Exact First-, Second-, and Third-Order Photocounting Cumulants for Lognormally Modulated Mixtures of Coherent and Chaotic Radiation*

S. Rosenberg† and M. C. Teich

Department of Electrical Engineering and Computer Science, Columbia University, New York, New York 10027

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In this paper we investigate the photocount statistics of mixed coherent plus Gaussian light that has suffered lognormal fading, such as produced by the turbulent atmosphere. The Gaussian component need not be stationary, may have an arbitrary spectral distribution, and the mean frequency of the coherent and Gaussian components need not coincide. The first three cumulants of the phototelectric counts for a single detector and for an array of detectors are obtained for arbitrary ratios of counting time to source coherence time, β, while assuming the fading to be fully time correlated over the detection interval. In particular, the cumulants are evaluated for Gaussian light of Lorentzian spectrum. The variation of the cumulants with degree of turbulence and detector separation is exhibited graphically, for several values of β, and ratio of coherent to chaotic component, y. The effect of increasing the degree of turbulence is shown to cause the ratio of the second-order cumulant for Gaussian light to the same cumulant for coherent light to approach a nominal value of 2, indicating the extent to which the fading dominates the counting statistics. As the detector separation is varied, the twofold cumulant exhibits the spatial correlations of the turbulence, when the source radiation alone is assumed to be approximately spatially coherent at the detector array. Furthermore, the cumulants are shown to increase exponentially with the turbulence level σ, the log-intensity standard deviation.

I. INTRODUCTION

The study of photoelectron counting statistics for light that has propagated through atmospheric turbulence has recently received considerable attention.[1-4] In particular, several authors have evaluated the photoelectron counting distribution to be expected from optical sources with a variety of statistics for both the single-detector and multidetector cases.[1-3] Some aspects of the related optical communication problem have also recently been presented.[5] In work to date, the generally accepted assumption of lognormal statistics for the scintillation induced by the turbulence leads to problems which are difficult to handle analytically. Only approximate results are available for the counting distributions, and none for the generating function.[1-4] Because of this difficulty, we investigate some aspects of the photocounting statistics which are exact, regardless of the turbulence level σ and the ratio of counting time to source coherence time, T/τ_c. In particular we investigate the first three cumulants of the photocounts for a single detector, and the twofold and threefold cumulant for an array of detectors, for a radiation source consisting of a chaotic component mixed with a coherent component. Aside from the exact nature of these expressions, they give some indication as to the variation of the performance of a detection scheme, as measured by the probability of error in making a decision, as the shape of the counting distribution varies with the level of turbulence and the detector separation. The variance and the covariance of the photoelectron counts are presented graphically as a function of σ, and of detector separation for several values of T/τ_c. The magnitude of the variance and covariance of the photoelectron counts is shown to increase exponentially with the turbulence level σ. For moderate to severe turbulence the covariance of the counts is also shown to reflect the spatial covariance of the turbulent fluctuations.

II. THEORY

A. General

The source radiation is assumed to consist of a linearly polarized coherent component with an additive chaotic component represented by

$$V_s(\vec{r},t)=V(\vec{r},t)+U(\vec{r},t), \tag{1}$$

where $V(\vec{r},t)$ is a complex zero mean Gaussian random process with independent and identically distributed real and imaginary parts, and $U(\vec{r},t)$ is the coherent component, having constant amplitude and uniform phase. After traversing the atmosphere, the field at the detector is given by

$$V_d(\vec{r},t)=V_s(\vec{r},t)Z(\vec{r},t)=V_s(\vec{r},t)\left|Z(\vec{r},t)\right|e^{j\theta(\vec{r},t)}, \tag{2}$$

where $Z(\vec{r},t)$ is a complex lognormal process.[6] That is, $\ln Z(\vec{r},t)=\ln|Z(\vec{r},t)|+j\theta(\vec{r},t)$ is a complex Gaussian process completely specified by its mean and covariance functions.[7] This is a useful model for laser radiation and for some forms of scattered radiation.[1,2]

The photoelectron counting distribution for a field of arbitrary statistics is given by[8]

$$p(n_1,t_1,T_1;\ n_2,t_2,T_2;\cdots;n_k,t_k,T_k)=\left\langle\prod_i \frac{W_i^{n_i}e^{-W_i}}{n_i!}\right\rangle_{\{W_i\}}, \tag{3}$$

where the ensemble average is over the joint statistics of the $\{W_i\}$. The integrated intensity at the ith detector, W_i, is given by

$$W_i = \int_{A_{di}} \int_{t_i}^{t_i+T_i} \frac{\eta}{h\nu} I_d(\vec{r}, t)\, dt\, dA, \quad (4)$$

where η is the quantum efficiency of the detector, $I_d(\vec{r}_i t) = V_d(\vec{r}_i, t)V_d^*(\vec{r}_i, t)$, A_{di} is the area of the ith detector, dA is the differential area, and $h\nu$ is the photon energy. If we assume $A_{di} \ll A_c$, where A_c is the coherence area for the detected process, then with $I_i(t) \equiv I_d(\vec{r}_i, t)$, we obtain

$$W_i = \alpha_i \int_{t_i}^{t_i+T_i} I_i(t)\, dt, \quad (5)$$

where α_i is a constant proportional to detector area and to the quantum efficiency. In other words, one spatial mode of the field is intercepted by the detector. Since the incident radiation is a product of the source radiation process and the fading process, the effect of integrating over an area greater than A_c is equivalent to decreasing the variance of the scintillations,[9] while the statistics of W_i, for an amplitude-stabilized source, are shown to remain lognormal even for large apertures.[10] Thus, with little loss in generality, we assume $A_{di} \ll A_c$, and therefore neglect spatial integration effects over the detector aperture. The intensity function $I_i(t)$ is then given by

$$I_i(t) = |Z_i(t)|^2 |V_{si}(t)|^2 = I_{zi}(t) I_{si}(t). \quad (6)$$

Furthermore, unless we talk about heterodyne detection, we need only consider the statistics of the absolute value of the fading or the log absolute value $\ln|Z_i(t)|$.

The statistical quantities that are of interest here are the cumulants, in that these represent the true mth-order correlations of the radiation with all lower-order correlations removed.[11] We first summarize some of the general relationships between cumulants and moments which will be used in the sequel[12–14]:

$$k_1(x) = \langle x\rangle,$$
$$k_2(x) = \langle x^2\rangle - \langle x\rangle^2, \quad (7a)$$
$$k_3(x) = \langle x^3\rangle - 3\langle x^2\rangle\langle x\rangle + 2\langle x\rangle^3,$$

where $k_i(x)$ is defined as the ith-order cumulant of x, and

$$k_{11}(x_1, x_2) = \langle x_1 x_2\rangle - \langle x_1\rangle\langle x_2\rangle, \quad (7b)$$
$$k_{111}(x_1, x_2, x_3) = \langle x_1 x_2 x_3\rangle - \sum_{\substack{i=1\\ i\neq j\neq k}}^{3} \langle x_i\rangle\langle x_j x_k\rangle + 2\langle x_1\rangle\langle x_2\rangle\langle x_3\rangle.$$

Here $k_{11\ldots1}(x_1, x_2, \cdots, x_N)$ is the N-fold first-order cumulant of $(x_1, x_2, \ldots, x_N)$. The summation as indicated in (7b) is defined as the sum over all integers i, such that j and k are filled by *combinations*, as opposed to permutations, of the remaining integers that i can assume. Therefore, for the above sum we have for (i,j,k) the sets (1, 2, 3), (2, 1, 3), and (3, 1, 2). *The same definition applies in later sections*. The cumulant generating function is defined as the natural logarithm of the moment generating function.[13] The moment generating function of the counts $M([n]; [u])$, however, can be shown to be related to the moment generating function for the integrated intensities by

$$M([n]; [u]) = M([W]; [e^u - 1]), \quad (8)$$

where $[x] \equiv (x_1, x_2, \ldots, x_N)$, and u is a parameter. A similar relationship exists between the cumulant generating functions for $[n]$ and for $[W]$. It can be shown[12] that

$$K([n]; [u]) = K([W]; [e^u - 1]), \quad (9)$$

where K is now the cumulant generating function.

The general expression for the relationship between these cumulants is rather complicated, and we repeat here only those expressions which are used explicitly. The relationships between the cumulants of $[n]$ and $[W]$ are, for the first-, second-, and third-order cases, as follows:

$$k_1(n) = k_1(W), \qquad \text{first-order onefold}$$
$$k_2(n) = k_1(W) + k_2(W), \qquad \text{second-order onefold} \quad (10a)$$
$$k_3(n) = k_1(W) + 3k_2(W) + k_3(W); \qquad \text{third-order onefold}$$

$$k_{11}(n_1, n_2) = k_{11}(W_1, W_2), \qquad \text{twofold}$$
$$k_{111}(n_1, n_2, n_3) = k_{111}(W_1, W_2, W_3). \qquad \text{threefold} \quad (10b)$$

Furthermore, for the first-order N-fold case, it is always true[12] that $k_{11\ldots1}(n_1, n_2, \cdots, n_N) = k_{11\ldots1}(W_1, W_2, \cdots, W_N)$. Exact expressions can be found for other higher-fold cumulants, but these rapidly increase in complexity for any product of two random variables, such as the lognormal fading channel considered here. Thus we limit ourselves to those cases given in (10).

The general expression for the twofold and threefold intensity cumulants, for arbitrary radiation experiencing lognormal fading, is given by

$$k_{11}(I_1, I_2) = \langle I_{z1}(t)I_{z2}(s)\rangle\langle I_{s1}(t)I_{s2}(s)\rangle - \langle I_{z1}(t)\rangle\langle I_{z2}(s)\rangle\langle I_{s1}(t)\rangle\langle I_{s2}(s)\rangle,$$

$$k_{111}(I_1, I_2, I_3) = \langle I_{z1}(t)I_{z2}(s)I_{z3}(p)\rangle\langle I_{s1}(t)I_{s2}(s)I_{s3}(p)\rangle - \sum_{\substack{i=1\\ i\neq j\neq k}}^{3} \langle I_{zi}(t)\rangle\langle I_{si}(t)\rangle\langle I_{zj}(s)I_{zk}(p)\rangle\langle I_{sj}(s)I_{sk}(p)\rangle$$
$$+ 2\langle I_{z1}(t)\rangle\langle I_{z2}(s)\rangle\langle I_{z3}(p)\rangle\langle I_{s1}(t)\rangle\langle I_{s2}(s)\rangle\langle I_{s3}(p)\rangle. \quad (11a)$$

This expression derives from (7b), where $x_i \Rightarrow I_{zi}(t) \times I_{si}(t)$. The cumulants for the integrated intensity are then

$$k_{11}(W_1, W_2) = \alpha_1\alpha_2 \int_{t_1}^{t_1+T_1} \int_{t_2}^{t_2+T_2} k_{11}(I_1, I_2)\, dt\, ds,$$

$$k_{111}(W_1, W_2, W_3) = \alpha_1\alpha_2\alpha_3 \times \int_{t_1}^{t_1+T_1} \int_{t_2}^{t_2+T_2} \int_{t_3}^{t_3+T_3} k_{111}(I_1, I_2, I_3)\, dt\, ds\, dp. \quad (11b)$$

Using the above expressions, one can always obtain lower-fold cumulants by setting variables equal to each other. For example, in (11a) if we let $I_2 = I_3$, then $k_{111}(I_1, I_2, I_3) \Rightarrow k_{12}(I_1, I_2)$, and similarly if $I_1 = I_2 = I_3$, that is, a single space-time point is involved, then $k_{111}(I_1, I_2, I_3) \Rightarrow k_3(I_1)$. The reduction of the cumulants for the integrated intensities follows in a similar manner.

B. Twofold Intensity Cumulant

In Sec. II A we presented some of the pertinent relationships between the cumulants of the counts and the cumulants of the integrated intensity. Since

$$W_i = \alpha_i \int_{t_i}^{t_i+T_i} I_i(t)\,dt,$$

we first evaluate the cumulants of the intensity, and then perform the appropriate time integration to obtain the cumulants for the integrated intensity.

From Eqs. (1), (2), and (6) we have

$$I_i(t) = \{I_i^T(t) + I_i^c\} I_{si}(t) + 2I_{si}(t)\,\mathrm{Re}[V_i(t)U_i^*(t)], \tag{12}$$

where $I_i^T(t) = V_i(t)V_i^*(t)$ represents the intensity contribution from the chaotic component and $I_i^c = U_i(t)\,U_i^*(t)$ represents the coherent component. The twofold cumulant of $I_i(t)$ and $I_j(s)$ is given by

$$k_{11}(I_i, I_j) = \langle I_i(t)I_j(s)\rangle - \langle I_i(t)\rangle\langle I_j(s)\rangle, \tag{13}$$

where the brackets indicate an ensemble average over the joint statistics of the relevant quantities within the brackets. The first quantity on the right-hand side in (13) is, using (12),

$$\langle I_i(t)I_j(s)\rangle = \{\langle I_i^T(t)I_j^T(s)\rangle + I_i^c I_j^c + I_j^c\langle I_i^T(t)\rangle + I_i^c\langle I_j^T(s)\rangle\} C_{ij}(t,s) + 2C_{ij}(t,s)\,\mathrm{Re}\{\Gamma_{ij}(t,s)\,U_i^*(t)U_j(s)\}, \tag{14}$$

where the fading intensity correlation $C_{ij}(t,s) \equiv \langle I_{si}(t) \times I_{sj}(s)\rangle$, and $\Gamma_{ij}(t,s) = \langle V_i(t)\,V_j^*(s)\rangle$ is the usual mutual coherence function.

From the Gaussian moment theorem,[15]

$$\begin{aligned}\langle I_i^T(t)I_j^T(s)\rangle &= \langle I_i^T(t)\rangle\langle I_j^T(s)\rangle + \Gamma_{ij}(t,s)\Gamma_{ji}(s,t)\\ &= \Gamma_{ii}(t,t)\Gamma_{jj}(s,s) + |\Gamma_{ij}(t,s)|^2,\end{aligned} \tag{15}$$

and from the assumed properties of lossless fading $\langle I_{si}(t)\rangle = \langle I_{sj}(s)\rangle = 1$, combining (13)–(15) we get

$$\begin{aligned}k_{11}(I_i, I_j) &= |\Gamma_{ij}(t,s)|^2 C_{ij}(t,s) + 2C_{ij}(t,s)\\ &\times \mathrm{Re}[\Gamma_{ij}(t,s)U_i^*(t)U_j(s)] + [C_{ij}(t,s) - 1]\\ &\times \{[I_i^c + \Gamma_{ii}(t,t)][I_j^c + \Gamma_{jj}(s,s)]\}.\end{aligned} \tag{16}$$

The joint cumulant of the integrated intensities, W_i and W_j, is then

$$k_{11}(W_i, W_j) = \alpha_i\alpha_j \int_{t_i}^{t_i+T_i}\int_{t_j}^{t_j+T_j} k_{11}(I_i(t), I_j(s))\,dt\,ds. \tag{17}$$

In the absence of fading, this expression reduces to the twofold cumulant for Gaussian plus coherent light given by Cantrell in Ref. 15b.

Expressions (16) and (17) are valid for arbitrary values of T_i/τ_c and T_i/τ_a, where τ_c and τ_a are the coherence times of the source and the fading, respectively, and reflect the space-time coherence of both the source statistics and the atmospheric turbulence. For a single detector, Eqs. (16) and (17) are still valid with the subscripts dropped on all variables, and reduce to $k_2(W)$. The cumulant then represents the time correlations of the combined source and fading statistics.

If, as in most cases of interest $T_i \ll \tau_a$, then the correlation function for the fading is constant over the detection time interval of interest and (17) becomes

$$\begin{aligned}k_{11}(W_i, W_j) &= \alpha_i\alpha_j C_{ij}\int_{t_i}^{t_i+T_i}\int_{t_j}^{t_j+T_j}\{|\Gamma_{ij}(t,s)|^2\\ &+ 2\,\mathrm{Re}[\Gamma_{ij}(t,s)U_i^*(t)U_j(s)]\}\,dt\,ds\\ &+ \alpha_i\alpha_j(C_{ij}-1)\int_{t_i}^{t_i+T_i}\int_{t_j}^{t_j+T_j}\{[I_i^c + \Gamma_{ii}(t,t)]\\ &\times [I_j^c + \Gamma_{jj}(s,s)]\}\,dt\,ds.\end{aligned} \tag{18a}$$

In (18a), the correlation function of the fading intensity is given by

$$C_{ij} \equiv \exp[4C_l(\rho)] = \exp[C_{\ln I}(\rho)], \tag{18b}$$

where $C_l(\rho)$ is the log-amplitude covariance function and $C_{\ln I}(\rho)$ is the log-intensity covariance function.[7] For radiation that is cross-spectrally pure, the two-point mutual coherence function can be written as a product of two separate coherence functions, one representing spatial coherence and the other time coherence.[16] Thus, with

$$\Gamma_{ij}(t,s) = \Gamma_{ij}(0)\gamma_{ii}(t-s), \tag{19}$$

where

$$\gamma_{ii}(t-s) \equiv \frac{\langle V_i(t)V_i^*(s)\rangle}{\langle I_i^T\rangle} \equiv \gamma(t-s), \tag{20}$$

and with

$$U_i(t) = (I_i^c)^{1/2}\exp[-j(\omega_c t - \psi)], \quad \text{and} \quad T_i = T_j,$$

the twofold cumulant is then given by

$$\begin{aligned}k_{11}(W_i, W_j) &= C_{ij}\langle n_i^T\rangle\langle n_j^T\rangle\,|\gamma_{ij}(0)|^2 B_2\\ &+ 2C_{ij}[\langle n_i^T\rangle\langle n_j^T\rangle\langle n_i^c\rangle\langle n_j^c\rangle]^{1/2}\,\mathrm{Re}[\gamma_{ij}(0)B_2']\\ &+ (C_{ij}-1)[(\langle n_i^T\rangle + \langle n_i^c\rangle)(\langle n_j^T\rangle + \langle n_j^c\rangle)].\end{aligned} \tag{21}$$

Here, $\gamma_{ij}(0) = \Gamma_{ij}(0)/(\langle I_i^T\rangle\langle I_j^T\rangle)^{1/2}$. The mean count at each detector due to the coherent and chaotic parts, is, respectively, $\langle n_i^c\rangle = \alpha_i I_i^c T$ and $\langle n_i^T\rangle = \alpha_i\langle I_i^T\rangle T$, and B_2 and B_2' are given by[17]

$$B_m = \frac{1}{T^m}\int_0^T dt_1 \cdots \int_0^T dt_m\,\gamma(t_1 - t_2)\gamma(t_2 - t_3)\cdots\gamma(t_m - t_1), \tag{22}$$

$$B_m' = \frac{1}{T^m}\int_0^T dt_1 \cdots \int_0^T dt_m \exp[j\omega_c(t_1 - t_m)] \times \gamma(t_1 - t_2)\gamma(t_2 - t_3)\cdots\gamma(t_{m-1} - t_m), \tag{23}$$

with $m = 2$.

For chaotic radiation of Lorentzian spectrum of half-width $\Lambda = 1/2\tau_c$ and mean frequency ω_T, B_2 and B_2' have been evaluated by others,[17] and are repeated here for convenience:

$$\begin{aligned}B_2 &= (e^{-2\beta} + 2\beta - 1)/2\beta^2, \quad \beta = \Lambda T,\\ B_2' &= 2(\beta^2 + \Omega^2)^{-2}\{e^{-\beta}[(\beta^2 - \Omega^2)\cos\Omega - 2\beta\Omega\sin\Omega]\\ &- (\beta^2 - \Omega^2) + \beta(\beta^2 + \Omega^2)\}, \quad \Omega = (\omega_c - \omega_T)T.\end{aligned} \tag{24}$$

The twofold cumulant of the integrated intensity can be written in terms of the twofold cumulant of the source radiation in the absence of turbulence and a term containing the twofold cumulant of the turbulence in the absence of source fluctuations. Letting $\langle n_i\rangle=\langle n_i^T\rangle+\langle n_i^c\rangle$, then

$$k_{11}(W_i, W_j)=C_{ij}k_{11}(W_i, W_j)_0+k_{11}(I_{si}, I_{sj})\ \langle n_i\rangle\langle n_j\rangle, \tag{25}$$

where $k_{11}(W_i, W_j)_0$ is the twofold cumulant for Gaussian plus coherent light in the absence of turbulence, and $k_{11}(I_{si}, I_{sj})$ is the twofold cumulant of the fading intensity alone. For a single detector, $i=j$, and (25) reduces to the second-order cumulant for the integrated intensity:

$$k_2(W_i)=C_{ii}k_2(W_i)_0+(C_{ii}-1)\langle n_i\rangle^2, \tag{26}$$

where $k_2(W_i)_0$ is the second-order cumulant of the source radiation in the absence of turbulence. Thus, for a single detector with $T\ll\tau_a$, the second-order cumulant of the integrated intensity reflects time correlations of only the source radiation.

C. Threefold Intensity Cumulant

We now obtain the threefold cumulant for Gaussian plus coherent light in the presence of fading. $k_{111}(I_1, I_2, I_3)$ is given by

$$k_{111}(I_1, I_2, I_3)=\langle I_1I_2I_3\rangle-\sum_{\substack{i=1\\ i\neq j\neq k}}^{3}\langle I_i\rangle\langle I_jI_k\rangle+2\prod_{i=1}^{3}\langle I_i\rangle, \tag{27}$$

where I_i is given by (12). All the terms in (27) except the first have been evaluated above, and thus we need only obtain the first quantity, that is, $\langle I_1I_2I_3\rangle$. Since the complex Gaussian component has zero mean, only terms containing products of an even number of $V_iV_j^*$ will not vanish in the ensemble average. Therefore we can write

$$\frac{\langle I_1I_2I_3\rangle}{\langle I_{s1}I_{s2}I_{s3}\rangle}=\left\langle\prod_{i=1}^{3}(I_i^T+I_i^c)\right\rangle + \sum_{\substack{i=1\\ i\neq j\neq k}}^{3}\langle(I_i^T+I_i^c)(U_j^*U_kV_jV_k^*+U_jU_k^*V_j^*V_k)\rangle. \tag{28}$$

The first term on the right-hand side (rhs) of (28) reduces to

$$\left\langle\prod_{i=1}^{3}(I_i^T+I_i^c)\right\rangle=\langle I_1^TI_2^TI_3^T\rangle+\prod_{i=1}^{3}I_i^c+\sum_{\substack{i=1\\ i\neq j\neq k}}^{3}I_i^c\langle I_j^TI_k^T\rangle+\sum_{\substack{k=1\\ i\neq j\neq k}}^{3}I_i^cI_j^c\langle I_k^T\rangle \tag{29}$$

which, with the help of the Gaussian moment theorem, becomes

$$\left\langle\prod_{i=1}^{3}(I_i^T+I_i^c)\right\rangle=\sum_{P_i}\prod_{i=1}^{3}\Gamma_{i,P_i}+\prod_{i=1}^{3}I_i^c+\sum_{\substack{i=1\\ i\neq j\neq k}}^{3}I_i^c(\Gamma_{jj}\Gamma_{kk}+\Gamma_{jk}\Gamma_{kj}) + \sum_{\substack{k=1\\ i\neq j\neq k}}^{3}I_i^cI_j^c\Gamma_{kk}, \tag{30}$$

where $\sum_{P_i}$ is the sum over all permutations of the integers $\{P_i\}\in\{1, 2, 3\}$. The second term on rhs of (28) can be reduced using the following identities:

$$\begin{aligned}\langle(I_i^T+I_i^c)2\,\mathrm{Re}(U_j^*U_kV_jV_k^*)\rangle&=2\,\mathrm{Re}\,U_j^*U_k(\langle I_i^TV_jV_k^*\rangle+I_i^c\langle V_jV_k^*\rangle)\\ &=2\,\mathrm{Re}\{U_j^*U_k[(\Gamma_{ii}\Gamma_{jk}+\Gamma_{ik}\Gamma_{ji})+I_i^c\Gamma_{jk}]\}.\end{aligned} \tag{31}$$

Then we have, for the second term on rhs of (28),

$$\sum_{\substack{i=1\\ i\neq j\neq k}}^{3}2\,\mathrm{Re}\{(U_j^*U_k[\Gamma_{ii}\Gamma_{jk}+\Gamma_{ik}\Gamma_{ji})+I_i^c\Gamma_{jk}]\}, \tag{32}$$

and with $C_{123}\equiv\langle I_{s1}I_{s2}I_{s3}\rangle$, we obtain

$$\begin{aligned}\langle I_1I_2I_3\rangle=C_{123}\Bigg(&\sum_{P_i}\prod_{i=1}^{3}\Gamma_{i,P_i}+\prod_{i=1}^{3}I_i^c+\sum_{\substack{i=1\\ i\neq j\neq k}}^{3}\\ &\times I_i^c[\Gamma_{jj}\Gamma_{kk}+\Gamma_{jk}\Gamma_{kj}]+\sum_{\substack{k=1\\ i\neq j\neq k}}^{3}I_i^cI_j^c\Gamma_{kk}\\ &+\sum_{\substack{i=1\\ i\neq j\neq k}}^{3}2\,\mathrm{Re}\{U_j^*U_k[(\Gamma_{ii}\Gamma_{jk}+\Gamma_{ik}\Gamma_{ji})+I_i^c\Gamma_{jk}]\}\Bigg).\end{aligned} \tag{33}$$

The second term in (27) is, from (14),

$$\begin{aligned}\sum_{\substack{i=1\\ i\neq j\neq k}}^{3}\langle I_i\rangle\langle I_jI_k\rangle=\sum_{\substack{i=1\\ i\neq j\neq k}}^{3}&C_{jk}(\Gamma_{ii}+I_i^c)\{[\Gamma_{jj}\Gamma_{kk}+\Gamma_{jk}\Gamma_{kj}]+I_j^cI_k^c\\ &+I_j^c\Gamma_{kk}+I_k^c\Gamma_{jj}+2\,\mathrm{Re}(U_j^*U_k\Gamma_{jk})\},\end{aligned} \tag{34}$$

and the last term in (27) is just

$$2\prod_{i=1}^{3}(\langle I_i^T\rangle+I_i^c)=2\prod_{i=1}^{3}(\Gamma_{ii}+I_i^c). \tag{35}$$

With the following identities:

$k_{111}(I_1^T, I_2^T, I_3^T)=2\,\mathrm{Re}(\Gamma_{12}\Gamma_{23}\Gamma_{31})$ and $k_{11}(I_1^T, I_2^T)=\Gamma_{12}\Gamma_{21}$,

we can write the threefold cumulant in the following form:

$$\begin{aligned}k_{111}(I_1, I_2, I_3)=&\,C_{123}k_{111}(I_1^T, I_2^T, I_3^T)\\ &+C_{123}\sum_{\substack{i=1\\ i\neq j\neq k}}^{3}2\,\mathrm{Re}[U_j^*U_k(\Gamma_{ii}\Gamma_{jk}+\Gamma_{ik}\Gamma_{ji})]\\ &+\sum_{\substack{i=1\\ i\neq j\neq k}}^{3}(C_{123}-C_{jk})[(\langle I_i^T\rangle+I_i^c)k_{11}(I_j^T, I_k^T)\\ &+2\,\mathrm{Re}(U_j^*U_kI_i^c\Gamma_{jk})]\\ &-\sum_{\substack{i=1\\ i\neq j\neq k}}^{3}2C_{jk}\,\mathrm{Re}(U_j^*U_k\langle I_i^T\rangle\Gamma_{jk})\\ &+\Bigg(C_{123}+2-\sum_{\substack{1\\ j\neq k}}^{3}C_{jk}\Bigg)\prod_{i=1}^{3}(\langle I_i^T\rangle+I_i^c).\end{aligned} \tag{36}$$

In the last term in (36), the sum $\sum_{1,j\neq k}^{3}$ means the sum of C_{jk} over all combinations of j,k, from the integers (1, 2, 3). In the absence of fading, (36) reduces to (13) in Cantrell (Ref. 15b). When the coherent component is zero, this reduces in complexity to a more simple form, given by

$$\begin{aligned}k_{111}(I_1, I_2, I_3)=&\,C_{123}k_{111}(I_1^T, I_2^T, I_3^T)\\ &+\sum_{\substack{i=1\\ i\neq j\neq k}}^{3}(C_{123}-C_{jk})\langle I_i^T\rangle k_{11}(I_j^T, I_k^T)\\ &+\Bigg(C_{123}+2-\sum_{\substack{1\\ j\neq k}}^{3}C_{jk}\Bigg)\prod_{i=1}^{3}\langle I_i^T\rangle.\end{aligned} \tag{37}$$

Recognizing that the coefficient of the last term,

$$C_{123}-\sum_{\substack{1\\ j\neq k}}^{3}C_{jk}+2\equiv\langle I_{s1}I_{s2}I_{s3}\rangle-\sum_{\substack{i=1\\ i\neq j\neq k}}^{3}\langle I_{sj}I_{sk}\rangle\langle I_{si}\rangle+2,$$

is merely the threefold cumulant of the fading, $k_{111}(I_{\varepsilon 1}, I_{\varepsilon 2}, I_{\varepsilon 3})$, then

$$k_{111}(I_1, I_2, I_3) = C_{123} k_{111}(I_1^T, I_2^T, I_3^T) + \sum_{\substack{i=1 \\ i \neq j \neq k}}^{3} (C_{123} - C_{jk}) \langle I_i^T \rangle k_{11}(I_j^T, I_k^T) + k_{111}(I_{\varepsilon 1}, I_{\varepsilon 2}, I_{\varepsilon 3}) \prod_{i=1}^{3} \langle I_i^T \rangle, \qquad (38)$$

again for chaotic radiation alone. The threefold cumulant of the integrated intensity is given by

$$k_{111}(W_1, W_2, W_3) = \alpha_1 \alpha_2 \alpha_3 \times \int_{t_1}^{t_1+T_1} \int_{t_2}^{t_2+T_2} \int_{t_3}^{t_3+T_3} k_{111}(I_1, I_2, I_3)\, dt\, ds\, dp. \qquad (39)$$

If we consider the case where $\{T_i\} \ll \tau_a$, then, as in the twofold case, C_{123} can be removed from the integral and, for the usually assumed homogeneous fading, is given by

$$C_{123} = \exp(\vec{A}^\dagger \vec{M} + \tfrac{1}{2} \vec{A}^\dagger \overleftrightarrow{\Lambda} \vec{A}) = \exp\left(\sum_{\substack{1 \\ j \neq k}}^{3} \lambda_{jk}\right) \qquad (40)$$

where

$$\vec{A}^\dagger = [1, 1, 1], \quad \lambda_{ij} = C_{lnI}(\rho_{ij}), \quad \rho_{ij} = |\vec{r}_i - \vec{r}_j|,$$

and $\vec{M}^\dagger = [-\frac{1}{2}\sigma^2, -\frac{1}{2}\sigma^2, -\frac{1}{2}\sigma^2]$. The quantity C_{12} is then the twofold equivalent of (40) above. For a single detector, and with a Lorentzian spectrum for the chaotic component, setting $i=j=k$ in (36) we obtain

$$\begin{aligned} k_3(W) = {} & 2C_3 \langle n^T \rangle^3 B_3 + 6C_3[\langle n^T \rangle^2 \langle n^c \rangle B_2' + \langle n^T \rangle^2 \langle n^c \rangle B_3'] \\ & + 3(C_3 - C_2)[(\langle n^T \rangle + \langle n^c \rangle) \langle n^T \rangle^2 B_2 + 2\langle n^c \rangle^2 \langle n^T \rangle B_2'] \\ & - 6C_2 \langle n^T \rangle^2 \langle n^c \rangle B_2' + (C_3 - 3C_2 + 2)(\langle n^T \rangle + \langle n^c \rangle)^3, \end{aligned} \qquad (41)$$

where $C_N = \exp[\frac{1}{2}\sigma^2 N(N-1)]$.

III. PHOTOCOUNTING CUMULANTS

Using the preceding results for the cumulants of the integrated intensity, and the relationships of Sec. II between the cumulants of the counts and the cumulants of the integrated intensity, we obtain

$$k_{11}(n_1, n_2) = k_{11}(W_1, W_2), \qquad (42)$$

as derived in (18);

$$k_{111}(n_1, n_2, n_3) = k_{111}(W_1, W_2, W_3)$$

as derived in (39); and

$$\begin{aligned} k_2(n) &= C_2 k_2(W)_0 + (C_2 - 1)\langle n \rangle^2 + \langle n \rangle, \\ k_3(n) &= k_3(W) + 3k_2(W) + k_1(W), \end{aligned} \qquad (43)$$

where $k_3(W)$ is given by (41).

We now evaluate (42) and (43) for a coherent component, along with a chaotic component of Lorentzian spectrum, with $\omega_c = \omega_T$. $k_2(n)$ is then given by

$$k_2(n) = e^{\sigma^2} \langle n^T \rangle^2 B_2 + 2e^{\sigma^2} \langle n^T \rangle \langle n^c \rangle B_2' + (e^{\sigma^2} - 1)(\langle n^T \rangle + \langle n^c \rangle)^2 + \langle n^T \rangle + \langle n^c \rangle. \qquad (44)$$

This expression is seen to contain terms proportional to the Gaussian component, a mixing term, a term representing an effective coherent component of mean value $\langle n^T \rangle + \langle n^c \rangle$, and the ordinary term that would appear for a shot noise process alone, in the absence of all other statistical fluctuations. Each of these terms, then, can be viewed as an "excess correlation" due to each of the contributing statistical fluctuations in the detected field. Figure 1 exhibits the variation of $k_2(n)$ for a chaotic source alone as a function of the level of turbulence σ, for several values of $\beta = T/2\tau_c$, where T is the counting time and τ_c is the source coherence time. As the value of β ranges from $\beta \ll 1$ to $\beta \gg 1$, $k_2(n)$ goes from that of a source with Bose-Einstein statistics to that of a stable source with Poisson statistics, in the absence of turbulence. As the turbulence increases, the cumulant increases exponentially from its quiescent value. If we consider the ratio $k_2(n; \beta = 0.01)/k_2(n; \beta = \infty)$ as a function of σ, we note that it decreases from 5 to 2 as σ increases from 0 to 1.4, thus indicating that the fading rapidly dominates the excess fluctuations regardless of the source statistics, as pointed out in Refs. 1 and 2. Analytically this ratio is, from (44), where $B_2 = 1$ for $\beta \to 0$ and $B_2 = 0$ for $\beta \to \infty$,

$$\frac{k_2(n;\beta=0)}{k_2(n;\beta=\infty)} = \frac{e^{\sigma^2}\langle n \rangle^2 + (e^{\sigma^2}-1)\langle n \rangle^2 + \langle n \rangle}{(e^{\sigma^2}-1)\langle n \rangle^2 + \langle n \rangle} = 1 + \frac{e^{\sigma^2}\langle n \rangle^2}{(e^{\sigma^2}-1)\langle n \rangle^2 + \langle n \rangle}. \qquad (45)$$

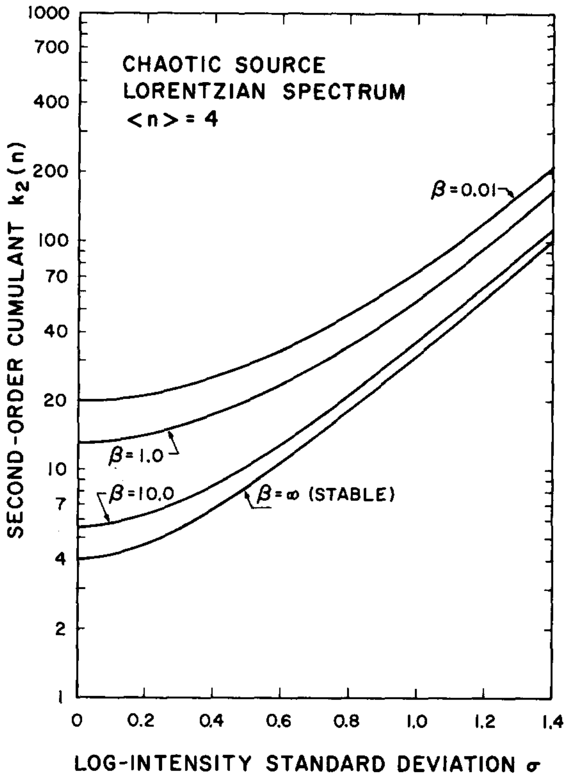

FIG. 1. Second-order photocounting cumulant for a single detector $k_2(n)$ as a function of the degree of turbulence σ, for a chaotic source with Lorentzian spectrum. The ratio of counting time to source coherence time is indicated by $\beta = T/2\tau_c$, and the mean photoelectron count is $\langle n \rangle = 4$. The lowest curve, $\beta = \infty$, corresponds to an amplitude-stabilized source as well.

For values of σ near saturation,[7] $(e^{\sigma^2}-1)\langle n\rangle^2 \gg \langle n\rangle$, and the ratio in (45) can be approximated as $1+[e^{\sigma^2}/(e^{\sigma^2}-1)]$. At the saturation value of $\sigma \simeq 1.5$, this quantity approaches a limiting value of 2.1. Thus for severe turbulence, and for $\langle n\rangle > 2$, the ratio of the cumulants converges to a nominal value of 2, independent of the mean count, whereas the same ratio can be shown to increase linearly with $\langle n\rangle$ in the absence of fading. Thus, the over-all fluctuation of the counts becomes dominated by the effects of the turbulence.

Curves for a chaotic plus coherent source lie below those for the chaotic source, for corresponding values of β. They possess the same general variation displayed in Fig. 1. Recalling that $k_2(n)$ is just the variance of the single-detector counts, the curves exhibit a drastic broadening of the counting distribution for $\sigma > 0$. This has already been graphically presented both for the single-detector and multidetector cases.[1-3]

In that $k_2(n)$ contains information from a single-point detector, it cannot contain any information about the spatial correlations of the field. The quantities that do contain spatial information are the joint statistics as expressed by the higher-fold cumulants; we examine specifically the twofold case. In order to exhibit the spatial variation due to the turbulence, we assume that over the region of interest $|\gamma_{12}(0)| \simeq 1$. That is, the field is taken to be spatially coherent with respect to the source statistics. For the spatial covariance of the log-intensity, we choose for simplicity the expression due to Tatarski[6] for the plane-wave case. Over the region of validity, $l_0 \ll \rho \ll (\lambda L)^{1/2}$, this covariance function is given by[7]

$$\frac{C_{\ln I}(\rho)}{C_{\ln I}(0)} = (1 - 2.36R^{5/6} + 1.71R - 0.024R^2 + \cdots), \qquad (46)$$

where $R = 2\pi\rho^2/\lambda L$. Here, λ is the optical wavelength, L is the pathlength through the turbulence, ρ is the detector spacing, and l_0 is the inner scale of the turbulence. The expression for the twofold cumulant of the counts is, from (42), (21), and (18b),

$$k_{11}(n_1, n_2) = e^{C_{\ln I}(\rho)}\langle n^T\rangle^2 B_2 + 2e^{C_{\ln I}(\rho)}\langle n^T\rangle\langle n^c\rangle B_2' + (e^{C_{\ln I}(\rho)} - 1)\langle n\rangle^2, \qquad (47)$$

where we have set $\langle n_1^T\rangle = \langle n_2^T\rangle$, $\langle n_1^c\rangle = \langle n_2^c\rangle$, and $\langle n\rangle = \langle n^T\rangle + \langle n^c\rangle$.

In Fig. 2 the twofold cumulant given by (47) is exhibited as it varies with normalized detector spacing $\rho(2\pi/\lambda L)^{1/2}$, and as a function of σ, β, and $y \equiv \langle n^c\rangle/\langle n^T\rangle$. When the turbulence is weak, as indicated by the curves for $\sigma = 0.1$, the effect of detector spacing is seen to be minimal. This is to be expected because of our assumption that the source radiation alone is fully correlated spatially. The values which $k_{11}(n_1, n_2)$ takes on as β and y vary correspond to those they would have in the absence of turbulence. As the turbulence increases to $\sigma = 1.0$, the effect of the spatial covariance of the fading becomes prominent. For a given value of β and a given ratio of coherent-to-chaotic component y, the curve decreases to and crosses below its turbulence-free value and then returns to its turbulence-free value for larger values of ρ.[18] This latter effect is not shown because of the limited range of validity of the expression for $C_{\ln I}(\rho)$ given in (46), which was used because of its simplicity. To exhibit the variation of $k_{11}(n_1, n_2)$ for a wider range of ρ would require the exact evaluation of $C_{\ln I}(\rho)$, which is given by an integral that must be evaluated numerically, for every value of ρ.[18]

For various values of the coherent-to-chaotic ratio, with all other parameters constant, $k_{11}(n_1, n_2)$ exhibits similar variation with y as does $k_2(n)$. This statement also applies as β varies, the curves for $y = \infty$ and $\beta = \infty$ being identical.

Finally, we present the threefold cumulant $k_3(n)$ vs σ in Fig. 3 for a chaotic source alone. In analogy with

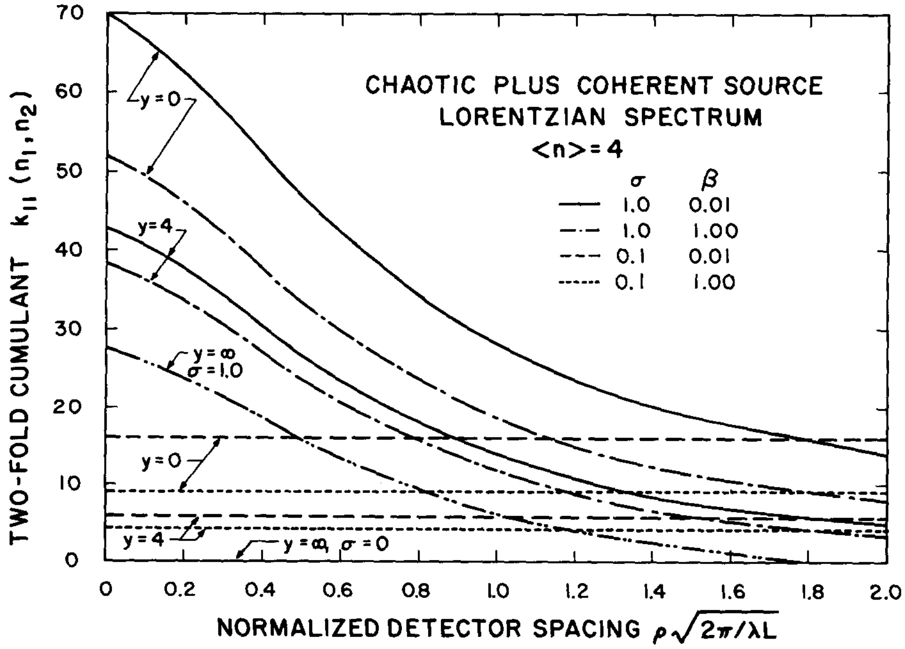

FIG. 2. Twofold photocounting cumulant $k_{11}(n_1, n_2)$ for two detectors, as a function of normalized detector spacing $\rho(2\pi/\lambda L)^{1/2}$ for various values of σ, β, and y. The total mean count at each detector, $\langle n\rangle = \langle n^T\rangle + \langle n^c\rangle$, is equal to 4. As in Fig. 1, the chaotic component has a Lorentzian spectrum, $\omega_c = \omega_T$, and $|\gamma_{12}(0)| \simeq 1$ over the region of interest.

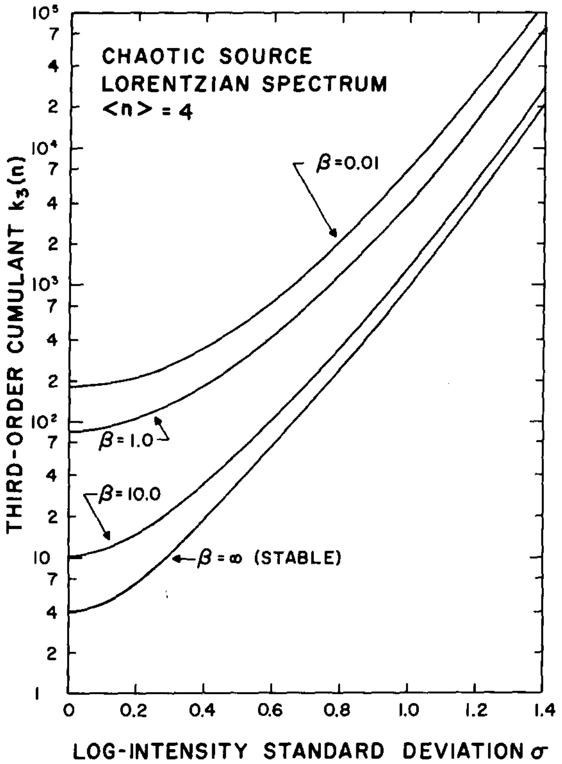

FIG. 3. Third-order photocounting cumulant for a single detector, $k_3(n)$, as a function of the degree of turbulence σ, for a chaotic source with Lorentzian spectrum. The ratio of counting time to source coherence time is indicated by $\beta = T/2\tau_c$, and the mean photoelectron count is $\langle n\rangle = 4$. The lowest curve, $\beta=\infty$, corresponds to an amplitude-stabilized source as well.

$k_2(n)$ as presented in Fig. 1, it has been assumed that $\langle n\rangle = 4$ and that the spectrum is Lorentzian. It is seen that the behavior of $k_3(n)$ is similar to that of $k_2(n)$ for the various values of β. Again, the curves converge as σ increases, demonstrating that the strong log-normal fluctuations arising from the atmosphere overpower the less violent source fluctuations. Since the curves represent nonnormalized cumulants, they are larger in magnitude for $k_3(n)$ than for $k_2(n)$. The finite value of β for the uppermost curve ($\beta = 0.01$) makes it just barely lower than the curve for $\beta \equiv 0$ would be; the two curves cannot be distinguished within the resolution of the figure.

IV. SUMMARY

We have presented here exact expressions for the first three cumulants of the photoelectron counts to be expected for arbitrary radiation that has suffered log-normal fading, such as induced by the turbulent atmosphere. In particular, we evaluated the cumulants for a radiation source consisting of a coherent component mixed with a chaotic component, such as is representative of a noisy single-mode laser above threshold, or of some scattered signals. The expressions derived are exact regardless of the ratio T/τ_c and the chaotic-component spectral distribution. The mean frequencies of the two need not coincide.

For light of Lorentzian spectrum, and for $T \ll \tau_a$, the expressions were evaluated explicitly and then presented graphically for $k_2(n)$, $k_{11}(n_1, n_2)$, and $k_3(n)$. The effect of increasing the turbulence level, as measured by σ, was shown to increase the magnitude of the cumulants exponentially up to the saturation value of σ, where the ratio $k_2(n,$ chaotic$)/k_2(n,$ coherent$) \to 2$. The effect of severe turbulence makes the photoelectron counting distribution relatively insensitive to the source radiation statistics. Furthermore, for moderate to severe turbulence, $k_{11}(n_1, n_2)$ shows the effect of spatial correlations induced by the turbulence, as the detector separation is varied. The behavior of $k_3(n)$ was found to be similar to that of $k_2(n)$. It may be concluded that increasing y and β, and decreasing σ, results in a lowering of the magnitude of all cumulants.

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†Present address: Bell Laboratories, Whippany, N. J. 07981.
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