CONSTRUCTION B.4: $t = 2, k = 7, r = 13$

weights in $\mathcal{M}$	w, 2 bits appended	no. of subcodes	11 bits constant weight code of distance 4	no. of tails
0	0	1	11	1
1				
2	3	28	7	35
3				
4				
5	6	28	4	35
6				
7	9	1	0	1

to make their weights become $0, 3, 6$, and 9. We use the following lemma by Baranyai [3].

Lemma 6 (Baranyai): Suppose $n = wi$ and $w < i$, then Ω_w^n can be partitioned into $\binom{n}{w}/i$ subsets of "*disjoint*" words. That means if $\boldsymbol{a}$ and $\boldsymbol{b}$ are in the same subset then $\boldsymbol{a} * \boldsymbol{b} = \boldsymbol{0}$, or equivalently, $M(\boldsymbol{a}, \boldsymbol{b}) = (w, w)$.

- 3 divides 9, so Ω_3^9 can be partitioned into $\binom{9}{3}/3 = 28$ subsets with $M(\boldsymbol{a}, \boldsymbol{b}) = (3, 3)$. Each of the subset is a 2EC/AUED subcode. Similarly, Ω_6^9 can be partitioned into 28 2EC/AUED subcodes since they are the inverses of sequences of weight 3.
- For the tail part, we use the constant weight codes with minimum distance 4. For the existence of these constant weight codes see [8].
- This code has the same parameteras the code designed in Example 2. However, it lacks efficient encoding/decoding procedure.

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REFERENCES

[1] C.M. Melas, "A cyclic code for double error correction," *IBM J. Res. Devel.*, vol. 4, pp. 364–366, 1960.
[2] T. M. Cover, "Enumerative Source Coding," *IEEE Trans. Inform. Theory*, vol. IT-19, pp. 73-77, Jan. 1973.
[3] Z. Baranyai, "On the factorization of the complete uniform hypergraph," in *Infinite and Finite Sets*, A. Hajnal, R. Rado, and Vera T. Sós, Eds. Amsterdam, The Netherlands, and London: North-Holland, 1975.
[4] S.D. Constantin and T.R.N. Rao, "Group theoretic codes for binary asymmetric channels," *Inform. Contr.*, vol. 40, pp. 20–26, Jan. 1979.
[5] D.K. Pradhan, "A new class of error-correcting/detecting codes for fault-tolerant computer applications," *IEEE Trans. Comput.*, vol. C-29, pp. 471–481, June 1980.
[6] B. Bose and T. Rao, "Theory of unidirectional error correcting/detecting codes," *IEEE Trans. Comput.*, vol. C-31, pp. 521–530, June 1982.
[7] M. Blaum and H. Van Tilborg, "On t-error correcting/all unidirectional error detecting codes," *IEEE Trans. Comput.*, vol. 38, pp. 1493–1501, Nov. 1989.
[8] A. E. Brouwer, J. B. Shearer, N. J. A. Sloane, and W. D. Smith, "A new table of constant weight codes," *IEEE Trans. Inform. Theory*, vol. 36, pp. 1334–1380, Nov. 1990.
[9] F. Böinck and H. Van Tilborg, "Constructions and bounds for systematic tEC/AUED codes," *IEEE Trans. Inform. Theory*, vol. 36, pp. 1381–1390, Nov. 1990.
[10] D. Nikolos, "Theory and design of t-error correcting/d-error detecting ($d > t$) and all unidirectional error detecting codes", *IEEE Trans. Comput.*, vol. 40, pp. 132–142, Feb. 1991.
[11] J. Bruck and M. Blaum, "New techniques for constructing EC/AUED codes," *IEEE Trans. Comput.*, vol. 41, pp. 1318–1324, Oct. 1992.
[12] Z. Zhang and X.G. Xia, "LYM-type inequalities for tEC/AUED codes," *IEEE Trans. Inform. Theory*, vol. 39, Jan. 1993.

Fractal Renewal Processes

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***Abstract*—Two relatively simple renewal processes whose power spectral densities vary as $1/f^D$ are constructed: 1) a standard renewal point process, with $0 < D < 1$; and 2) a finite-valued alternating renewal process, with $0 < D < 2$. The resulting event number statistics, coincidence rates, minimal coverings, and autocorrelation functions are shown also to follow power-law forms. These fractal characteristics derive from interevent-time probability density functions which themselves decay in a power-law fashion.**

***Index Terms*—Fractal process, renewal process, $1/f$ noise.**

Noise with a power spectral density that varies as an inverse power of frequency is called $1/f^D$ noise. Mathematical models generating continuous-time $1/f^D$ noise include fractal shot noise [1]–[3], suitably filtered white Gaussian noise [4], fractionally integrated white noise [5], fractal Brownian motion [6], [7], and a superposition of relaxation processes with an appropriate distribution of time constants [8], [10]. Mandelbrot modeled burst noise in communication systems with a form of fractal renewal process [11]. The fractal-shot-noise-driven doubly stochastic Poisson point process is another discrete (point) process which yields $1/f^D$ noise [12].

In this correspondence, we develop two relatively simple fractal renewal point processes (FRP's) that provide plausible models for a number of physical and biological processes. Both generate $1/f^D$ noise, in the ranges $0 < D < 1$ and $0 < D < 2$ respectively. The first is a standard fractal renewal point process (SFRP), with events represented as points distributed on a line; the second is an alternating fractal renewal process (AFRP), where the process switches between two states (see Fig. 1). Both of our processes exhibit power-law scaling in many of their statistics, and are therefore fractal. This fractal behavior derives from interevent times which have power-law-varying probability density functions. We consider the case where the processes have reached equilibrium so that the renewal density, or expected rate of events, is constant in time, and thus the processes are stationary.

Possibly the simplest example is the abrupt-cutoff power-law density

$$p(t) = \frac{D}{A^{-D} - B^{-D}} \times \begin{cases} t^{-(D+1)}, & \text{for } A < t < B, \\ 0, & \text{otherwise.} \end{cases} \qquad (1)$$

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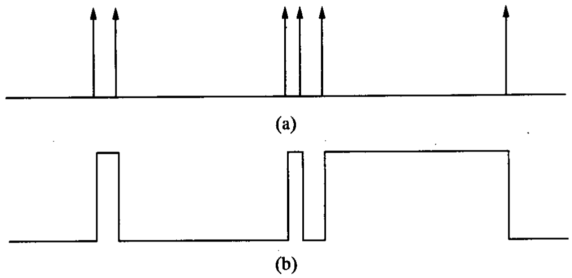

Fig. 1. Sample functions of fractal renewal processes. Interevent times are power-law distributed. (a) The standard fractal renewal process (SFRP) consists of Dirac delta functions and is zero-valued elsewhere. (b) The alternating fractal renewal process (AFRP) switches between values of zero and unity. The symmetric case is shown here.

The characteristic function of the interevent time is

$$\phi(\omega) = \frac{D(-j\omega)^D}{A^{-D} - B^{-D}} \int_{-j\omega A}^{-j\omega B} e^{-x} x^{-(D+1)}\, dx, \tag{2}$$

and $\langle T \rangle$ denotes the average interevent time. The following results apply for power-law densities with cutoffs of arbitrary shape; the abrupt-cutoff form in (1) is most convenient.

These fractal probability densities may be used to construct well-defined FRP's, since the densities are zero for nonpositive arguments.

For the SFRP [labeled $N(t)$] in the medium-frequency limit $B^{-1} \ll \omega \ll A^{-1}$, the power spectral density is given by (3) (see the equation at the bottom of the page) [13]. Thus, for $0 < D < 1$, the power spectral density varies as $1/f^D$ over a substantial range of frequencies $B^{-1} \ll \omega = 2\pi f \ll A^{-1}$, where D corresponds to the power-law exponent in the interevent-time density. However, this power-law exponent never reaches unity, and no new exponents are introduced by considering $D < 0$ or $D > 1$. A different point process will result when several SFRP's are superposed, but the overall power spectral density will still be $1/f^D$ [13].

An approximation for the coincidence rate of the abrupt-cutoff power-law process for $0 < D < 1$ is [13]

$$G_N(\tau) \approx (\pi D)^{-1} B^{D-1} A^{-2D} \sin(\pi D) |\tau|^{D-1} \tag{4}$$

in the range $A \ll |\tau| \ll B$.

For the case $A \ll |\tau| \ll B$ and $0 < D < 1$, but with arbitrary cutoffs in the interevent-time probability density, we have the renewal density $u(t) \sim t^{D-1}$, so that $u^{*k}(t) \sim t^{kD-1}$, and [13]

$$\mathrm{E}\left\{ \frac{[N(t)+k]!}{[N(t)-1]!} \right\} \sim t^{kD+1}. \tag{5}$$

In particular, $\mathrm{E}\{N(t)\} = t/\langle T \rangle$ and

$$\mathrm{Var}\{N(t)\} \approx \left[\pi D^2 (D+1)\right]^{-1} B^{D-1} A^{-2D} \sin(\pi D) t^{D+1}. \tag{6}$$

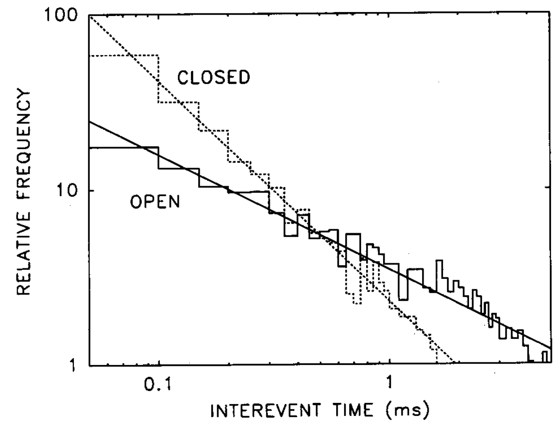

Fig. 2. Double logarithmic plot of open- and closed-time probability density functions $p(t)$ vs. t for a K^+ ion channel at 50 mV bias, replotted from [17]. Note the power-law behavior for both functions.

For the parameter ranges $0 < D < 1$ and $0 < A \ll B < \infty$, the set of points generated by the SFRP is itself fractal with dimension D. Consider a realization of the process, and a minimal covering of it using segments of length L. For the range $A \ll L \ll B$, the expected number of intervals required to cover the SFRP will scale as L^{-D}, and thus the capacity dimension is D [13].

The AFRP [labeled $X(t)$] has two interesting domains: symmetric and extreme asymmetric. In the latter, the times spent in one state are much longer than the times spent in the other state, where the longer times have a fractal distribution. For the regime $D \geq 0$, the power spectral density and the coincidence rate are proportional to the SFRP results except for the high-frequency and short-time limits. In the regime $-1 < D < 0$, however, we obtain

$$S_X(\omega) \to 2\langle T_S \rangle^2 (-D)^{-1} \Gamma(2-D) \cos(\pi D/2) B^{D-1} \omega^D, \tag{7}$$

where $\langle T_S \rangle$ is the average time spent in the short-time state.

$$\langle T \rangle S_N(\omega) \to \begin{cases} 1, & \text{for } -1 \leq D \leq 0, \\ 2\left[\Gamma(1-D)\right]^{-1} \cos(\pi D/2)(\omega A)^{-D}, & \text{for } 0 < D < 1, \\ \pi\left[\ln(\omega A)\right]^{-2} (\omega A)^{-1}, & \text{for } D = 1, \\ 2D^{-2}(D-1)\Gamma(2-D)\left[-\cos(\pi D/2)\right](\omega A)^{D-2}, & \text{for } 1 < D < 2, \\ (1/2)\left[-\ln(\omega A)\right], & \text{for } D = 2, \\ D^{-1}(D-2)^{-1}(D-1)^2, & \text{for } D > 2. \end{cases} \tag{3}$$

$$4\langle T \rangle S_X(\omega) \to \begin{cases} 4\omega^{-2}, & \text{for } -1 \leq D \leq 0 \\ 2\Gamma(1-D)\cos(\pi D/2) A^D \omega^{D-2}, & \text{for } 0 < D < 1 \\ \pi A \omega^{-1}, & \text{for } D = 1 \\ 2(D-1)^{-1}\Gamma(2-D)\left[-\cos(\pi D/2)\right] A^D \omega^{D-2}, & \text{for } 1 < D < 2 \\ 2A^2\left[-\ln(\omega A)\right], & \text{for } D = 2 \\ D(D-2)^{-1} A^2, & \text{for } D > 2. \end{cases} \tag{8}$$

In the symmetric domain the dwell times of the two states are given by identical, abrupt-cutoff power-law distributions, and the power spectral density also varies as $1/f^D$, but with a different form from that of the SFRP. In the medium-frequency limit ($A^{-1} \ll \omega \ll B^{-1}$), we then obtain (8) (see the equation at the top of the page) [13]. The AFRP generates $1/f^D$ noise for the full range $0 < D < 2$ over a substantial range of frequencies $B^{-1} \ll \omega = 2\pi f \ll A^{-1}$. When several AFRP's are superposed, the overall power spectral density will still be $1/f^D$, and the resulting process will approach a Gaussian process in the limit of a large number of AFRP's [13].

For $1 < D < 2$, the autocorrelation function also shows power-law behavior [13]:

$$R_X(\tau) - \mathrm{E}\{X\}^2 \approx (2D)^{-1}\left[-\cos(\pi D/2)\right] A^{D-1}|\tau|^{1-D}, \quad (9)$$

for $A \ll |\tau| \ll B$.

FRP's apply to a wide variety of phenomena [13], including trapping in amorphous semiconductors [14], electronic burst noise, movement in systems with fractal boundaries, the digital generation of $1/f^D$ noise, and ionic currents in cell membranes. We focus briefly on this last application. Ion channels are openings in the membranes of cells which allow ions to diffuse into or out of a cell [15], and which alternate between open and closed states. Some ion channels may be modeled by a two-state Markov process, with one state representing the open channel, and the other representing the closed channel. This model generates exponentially distributed dwell times in both states. However, many ion channels exhibit power-law distributed dwell times [16].

Fig. 2 shows an example of a particular ion channel [17] for which the open- and closed-time probability density functions both follow power-law forms. The authors of [17] fit the data less accurately with a combination of three exponentials. The open times decay with a power-law exponent $D + 1 = 0.66$ and are much longer than the closed times, so that this channel is in the regime of (7). The symmetric AFRP model, in contrast, describes the activity of other ion channels for which the open and closed times are similar and fractal. Whole-cell ion currents exhibit spontaneous fluctuations due to the additive effects of large numbers of ion channels on the cell membrane. If these ion channels are independent of each other, then this model predicts that the whole-cell ion current will be Gaussian-distributed $1/f^D$ noise. Even for dependent ion channels, in fact, evidence exists that the overall effect will be the same, although with a higher variance than for the independent channel case [18]. Indeed, spontaneous voltage fluctuations of neurons often exhibit Gaussian-distributed $1/f^D$ noise [19].

REFERENCES

[1] S. B. Lowen and M. C. Teich, "Generalized $1/f$ shot noise," *Electron. Lett.*, vol. 25, pp. 1072–1074, 1989.

[2] ——, "Fractal shot noise," *Phys. Rev. Lett.*, vol. 63, pp. 1755–1759, 1989.

[3] ——, "Power-law shot noise," *IEEE Trans. Inform. Theory*, vol. 36, pp. 1302–1318, 1990.

[4] H. Takayasu, "$f^{-\beta}$ power spectrum and stable distribution," *J. Phys. Soc. Japan*, vol. 56, pp. 1257–1260, 1987.

[5] J. A. Barnes and D. W. Allan, "A statistical model of flicker noise," *Proc. IEEE*, vol. 54, pp. 176–178, 1966.

[6] B. B. Mandelbrot and I. W. Van Ness, "Fractional Brownian motions, fractional noises, and applications," *Soc. for Ind. and Appl. Math. (SIAM) Review*, vol. 10, pp. 422–437, 1968.

[7] B. B. Mandelbrot, *The Fractal Geometry of Nature*. New York: Freeman, 1983.

[8] M. B. Weissman, "$1/f$ noise and other slow, nonexponential kinetics in condensed matter," *Rev. Mod. Phys.*, vol. 60, pp. 537–571, 1988.

[9] A. van der Ziel, "On the noise spectra of semi-conductor noise and of flicker effect," *Physica*, vol. 16, pp. 359–372, 1950.

[10] A. L. McWhorter, "$1/f$ noise and related surface effects in germanium," in *Semiconductor Surface Physics*, R. H. Kingston, Ed. Philadelphia: University of Pennsylvania, 1956.

[11] B. B. Mandelbrot, "Self-similar error clusters in communication systems and the concept of conditional stationarity," *IEEE Trans. Commun. Technol.*, vol. 13, pp. 71–90, 1965.

[12] S. B. Lowen and M. C. Teich, "Doubly stochastic Poisson point process driven by fractal shot noise," *Phys. Rev. A*, vol. 43, pp. 4192–4215, 1991.

[13] S. B. Lowen, "Fractal renewal prcosses generate $1/f$ noise," *Phys. Rev. E*, vol. 47, pp. 992–1001, 1993; see also, "Fractal Stochastic Processes," Ph. D. dissertation, Columbia Univ., New York, NY, ch. 3, 1992.

[14] S. B. Lowen and M. C. Teich, "Fractal renewal processes as a model of charge transport in amorphous semiconductors," *Phys. Rev. B*, vol. 46, pp. 1816–1819, 1992.

[15] B. Sakmann and E. Neher, *Single-Channel Recording*. New York: Plenum, 1983.

[16] L. S. Liebovitch and T. I. Tóth, "Using fractals to understand the opening and closing of ion channels," *Ann. Biomed. Eng.*, vol. 18, pp. 177–194, 1990.

[17] A. H. Gitter, E. Frömter, and H. P. Zenner, "C-Type potassium channels in the lateral cell membrane of guinea-pig outer hair cells," *Hear. Res.*, vol. 60, pp. 177–194, 1992.

[18] L. S. Liebovitch and J. Fischbarg, "Membrane pores: a computer simulation of interacting pores analyzed by $g_1(\tau)$ and $g_2(\tau)$ correlation functions," *J. Theor. Biol.*, vol. 119, pp. 287–297, 1986.

[19] A. A. Verveen and H. E. Derksen, "Fluctuation phenomena in nerve membrane," *Proc. IEEE*, vol. 56, pp. 906–916, 1968.

Forward Collision Resolution — A Technique for Random Multiple-Access to the Adder Channel

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***Abstract*—Consider M-Choose-T communications: T users or less, out of M potential users, are chosen at random to simultaneously transmit binary data over a common channel. A method for constructing codes that achieve error-free M-Choose-T communication over the noiseless Adder Channel (AC), at a nominal rate of $1/T$ bits per channel symbol per active user, is described and an efficient decoding procedure is presented. The use of such codes is referred to as Forward Collision Resolution (FCR), as it enables correct decoding of collided messages without retransmissions. For any *given* T a code is available that yields a stable throughput arbitrarily close to 1 message/slot. Furthermore, if the occurrence of collisions is made known to the transmitters, such a throughput can be maintained for *arbitrary* $T, T \leq M$ as well. If such feedback is not available, and T is *random*, the probability of an unresolved collision is significantly smaller than the probability of a collision in an uncoded system, at comparable message-arrival and information rates.**

I. INTRODUCTION

The problem of sharing a common channel by several users has been mostly treated within the framework of either of the two categories described by Gallager in his review paper [1] and references cited therein.

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