poisson initial population.[10]

The nth ordinary moment of N_x is found to satisfy

$$\langle N_x^{n+1} \rangle = \langle N_x \rangle \sum_{k=0}^{n} \binom{n}{k} \langle N_x^{n-k} \rangle I_x^{(k+1)} , \tag{25}$$

where

$$I_x^{(1)} = 1 ,$$

$$I_x^{(k+1)} = \frac{(-1)^{k+1} e^{-2x}}{1 - e^{-x}} \sum_{l=1}^{\infty} \frac{l^{k+1}}{(1 - e^{-x})^l} .$$

The mean count is

$$\langle N_x \rangle = a e^x , \tag{26}$$

and the variance, which is readily obtained from (25), is given by

$$\mathrm{Var}[N_x] = a e^x (2e^x - 1) . \tag{27a}$$

The Fano factor therefore takes the particularly simple form

$$F_x = 2e^x - 1 , \tag{27b}$$

which is, of course, also obtainable from (18).

The probability (counting) distribution $p_x(n)$ of N_x may be determined from (24) or from the limit of (21). The result is

$$p_x(0) = e^{-a} ,$$
$$(n+1)\,p_x(n+1) = \langle N_x \rangle \sum_{k=0}^{n} p_x(n-k)\, J_x^{(k+1)} , \tag{28}$$

where

$$J_x^{(k+1)} = e^{-2x}(k+1)(1 - e^{-x})^k .$$

It is of interest to show the manner in which the distribution $p_m(n)$ approaches $p_x(n)$ as $m \to \infty$ and $\alpha = x/m \to 0$. In Fig. 2, we plot the counting distributions $p_m(n)$ for $m = 5$, 10, and 50, with fixed $m\alpha = x = 1.0$. We also plot $p_x(n)$ for $x = 1.0$, which is labeled Y–F (Yule–Furry). The final count mean of all distributions was kept constant at a value $\langle N_m \rangle = 10$ [this means that the initial mean a differs from curve to curve; see (16) and (26)]. The results demonstrate

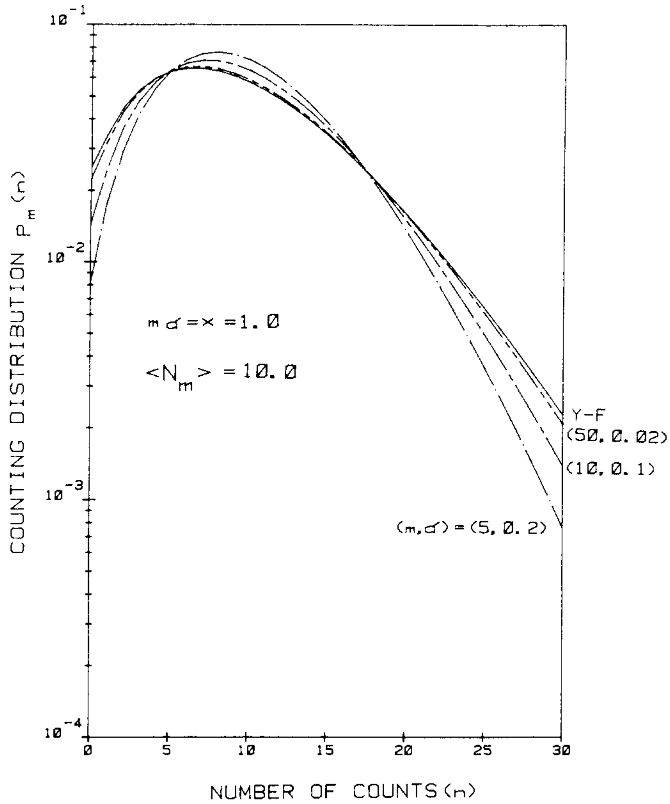

FIG. 2. Counting distributions $p_x(n)$ for the Poisson-driven Yule–Furry process (labeled Y–F) and $p_m(n)$ for the m-stage Poisson Galton–Watson branching processes with a Poisson initial population. $\langle N_m \rangle = 10.0$ and $m\alpha = x = 1.0$ for all cases. Note that $p_m(n)$ approaches $p_x(n)$ quite closely for $m = 50$.

that the limiting Yule–Furry distribution $p_x(n)$ is essentially attained (for this particular set of parameters) when $m \geqslant 50$.

III. POISSON BRANCHING POINT PROCESS

A. General branching point process

A generalization of the sequence of integers $N_0, N_1, N_2, \ldots$ discussed in Sec. IIA is the sequence of point processes $N_0(t)$, $N_1(t), N_2(t), \ldots$. Events now have times associated with them. The variable $N_m(t)$ represents the numbers of events of the mth generation which occur in the time interval $(-\infty, t]$. It is again assumed that the sequence $N_m(t)$ is Markov, i.e., given the point process $N_m(t)$, the statistics of the point process $N_{m+1}(t)$ are completely defined. The transition from the process $N_m(t)$ is obtained as follows. Each event of a given generation independently generates a point process. These point processes are statistically identical when each is measured from the occurrence time of the event that generated it. The following generation is comprised of the union of those point processes. For example, if the process $N_m(t)$ has occurrence (jump) times $t_1^m, t_2^m, t_k^m, \ldots$, the kth event of the mth generation, which occurs at time t_k^m, generates a point process $Z_k^m(t - t_k^m)$. The point processes $Z_1^m(t)$, $Z_2^m(t), \ldots$ are iid. The process $N_{m+1}(t)$ is the union of the processes $Z_k^m(t - t_k^m)$, $k = 1, 2, \ldots$; i.e.,

$$N_{m+1}(t) = \sum_{k=1}^{N_m(t)} Z_k^m(t - t_k^m).$$

The general branching point process $N_0(t), N_1(t), \ldots$ is completely defined once the point processes $Z^m(t)$ are defined for $m = 0, 1, \ldots$.

B. Poisson branching point process

We shall call a general branching point process Poisson if $Z^m(t)$ is the union of a Poisson point process of rate $h_m(t)$, with a process $u(t)$ $[u(t) = 0,\ t < 0;\ u(t) = 1,\ t \geqslant 0]$ containing only one count at $t = 0$. The initial process $N_0(t)$ also contains a single event at $t = 0$, i.e., $N_0(t) = u(t)$.

C. Poisson branching point process driven by an initial Poisson point process

Here we assume that the 1st generation $N_1(t)$ is described by an HPP counting process of rate μ. Subsequent branching follows a Poisson branching point process as described in Sec. IIIB. Because of the stationarity of the initial generation $N_1(t)$, the point processes of subsequent generations will remain stationary. This process shall be referred to as the Poisson-driven Poisson branching point process.

To understand the nature of the formation of this process, and its possible applicability to physical systems, we can think of it schematically as a cascade of systems T_m operating on random point signals. Consider an operator P representing a Poisson point generator that operates on a function $X(t)$ to produce a sequence of impulses $dN(t) = \Sigma_k \delta(t - t_k)$; $dN(t)$ represents a Poisson point process of rate $X(t)$. Consider also a unit system designated $h_m(t)$, representing a time-invariant linear system of impulse response $h_m(t)$, that operates on the signal $\Sigma_k \delta(t - t_k)$ to produce the signal $\Sigma_k h_m(t - t_k)$. The functions $h_m(t)$ are assumed nonnegative.

The Poisson branching point process with an initial Poisson population is formed as follows. The first generation $dN_1(t)$ is a homogeneous set of Poisson impulses of rate μ as shown in Fig. 3(a). This signal is modified by the system T_1 to produce a set of random impulses $dN_2(t)$ representing the second generation, and so on, as indicated in the figure. The system T_m, which is shown in Fig. 3(b), filters the stream of impulses provided to its input with a linear time-invariant filter of impulse response $h_m(t)$. The filtered signal $X_m(t)$ is a random continuous process, which in turn acts as the stochastic rate of a DSPP, represented by the set of impulses $dM_m(t)$. The union of this set of impulses with the input set $dN_m(t)$ constitutes the final output set of impulses $dN_{m+1}(t)$. [Figure 3(c) will be discussed subsequently.]

We now proceed to determine the statistical properties of the above-described Poisson-driven Poisson branching point process. The quantities we derive in this section include: (i) the moment generating functional for the process $N_m(t)$; (ii) the multifold and singlefold moment generating functions for the numbers of counts in L intervals $[t_j, t_j + T_j]$, $j = 1, 2, \ldots, L$; (iii) the moments of the number of counts $N_m(t)$ in the interval $[0, T]$; (iv) the counting probability distribution for $N_m(T)$ in $[0, T]$; and (v) the correlation function and power spectral density.

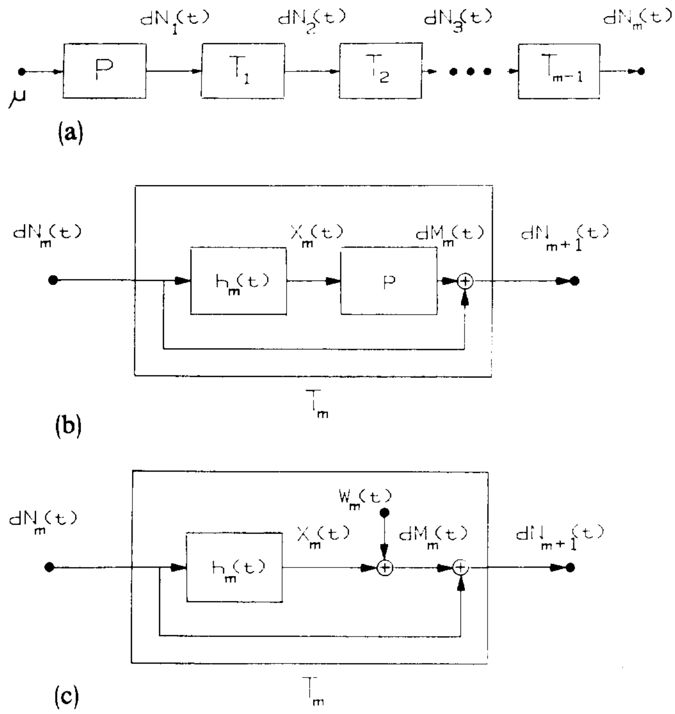

FIG. 3. (a) Schematic representation for the m-stage Poisson branching process excited by a homogeneous Poisson process with rate μ. P represents a Poisson point process generator whereas T_m represents a random point process transformation operator. (b) Point process transformation unit cell for each stage. The box $h_m(t)$ represents the impulse-response function for a time-invariant linear filter, and P is a Poisson point process generator. (c) Equivalent unit cell useful for calculating the count mean and variance. $W_m(t)$ is a stationary, zero-mean, white process.

1. Moment generating functional

The moment generating functional associated with a Poisson-driven Poisson branching point process $N_m(t)$, at the mth stage, is defined by the expectation

$$L_m(s) = \left\langle \exp\left(- \int_{-\infty}^{\infty} s(t)\, dN_m(t) \right) \right\rangle . \qquad (29)$$

It can be shown[6,7] that $L_m(s)$ satisfies the following recurrence relation:

$$L_m(s) = \left\langle \exp\left[- \int_{-\infty}^{\infty} \{ s(t) - h_{m-1}(-t) * [e^{-s(t)} - 1] \}\, dN_{m-1}(t) \right] \right\rangle$$
$$= L_{m-1}\, \{ s(t) - h_{m-1}(-t) * [e^{-s(t)} - 1] \} , \qquad (30)$$

where the symbol $*$ indicates convolution. The moment generating functional for the first stage is

$$L_1(s) = \exp\left\{ \mu \int_{-\infty}^{\infty} [e^{-s(t)} - 1]\, dt \right\} . \qquad (31)$$

For convenience, we define the following operator:

$$q_m(\cdot) = -(\cdot) + \int_{-\infty}^{\infty} h_m(\sigma - t)[\exp\{-(\cdot)\} - 1]\, d\sigma . \qquad (32)$$

Combining (29)–(32) then yields

$$L_m(s) = \exp\left\{ \mu \int_{-\infty}^{\infty} [\exp\{ q_1(q_2 \cdots q_{m-1}(s(t))) \} - 1]\, dt \right\} ,$$
$$m \geqslant 2 . \qquad (33)$$

For the case of identical impulse response functions at each stage [$h_m(t) = h(t)$ for all m], (33) can be expressed as

$$L_m(s) = \exp\left\{ \mu \int_{-\infty}^{\infty} [D_m(s,t) - 1]\, dt \right\}, \quad m \geqslant 1 , \qquad (34)$$

where

$$D_m(s,t) = D_1(s,t) \exp\left\{ h(-t) * \sum_{j=1}^{m-1} [D_j(s,t) - 1] \right\} ,$$
$$D_1(s,t) = e^{-s(t)} .$$

2. Multifold and singlefold moment generating function

The L-fold moment generating function for the numbers of counts in the intervals $[t_j, t_j + T_j]$, $j = 1,2,\ldots,L$, can be obtained from the moment generating functional $L_m(s)$ by the substitution

$$s(t) = \mathbf{s}\mathbf{v}^{\dagger}(t) , \qquad (35)$$

where $\mathbf{s}$ and $\mathbf{v}(t)$ are vectors defined by

$$\mathbf{s} = (s_1, s_2, \ldots, s_L) ,$$
$$\mathbf{v}(t) = (v_1(t), v_2(t), \ldots, v_L(t)) ,$$
$$v_j(t) = \begin{cases} 1, & t_j \leqslant t \leqslant t_j + T_j , \\ 0, & \text{otherwise} , \quad j = 1,2,3,\ldots,L . \end{cases}$$

The symbol $\dagger$ indicates vector transposition. This results in

$$Q_1(\mathbf{s}) = \exp\left\{ \mu \int_{-\infty}^{\infty} [\exp\{ -\mathbf{s}\mathbf{v}^{\dagger}(t) \} - 1]\, dt \right\} ,$$
$$Q_m(\mathbf{s}) = \exp\left\{ \mu \int_{-\infty}^{\infty} [\exp\{ q_1(q_2(q_3 \cdots q_{m-1}(\mathbf{s}\mathbf{v}^{\dagger}(t)))) \} - 1]\, dt \right\} , \quad m \geqslant 2 . \qquad (36)$$

For identical branching, it follows that

$$Q_m(\mathbf{s}) = \exp\left\{ \mu \int_{-\infty}^{\infty} [D_m(\mathbf{s},t) - 1]\, dt \right\}, \quad m \geqslant 1 , \qquad (37)$$

where

$$D_m(\mathbf{s},t) = D_1(\mathbf{s},t) \exp\left\{ h(-t) * \sum_{j=1}^{m-1} [D_j(\mathbf{s},t) - 1] \right\} ,$$
$$D_1(\mathbf{s},t) = \exp\left\{ - \sum_{j=1}^{L} s_j\, v_j(t) \right\} .$$

Equation (37) will be used to determine the correlation function and power spectral density for the process.

The statistical properties of $N_m(T)$, the number of counts in an interval $[0,T]$ at the mth stage, may be determined from the singlefold moment generating function, which is readily obtained from (36) by substituting $L = 1$:

$$Q_1(s) = \exp\left\{ \mu \int_{-\infty}^{\infty} [\exp\{ -s v(t) \} - 1]\, dt \right\} , \qquad (38)$$

$$Q_m(s) = \exp\left\{ \mu \int_{-\infty}^{\infty} [\exp\{ q_1(q_2(q_3 \cdots q_{m-1}(s v(t)))) \} - 1]\, dt \right\} , \quad m \geqslant 2.$$

This recurrence relation is difficult to use unless the branching stages are identical (homogeneous branching), in which case it reduces to

$$Q_m(s) = \exp\left\{\mu \int_{-\infty}^{\infty} [D_m(s,t) - 1]\, dt\right\}, \quad m \geqslant 1, \tag{39}$$

with

$$D_m(s,t) = D_1(s,t) \exp\left\{h(-t) * \sum_{j=1}^{m-1} [D_j(s,t) - 1]\right\},$$

$$D_1(s,t) = e^{-sv(t)},$$

$$v(t) = \begin{cases} 1, & 0 \leqslant t \leqslant T, \\ 0, & \text{otherwise}. \end{cases}$$

3. Moments

The nth ordinary moment of $N_m(T)$ follows directly from the singlefold mgf by means of the relation[11]

$$\langle N_m^n(T)\rangle = (-1)^n \frac{\partial^n}{\partial s^n} Q_m(s)\bigg|_{s=0}. \tag{40}$$

Using (39) and (40), the recurrence relation for the moments (in the special case of homogeneous branching) becomes

$$\langle N_m^{n+1}(T)\rangle = \langle N_m(T)\rangle \sum_{k=0}^{n} \binom{n}{k} \langle N_m^{n-k}(T)\rangle\, I_m^{(k+1)}, \quad m \geqslant 2, \tag{41}$$

where

$$I_m^{(1)} = 1,$$

$$I_m^{(k+1)} = \frac{1}{T(1+\alpha)^{m-1}} \int_{-\infty}^{\infty} D_m^{(k+1)}(t)\, dt,$$

$$D_m^{(k+1)}(t) = v(t)\, D_m^{(k)}(t) + \sum_{l=0}^{k} \binom{k}{l} D_m^{(k-l)}(t) \times \left[h(-t) * \sum_{j=1}^{m-1} D_j^{(l+1)}(t)\right],$$

$$D_m^{(0)}(t) = 1 \quad \text{for all } t,$$

$$D_1^{(k)} = v(t), \quad k \geqslant 1.$$

This should be compared with the expression for the instantaneous case given in (19).

For homogeneous branching, the mean number of counts is

$$\langle N_m(T)\rangle = \langle N_m^1(T)\rangle = \mu T(1+\alpha)^{m-1}, \tag{42}$$

and the variance of $N_m(T)$ is

$$\mathrm{Var}[N_m(T)] = \langle N_m(T)\rangle\, I_m^2, \quad m \geqslant 2, \tag{43}$$

with

$$I_m^2 = \frac{1}{T(1+\alpha)^{m-1}} \int_{-\infty}^{\infty} D_m^{(2)}(t)\, dt,$$

$$D_m^{(2)}(t) = \{D_m^{(1)}(t)\}^2 + h(-t) * \sum_{j=1}^{m-1} D_j^{(2)}(t),$$

$$D_m^{(1)}(t) = v(t) + h(-t) * \sum_{j=1}^{m-1} D_j^{(1)}(t),$$

$$D_1^{(1)}(t) = v(t).$$

In the limit of long counting times

$$\mathrm{Var}[N_m] = \mu T(1+\alpha)^{m-2}[(2+\alpha)(1+\alpha)^{m-1} - 1], \tag{44}$$

in accord with (17) for instantaneous branching. Though higher statistical properties are difficult to compute for nonhomogeneous branching, the count mean and variance can be obtained.

For this purpose, we consider the representation provided in Fig. 3(c), where $W_m(t)$ is a stationary, zero-mean, white random process, with a cross-correlation function given by

$$R_{W_i W_j}(\tau) = \langle W_i(t+\tau) W_j(t)\rangle = \langle X_j(t)\rangle^{1/2} \delta(t)\delta_{ij}. \tag{45}$$

$\delta(t)$ and δ_{ij} are the Dirac and Kronecker delta functions, respectively. The system in Fig. 3(c) turns out to be identically equivalent to the one in Fig. 3(b) as far as computation of the first and second moments are concerned.[7,12,13] A straightforward calculation provides

$$\langle N_m(T)\rangle = \mu T \prod_{k=1}^{m-1} (1+\alpha_k), \quad m \geqslant 2 \tag{46}$$

and

$$\mathrm{Var}[N_m(T)] = \mu \sum_{k=0}^{m-1} C_k \int_{-T}^{T} (T - |\tau|) \times \mathop{*}_{r=k+1}^{m-1} [\delta(\tau) + h_r(\tau) + h_r(-\tau) + g_r(\tau)]\, d\tau, \quad m \geqslant 2, \tag{47}$$

where

$$C_0 = 1,$$

$$C_1 = 1,$$

$$C_k = \alpha_k \prod_{r=1}^{k-1} (1+\alpha_r), \quad k \geqslant 2,$$

$$\alpha_r = \int_{-\infty}^{\infty} h_r(t)\, dt,$$

$$g_r(\tau) = h_r(\tau) * h_r(-\tau),$$

$$\mathop{*}_{r=i}^{j} [\delta(\tau) + h_r(\tau) + h_r(-\tau) + g_r(\tau)] = \delta(\tau) \text{ for } j < i.$$

The symbol $*_{k=1}^{n}$ indicates n-fold convolution. The Fano factor is therefore

$$F_m(T) = \left[\prod_{k=1}^{m-1} (1+\alpha_k)\right]^{-1} \sum_{k=0}^{m-1} C_k \int_{-T}^{T} \left(1 - \frac{|\tau|}{T}\right) \times \mathop{*}_{r=k+1}^{m-1} [\delta(\tau) + h_r(\tau) + h_r(-\tau) + g_r(\tau)]\, d\tau, \quad m \geqslant 2. \tag{48}$$

When all α_j are identical and equal to α, (46) and (47) reduce to (42) and (43), respectively. In the limit of long counting times, the process is effectively instantaneous and the above expressions for the mean, variance, and Fano factor become (13), (14), and (15), with $a = \mu T$, respectively. In the special case $m = 2$, (46)–(48) reproduce the previously obtained results for the Thomas point process.[6]

Because of the importance of the Fano factor as a simple measure characterizing the departure of a process from the HPP, we carry out a parametric study of its dependence in our branching process. For simplicity, we assume that the impulse response functions $h_m(t)$ are identical at each stage, and have the simple exponential form

$$h(t)=\begin{cases}(2\alpha/\tau_p)\exp(-2t/\tau_p), & t\geqslant 0,\\ 0, & t<0.\end{cases} \tag{49}$$

Here $\tau_p/2$ is the characteristic decay time of the filter and α is the area under the function.

In Fig. 4, we plot the Fano factor $F_m(T)$ versus the number of stages m, with $2T/\tau_p$ and α as parameters. All of the curves are monotonically increasing functions of m (as are the underlying mean and variance curves). This is to be contrasted with the results for the cascaded Poisson process that we studied earlier,[7] in which the mean and variance decay with increasing m if $\alpha<1$. The distinction arises because of the presence of the feed-forward path [shown in Fig. 3(b)], whch distinguishes the present model as a branching process, rather than as a simple cascade of stages. For $T/\tau_p \gg 1$, the curves will obey (18), which provides essentially exponential growth (straight-line behavior on a logarithmic ordinate). For $T/\tau_p \ll 1$, the particlelike clusters of the points in the process are chopped apart by the small sampling time, leading to the independence that is characteristic of the HPP.[6] Indeed, as the curves for $2T/\tau_p=0.01$ show, $F_m(T)$ remains essentially constant at unity, up to four stages. The small residual clustering is amplified as m increases above this value. Increasing values of α, of course, correspond to increased clustering.

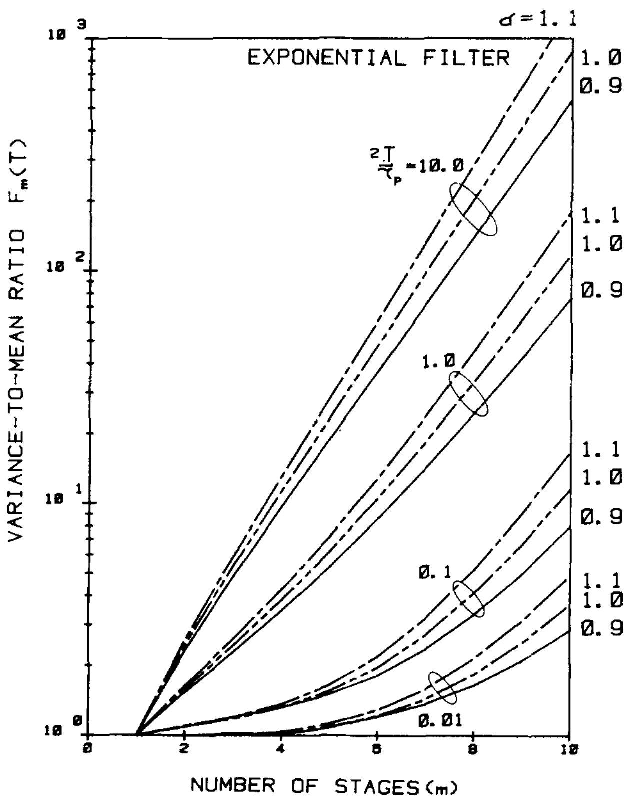

FIG. 4. Count variance-to-mean ratio (Fano factor) $F_m(T) = \mathrm{Var}[N_m(T)]/\langle N_m(T)\rangle$ vs number of stages m, with $2T/\tau_p$ and α as parameters. The impulse response functions $h_m(t)$ are all assumed to be identical, exponentially decaying functions with time constant $\tau_p/2$.

4. Counting probability distribution

The counting probability distribution function of $N_m(T)$ can be derived by using (11), (20), and (38) from which it follows that

$$p_m(0)=\exp\left\{\mu\int_{-\infty}^{\infty}\left[E_m^{(0)}(t)-1\right]dt\right\}, \tag{50}$$

$$(n+1)p_m(n+1)=\langle N_m(T)\rangle\sum_{k=0}^{n}p_m(n-k)J_m^{(k+1)},$$

where

$$J_m^{(k+1)}=\frac{(-1)^{k+1}}{T(1+\alpha)^{m-1}k!}\int_{-\infty}^{\infty}E_m^{(k+1)}(t)\,dt,$$

$$E_m^{(k+1)}(t)=E_m^{(1)}Y_m^{(k)}(t)+\sum_{l=0}^{k}\binom{k}{l}E_m^{(k-l)}(t)h(-t)*\sum_{j=1}^{m-1}E_j^{(l+1)}(t),$$

$$Y_m^{(k+1)}(t)=\sum_{l=0}^{k}\binom{k}{l}Y_m^{(k-l)}h(-t)*\sum_{j=1}^{m-1}E_j^{(l+1)}(t),$$

$$Y_m^{(0)}(t)=\exp\left\{h(-t)*\sum_{j=1}^{m-1}\left[E_j^{(0)}(t)-1\right]\right\},$$

$$E_m^{(0)}(t)=\begin{cases}0, & 0\leqslant t\leqslant T,\\ \exp\left\{h(-t)*\sum_{j=1}^{m-1}\left[E_j(t)-1\right]\right\}, & \text{otherwise},\end{cases}$$

$$E_1^{(0)}(t)=\begin{cases}0, & 0\leqslant t\leqslant T,\\ 1, & \text{otherwise},\end{cases}$$

$$E_1^{(1)}(t)=\begin{cases}-1, & 0\leqslant t\leqslant T\\ 0, & \text{otherwise},\end{cases}$$

$$E_1^{(k)}(t)=0 \quad \text{for all } t,\ k\geqslant 2.$$

Equation (50) reduces to (21) in the limit $T/\tau_p \gg 1$. As T/τ_p is reduced, $F_m(T)$ will decrease (see Fig. 4), and the counting distributions will narrow. The transition in $p_m(n)$ vs n will not be unlike that demonstrated for the cascaded Poisson process (see Ref. 7, Fig. 8).

5. Autocorrelation function and power spectral density

In this subsection, we derive the autocorrelation function and the power spectral density for the Poisson branching point process. The autocorrelation function $r_m(\tau)$ is defined as

$$r_m(\tau)=\lim_{\Delta t\to 0}\frac{1}{(\Delta t)^2}\langle \Delta N_m(t)\Delta N_m(t+\tau)\rangle, \tag{51}$$

where the quantity $\Delta N_m(t)$ represents the number of counts in the time interval $[t,t+\Delta t]$, at the mth stage. The equation for $\langle \Delta N_m(t)\Delta N_m(t+\tau)\rangle$ may be obtained from (37) by substituting

$$L=2,$$

$$v_1(t)=\begin{cases}1, & 0\leqslant t\leqslant \Delta t,\\ 0, & \text{otherwise},\end{cases}$$

$$v_2(t)=\begin{cases}1, & \tau\leqslant t\leqslant \tau+\Delta t,\\ 0, & \text{otherwise}.\end{cases}$$

Differentiating (37) with respect to s_1 and s_2, substituting $s_1=s_2=0$, and letting $\Delta t\to 0$ leads to (see Appendix)

$$r_m(\tau) = \{ \mu(1+\alpha)^{m-1}\}^2 + \mu \int_{-\infty}^{\infty} Y_m(\omega)\, e^{j\omega\tau} \frac{d\omega}{2\pi}, \quad (52a)$$

where

$$Y_m(\omega) = |1 + H(\omega)|^{2(m-1)} + \alpha(1+\alpha)^{m-2} \times \frac{1 - [|1+H(\omega)|^2/(1+\alpha)]^{m-1}}{1 - |1+H(\omega)|^2/(1+\alpha)}. \quad (52b)$$

$H(\omega)$ is the Fourier transform of $h(t)$. Substituting $\tau = 0$ into the second term of (52a) yields the variance

$$\mathrm{Var}\,[dN_m(t)] = \mu \int_{-\infty}^{\infty} Y_m(\omega) \frac{d\omega}{2\pi}, \quad (53)$$

which represents the power fluctuations of the process $dN_m(t)$ in the infinitesimal duration Δt.

The power spectral density $s_m(\omega)$ is defined as the Fourier transform of the autocorrelation function $r_m(\tau)$, which is clearly

$$s_m(\omega) = 2\pi \{ \mu(1+\alpha)^{m-1}\}^2 \delta(\omega) + \mu Y_m(\omega). \quad (54)$$

The first term of (54) represents the dc power of the process $dN_m(t)$, whereas the second term represents the frequency distribution of the ac power, which depends on the shape of the impulse response function $h(t)$ through $H(\omega)$.

The autocorrelation function between the number of counts in the interval T, separated by a time delay τ, is defined as

$$R_m(\tau) = \langle [N_m(t+T) - N_m(t)] \times [N_m(t+T+\tau) - N_m(t+\tau)] \rangle, \quad (55)$$

which can be easily obtained from (52a) by means of

$$R_m(\tau) = \int_0^T \int_0^T r_m(t_1 - t_2 + \tau)\, dt_1\, dt_2. \quad (56)$$

Substituting (52a) into (56) gives rise to

$$R_m(\tau) = \{ \mu T(1+\alpha)^{m-1}\}^2 + \mu T \int_{-\infty}^{\infty} Y_m(\omega) \Phi_T(\omega) e^{j\omega\tau} \frac{d\omega}{2\pi}, \quad (57a)$$

where

$$\Phi_T(\omega) = T\,[\sin(\omega T/2)/(\omega T/2)]^2. \quad (57b)$$

Substituting $\tau = 0$ into the second term of (57a) leads to the variance of the counting process,

$$\mathrm{Var}[N_m(T)] = \mu T \int_{-\infty}^{\infty} Y_m(\omega) \Phi_T(\omega) \frac{d\omega}{2\pi}, \quad (58)$$

which is the frequency-domain representation of (43). The power spectral density for the counts is easily obtained by taking the Fourier transform of (57a), which provides

$$S_m(\omega) = 2\pi \{ \mu T(1+\alpha)^{m-1}\}^2 \delta(\omega) + \mu T Y_m(\omega) \Phi_T(\omega). \quad (59)$$

IV. POISSON BRANCHING POINT PROCESS IN THE LIMIT OF CONTINUOUS BRANCHING

A. Introduction

In this section we investigate properties of the Poisson branching point process in the limit of an infinite number of branching stages, when the branching at each stage is infinitesimal. Thus we allow

$$m \to \infty, \quad (22a)$$

$$\alpha \to 0, \quad (22b)$$

with the product

$$m\alpha = x \quad (22c)$$

remaining finite. In this limit we replace the discrete index m, which has been used throughout Sec. III to indicate the branching stage number, with the continuous index x. Thus L_m, Q_m, N_m,... become L_x, Q_x, N_x, ..., respectively. Furthermore we define a normalized impulse response function $h_0(t)$ such that

$$h(t) = \alpha h_0(t) \quad (60)$$

and

$$\int_{-\infty}^{\infty} h_0(t)\, dt = 1.$$

By applying this limit to the expression derived in Sec. III, we obtain a number of results that form a simple generalization of the Yule–Furry process. Their application to the generation of cosmic ray showers is likely to be useful.

B. Results

We are able to obtain results for the moment generating functional and moment generating function in the case of instantaneous branching, when the initial process is Poisson. These are, of course, identical to those for the Poisson-driven Yule–Furry process, as provided in Sec. II C. General results, with arbitrary time dynamics, have been derived for the count mean, variance, and Fano factor, and for the autocorrelation function and power spectral density of the point process. It will be evident in the following that the count mean and variance depend critically on m. The results below should be compared with those provided in Secs. II C and III C.

1. Moment generating functional

The moment generating functional (34) becomes

$$L_x(s) = \exp\left\{ \mu \int_{-\infty}^{\infty} [D_x(s,t) - 1]\, dt \right\}, \quad (61a)$$

where $D_x(s,t)$ satisfies the nonlinear integro-differential functional equation

$$\frac{\partial}{\partial x} D_x(s,t) = D_x(s,t)\, \{ h_0(-t) * [D_x(s,t) - 1] \}, \quad (61b)$$

with the initial condition

$$D_0(s,t) = e^{-s(t)}. \quad (61c)$$

We are unable to obtain a general solution to (61b). However, in the simple special case where

$$h_0(t) = \delta(t), \quad (62)$$

(61b) can be shown to have the solution

$$D_x(s,t) = \frac{e^{-x} e^{-s(t)}}{1 - (1 - e^{-x}) e^{-s(t)}}. \quad (63)$$

The moment generating functional is then

$$L_x(s) = \exp\left\{ -\mu \int_{-\infty}^{\infty} \frac{1 - e^{-s(t)}}{1 - (1 - e^{-x}) e^{-s(t)}}\, dt \right\}. \quad (64)$$

2. Moment generating function

The moment generating function $Q_x(s)$ of the random variable $N_x(T)$ may be obtained from the moment generating functional $L_x(x)$ by setting $s(t) = sv(t)$. Equation (61b) is then a nonlinear integro-differential equation which is difficult to solve. In the special case of instantaneous branching, we can use (64) to obtain

$$Q_x(s) = \exp\left\{-\mu T \frac{1-e^{-s}}{1-(1-e^{-x})e^{-s}}\right\}, \tag{65}$$

which is identical to (24) with $a = \mu T$, as it should be. Equations (24) and (65) are identified as the moment generating function of a Yule–Furry process driven by a homogeneous Poisson point process, as mentioned above.

3. Moments

It is possible to obtain expressions for the mean and variance of $N_x(T)$ for an arbitrary impulse response function $h_0(t)$. Applying the limits of (22) on (42) leads to

$$\langle N_x(T)\rangle = \mu T e^x. \tag{66}$$

Note that (66) is identical to (26) with $a = \mu T$. A similar operation on (52b) yields

$$Y_x(\omega) = \lim_{m\to\infty} Y_m(\omega) = \frac{[H(\omega)+H(-\omega)]e^{x[H(\omega)+H(-\omega)]} - e^x}{H(\omega)+H(-\omega)-1}, \tag{67}$$

so that the count variance is [see (58)]

$$\mathrm{Var}[N_x(T)] = \mu T \int_{-\infty}^{\infty} Y_x(\omega)\Phi_T(\omega)\frac{d\omega}{2\pi}. \tag{68}$$

Here $H(\omega)$ is the Fourier transform of $h_0(t)$ (the transfer function of the filtering system), $H(-\omega)$ is the complex conjugate of $H(\omega)$, and the function $\Phi_T(\omega)$ is given in (57b). Using Eqs. (66) and (68), the Fano factor becomes

$$F_x(T) = \int_{-\infty}^{\infty} \Phi_T(\omega) \times \frac{[H(\omega)+H(-\omega)]\,e^{x[H(\omega)+H(-\omega)-1]} - 1}{H(\omega)+H(-\omega)-1}\frac{d\omega}{2\pi}. \tag{69}$$

For the case of instantaneous multiplication, $H(\omega) = 1$ for all ω so that (68) and (69) reduce to the Poisson-driven Yule–Furry results

$$\mathrm{Var}[N_x(T)] = \mu T e^x(2e^x - 1) \tag{70}$$

and

$$F_x = 2e^x - 1, \tag{71}$$

respectively. Of course, (70) and (71) are then identical with (27a) and (27b) with $a = \mu T$.

To assess the effects of the characteristic decay time τ_p of the filter $h_0(t)$ on the fluctuation properties of the counting process $N_x(T)$, we consider a simple example. We make use of the ideal low-pass filter transfer function

$$H(\omega) = \begin{cases} 1, & |\omega| \leqslant \omega_c, \\ 0, & \text{otherwise}, \end{cases} \tag{72}$$

where $\omega_c/2 = 1/\tau_p$. It can be shown that (69) then leads to

$$F_x = [2e^x - 1]\,\xi(T/\tau_p) + [1 - \xi(T/\tau_p)], \tag{73a}$$

where

$$\xi(\beta) = (2/\pi)\,\mathrm{Si}(\beta/2) - (4/\beta)[1 - \cos(\beta/2)] \tag{73b}$$

and

$$\mathrm{Si}(\beta) = \int_0^\beta \frac{\sin(y)}{y}\,dy. \tag{73c}$$

In Fig. 5 we plot the Fano factor $F_x(T)$ as a function of the branching parameter x, with the ratio $\beta = T/\tau_p$ as a parameter. In the limit $T \gg \tau_p$, $\xi(T/\tau_p) \to 1$, and

$$F_x = 2e^x - 1 \quad \text{for } T \gg \tau_p, \tag{73d}$$

in accord with the (instantaneous) results presented in (71). In the opposite limit $(T \ll \tau_p)$, $\xi(T/\tau_p) \to 2T/\tau_p$, corresponding to a reduced Fano factor

$$F_x = [2e^x - 1](2T/\tau_p) + 1 - (2T/\tau_p) \quad \text{for } T \ll \tau_p. \tag{74}$$

It is apparent from (74) and from Fig. 5 that as T/τ_p decreases, the Fano factor, and therefore the degree of fluctuation, decreases. The reason for this, once again, is the cutting apart of the particlelike clusters of multiplied events.

4. Autocorrelation function and power spectral density

The autocorrelation function and power spectral density for the process $dN_x(t)$ may be determined by taking the limit of (52a) and (54), respectively. The results are

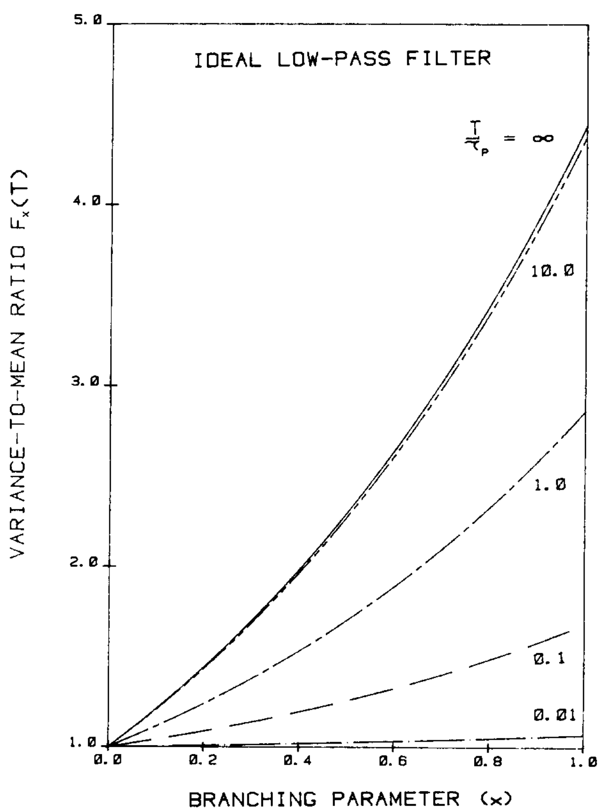

FIG. 5. Fano factor $F_x(T)$ as a function of the branching parameter x, with T/τ_p as a parameter. In this example of continuous branching, the time dependence of the process is represented by an impulse-response function whose Fourier transform is an ideal low-pass filter.

$$r_x(\tau) = \mu^2 e^{2x} + \mu \int_{-\infty}^{\infty} Y_x(\omega)\, e^{j\omega\tau} \frac{d\omega}{2\pi} \tag{75}$$

and

$$s_x(\omega) = 2\pi\mu^2 e^{2x}\delta(\omega) + \mu Y_x(\omega), \tag{76}$$

where $Y_x(\omega)$ is given by (67).

The autocorrelation function of the counts $N_x(T)$ for the infinite branching case is obtained from (75) by using (56). This provides

$$R_x(\tau) = (\mu T)^2 e^{2x} + \mu T \int_{-\infty}^{\infty} Y_x(\omega)\Phi_T(\omega)\, e^{j\omega\tau} \frac{d\omega}{2\pi}. \tag{77}$$

The power spectral density in this case is

$$S_x(\omega) = 2\pi(\mu T)^2 e^{2x}\delta(\omega) + \mu T Y_x(\omega)\Phi_T(\omega), \tag{78}$$

corresponding to (59).

In Fig. 6, we present the power spectral density for the Poisson branching point process $s_m(\omega\tau_p)$ versus normalized frequency $\omega\tau_p$ [see (54) and (76)] with m as a parameter. For the purposes of illustration, we have chosen an exponential impulse response function [see (49)] and ignored the delta function at $\omega\tau_p = 0$. The product $m\alpha = x$ was maintained constant for each plot [$m\alpha = x = 1.0$ in Fig. 6(a); $m\alpha = x = 4.0$ in Fig. 6(b)]. This enables us to follow the behavior of $s_m(\omega\tau_p)$ as m increases toward the continuous limit ($m = \infty$). The driving rate was adjusted in all cases to be $\mu = (1+\alpha)^{-m} = (1 + 1/m)^{-m}$ so that the rate of the final point processes is unity. For the parameters shown, it is evident that the curves are of very similar shape, although their absolute and relative magnitudes are strongly dependent on m and on $m\alpha = x$.

Finally, we note that while we generally think of x as position in a continuum of branching stages, and t as time, it may be more appropriate in some applications to regard the variable x as time along which branching progresses, and t as position. In such an interpretation, $h(t)$ will indicate diffusion or migration of particles in space, and $N_x(T)$ the number of particles in the space $[0,T]$ at the time x.

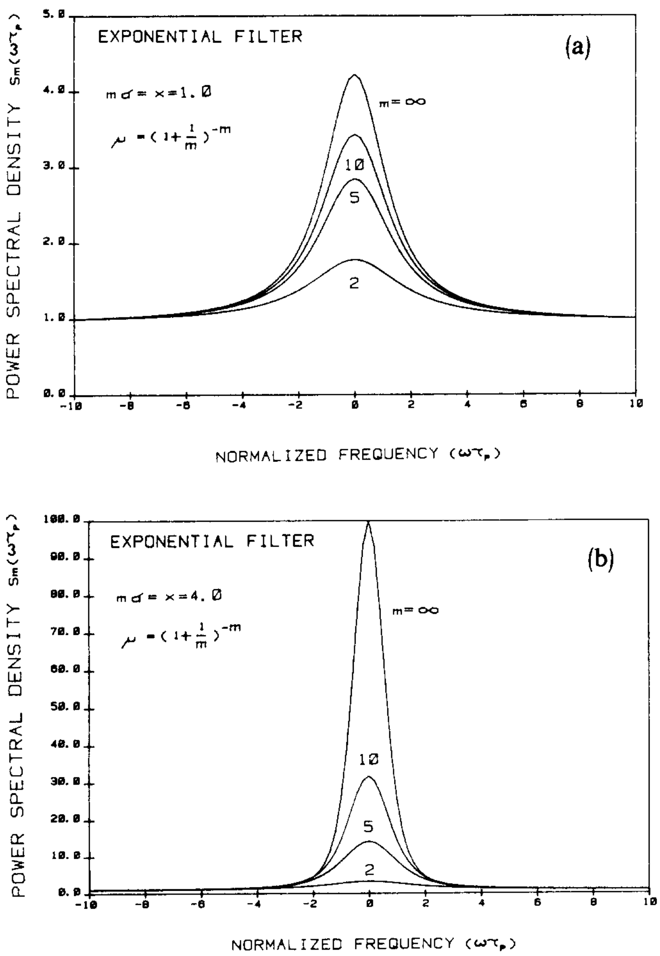

FIG. 6. Power spectral density for the Poisson branching point process $s_m(\omega\tau_p)$ vs normalized frequency $\omega\tau_p$, with m as a parameter. For the purposes of this illustration, we have chosen an exponential impulse response function, and eliminated the delta function at $\omega\tau_p = 0$. The driving rate $\mu = (1+\alpha)^{-m} = (1+1/m)^{-m}$ in all cases. (a) $m\alpha = x = 1.0$; (b) $m\alpha = x = 4.0$.

ACKNOWLEDGMENTS

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APPENDIX: DERIVATION OF THE CORRELATION FUNCTION $r_m(\tau)$ FOR THE POISSON BRANCHING POINT PROCESS

Differentiating (37) with respect to s_1 and s_2, and setting $s_1 = s_2 = 0$, provides

$$\begin{aligned} r_m(\tau) &= \lim_{\Delta t \to 0} \frac{1}{(\Delta t)^2} \langle \Delta N_m(t) \Delta N_m(t+\tau) \rangle \\ &= \mu^2 \int_{-\infty}^{\infty} \Phi_m^{(1)}(t)dt \int_{-\infty}^{\infty} \Phi_m^{(2)}(t)dt \\ &\quad + \mu \int_{-\infty}^{\infty} \Phi_m^{(3)}(t)dt, \end{aligned} \tag{A1}$$

where

$$\begin{aligned} \Phi_m^{(1)}(t) &= \Phi_1^{(1)}(t) + h(-t) * \sum_{j=1}^{m-1} \Phi_j^{(1)}(t), \\ \Phi_m^{(2)}(t) &= \Phi_1^{(2)}(t) + h(-t) * \sum_{j=1}^{m-1} \Phi_j^{(2)}(t), \\ \Phi_m^{(3)}(t) &= \Phi_1^{(3)}(t) + \Phi_1^{(1)}(t)\left[h(-t) * \sum_{j=1}^{m-1} \Phi_j^{(2)}(t)\right] \\ &\quad + \Phi_1^{(2)}(t)\left[h(-t) * \sum_{j=1}^{m-1} \Phi_j^{(1)}(t)\right] \\ &\quad + h(-t) * \sum_{j=1}^{m-1} \Phi_j^{(3)}(t) \\ &\quad + \left[h(-t) * \sum_{j=1}^{m-1} \Phi_j^{(1)}(t)\right] \\ &\quad \times \left[h(-t) * \sum_{j=1}^{m-1} \Phi_j^{(2)}(t)\right], \end{aligned} \tag{A2}$$

with the initial conditions

$$\begin{aligned} \Phi_1^{(1)}(t) &= -\delta(t), \\ \Phi_1^{(2)}(t) &= -\delta(t-\tau), \\ \Phi_1^{(3)}(t) &= \delta(t)\delta(\tau). \end{aligned} \tag{A3}$$

Taking the Fourier transform of (A2) and (A3) to obtain the frequency-domain equivalent of Eq. (A1) provides

$$r_m(\tau) = \mu^2 \widetilde{\Phi}_m^{(1)}(0)\widetilde{\Phi}_m^{(2)}(0) + \mu\widetilde{\Phi}_m^{(3)}(0), \tag{A4}$$

where $\widetilde{\Phi}_m^{(i)}(0)$ is the Fourier transform of $\Phi_m^{(i)}(t)$ evaluated at $\omega = 0$. A simple calculation shows that the first term in (A4) is

$$\mu^2\widetilde{\Phi}_m^{(1)}(0)\widetilde{\Phi}_m^{(2)}(0) = \{\mu(1+\alpha)^{m-1}\}^2, \tag{A5}$$

whereas the second term of (A4) is

$$\mu\widetilde{\Phi}_m^{(3)}(0) = \mu\int_{-\infty}^{\infty} Y_m(\omega)e^{j\omega\tau}\frac{d\omega}{2\pi}, \tag{A6}$$

with $Y_m(\omega)$ as given in (52b).

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