TWO-DIMENSIONAL FRACTIONAL BROWNIAN MOTION: WAVELET ANALYSIS AND SYNTHESIS

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ABSTRACT

Two-dimensional fractional Brownian motion (2D FBM) is a non-stationary random process that displays fractal (or self-similar properties). Its correlation function and power spectral density both follow power-law forms. Its fractal nature is characterized by a self-similarity parameter, termed the Hurst exponent H [1]. We first demonstrate how a power-law spectral density arises from the definition of 2D FBM. We then consider the wavelet transform of 2D FBM, and show how it can be used to estimate H. Finally, we consider the converse problem and demonstrate how wavelets can be used to synthesize processes with power spectral densities close to those of 2D FBM.

I. INTRODUCTION

Fractional Brownian motion (FBM) in one dimension has proved useful for describing a range of phenomena with long-range dependence [2]. FBM exhibits a power spectral density (PSD) proportional to $1/\omega^\gamma$, where ω typically represents the temporal angular frequency, and γ is a fractal exponent related to the Hurst exponent H.

We consider two-dimensional fractional Brownian motion with a 2D PSD that assumes the form

$$S(\vec{\omega}) = S(\omega_x, \omega_y) = \frac{C_1}{(\omega_x^2 + \omega_y^2)^{\gamma/2}}; \qquad (1)$$

the vector $\vec{\omega}$ (with components ω_x and ω_y) represents spatial frequency in the x and y directions, respectively, and C_1 is an arbitrary constant. This is an isotropic PSD, since it is a function only of the radial spatial frequency $|\omega| = \sqrt{(\omega_x^2 + \omega_y^2)}$. The motivation for this generalization is provided by the many kinds of processes displaying this type of PSD. Applications include:

- models of natural (fractal) landscapes [3],
- texture discrimination [4, 5]
- human discrimination of fractal images [6].

Furthermore, since the fixed-pattern image noise in some focal-plane array cameras exhibits a power spectral density that obeys Eq. (1) [7, 8], we expect that 2D FBM will provide a robust framework for modeling the noise properties of these devices as well.

The development of techniques for combating this type of fractal image noise requires not only a plausible noise model but also reliable methods for estimating the value of γ (as has already been demonstrated for the one-dimensional case [9, 10]). The work presented here lays the groundwork for exploring such approaches. In particular, we show how wavelet theory can be used in the estimation of the self-similarity parameter of 2D FBM [4, 11].

Moreover, 2D FBM processes provide useful models for synthesizing natural looking landscapes [2, 3, 12], and a variety of techniques for generating such processes has already been developed [3]. In this paper, we also show how wavelets and wavelet theory can be used in the synthesis of 2D processes with power-law spectra.

II. DEFINITION OF TWO-DIMENSIONAL FRACTIONAL BROWNIAN MOTION

The generalization of Brownian motion to more than one dimension was first considered by Lévy [13]; the generalization of fractional Brownian motion follows along similar lines. Let $B(\vec{u})$ be a fractional

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Brownian motion (also called a fractional Brownian surface [2]), where $\vec{u}$ denotes the position vector (u_x, u_y) of a point in the process. The properties of FBM can be characterized as obeying the following rules [3]:

1) The process increments $\Delta B(\vec{u}) = B(\vec{u}+\Delta\vec{u}) - B(\vec{u})$ form a stationary zero-mean Gaussian process;

2) The variance of the increments $\Delta B(\vec{u})$ depends only on the distance $\Delta u = \sqrt{u_x^2 + u_y^2}$ so that

$$\mathrm{E}[|\Delta B(\vec{u})|^2] = 2C_2 \Delta u^H \quad (2)$$

where E is the expectation operator, H is the Hurst exponent, and C_2 is a function of H [2];

3) The two-dimensional correlation function of FBM is defined as

$$\begin{aligned} r_B(\vec{u},\vec{v}) &= \mathrm{E}[B(\vec{u})B(\vec{v})] \\ &= C_2\left[|\vec{u}|^{2H} + |\vec{v}|^{2H} - |\vec{u}-\vec{v}|^{2H}\right]. \end{aligned} \quad (3)$$

Since the correlation function in Eq. (3) is not simply a function of $(\vec{u} - \vec{v})$, 2D FBM is nonstationary and the PSD no longer has a clear definition. Rather it depends on the techniques used in its construction. The Wigner-Ville spectrum (WVS) has proved to be a useful time dependent spectral analysis tool in one dimension [14, 15], and it may readily be extended to two dimensions:

$$W_B(\vec{u},\vec{\omega}) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} r_B\left(\vec{u}+\vec{\xi}/2, \vec{u}-\vec{\xi}/2\right)\exp(-j\vec{\omega}\cdot\vec{\xi})d\xi_x d\xi_y, \quad (4)$$

where $\vec{\xi}$ is a position vector of the form (ξ_x, ξ_y), $j = \sqrt{-1}$, and '$\cdot$' denotes the standard vector inner product. Substituting Eq. (3) into Eq. (4), we obtain for the WVS of 2D FBM:

$$\begin{aligned} W_B(\vec{u},\vec{\omega})/C_2 = \\ \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left|\vec{u}+\vec{\xi}/2\right|^{2H} \exp(-j\vec{\omega}\cdot\vec{\xi})d\xi_x d\xi_y + \\ \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left|\vec{u}-\vec{\xi}/2\right|^{2H} \exp(-j\vec{\omega}\cdot\vec{\xi})d\xi_x d\xi_y - \\ \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left|\vec{\xi}\right|^{2H} \exp(-j\vec{\omega}\cdot\vec{\xi})d\xi_x d\xi_y \end{aligned} \quad (5)$$

By carrying out the substitutions

$$\begin{aligned} \vec{r} &= \vec{u} + \vec{\xi}/2, \\ \vec{s} &= \vec{u} - \vec{\xi}/2, \end{aligned} \quad (6)$$

and using

$$I = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left(\xi_x^2 + \xi_y^2\right)^H \exp(-j\vec{\omega}\cdot\vec{\xi})d\xi_x d\xi_y, \quad (7)$$

we can obtain an analytical result for the WVS. After a series of substitutions, Eq. (7) becomes:

$$I = \frac{C_3}{(\omega_x^2 + \omega_y^2)^{H+1}} \quad (8)$$

in the range where the integral converges ($-1 < H < -1/4$), and where C_3 is a function of H. Finally, we obtain

$$W_B(\vec{u},\vec{\omega}) = \left[2^{1-2H}\cos(2\vec{\omega}\cdot\vec{u}) - 1\right]\frac{C_2 C_3}{(\omega_x^2+\omega_y^2)^{H+1}}, \quad (9)$$

valid in the range $-1 < H < -1/4$. A meaningful average power spectral density can be obtained by averaging W_B over $\vec{u}$, whereupon we reach

$$\begin{aligned} \overline{S}(\vec{\omega}) &= \lim_{L\to\infty}(1/L^2)\int_0^L\int_0^L W_B(\vec{u},\vec{\omega})du_x du_y \\ &= C_1/(\omega_x^2+\omega_y^2)^{H+1} \end{aligned} \quad (10)$$

which is a power spectral density of the form given in Eq. (1) with $H + 1 = \gamma/2$. Since H lies in the range, $[-1, -1/4]$, γ is restricted to the range $0 < \gamma < 3/2$.

In practice, the power spectral density estimate obtained from any finite sample of $B(\vec{u})$ will approach the limit given in Eq. (10), provided the sample is sufficiently large. Accordingly, 2D FBM with a correlation function in the form given by Eq. (3) provides a possible model for processes in which an isotropic power-law PSD with $0 < \gamma < 3/2$ is observed.

III. WAVELET TRANSFORM OF 2D FRACTIONAL BROWNIAN MOTION

Wavelet-based techniques have emerged as a natural framework for the analysis and synthesis of multiscale processes, since wavelet theory itself is built on the notion of scaling (i.e., the wavelet series representation uses the same basis function, taken at many different scales, to approximate a target function). An excellent overview of wavelet theory can be found in [16]. Properties of the wavelet transform of 1D FBM have already been considered by several authors [11, 17].

Let us first define the continuous wavelet transform (CWT) of a function in two dimensions [18], and show how it can be used in the analysis of 2D FBM. The two-dimensional CWT of the function $B(\vec{u})$ is defined as

$$CWT_\Psi^B(a,\vec{b},\theta) = \frac{1}{a}\int_{\vec{u}} \Psi^*\left(R^\theta\left[\frac{\vec{u}-\vec{b}}{a}\right]\right)B(\vec{u})d\vec{u}, \quad (11)$$

where $\Psi(\vec{u})$ is the two-dimensional spatial wavelet basis (or function), the asterisk denotes complex conjugation, a is a scaling (or dilation) factor, $\vec{b}$ is a two-dimensional translation vector, and R^θ denotes a rotation by the angle θ in the two-dimensional plane.

In general, all wavelet bases obey the admissibility condition [16]

$$\int_{\vec{u}} \Psi(\vec{u})d\vec{u} = 0. \quad (12)$$

If we are considering an isotropic signal (or isotropic wavelet), the two-dimensional continuous wavelet transform of $B(\vec{u})$ will have no functional dependence on θ and can be simply written as $CWT^B_\Psi(a,\vec{b})$. As noted previously by Peyrin *et al.* [4], it is not difficult to demonstrate that the CWT of 2D FBM retains the scaling properties of the original signal, and can be conveniently used to estimate those properties.

Specifically, if we consider the expected value of the square of the wavelet transform (which is equal to its variance since the expected value of the transform itself is zero), we obtain:

$$\mathrm{E}\left[|CWT^B_\Psi(a,\vec{b})|^2\right] = \frac{1}{a^2}\int_{\vec{u}}\int_{\vec{v}}\Psi^*\left(\frac{\vec{u}-\vec{b}}{a}\right)\Psi\left(\frac{\vec{v}-\vec{b}}{a}\right)\mathrm{E}\left[B(\vec{u})B(\vec{v})\right]d\vec{u}d\vec{v}. \tag{13}$$

Using the expression in Eq. (3) for $\mathrm{E}[B(\vec{u})B(\vec{v})]$ in conjunction with Eq. (12) leads to

$$\mathrm{E}\left[|CWT^B_\Psi(a,\vec{b})|^2\right] = (C_2/a^2)\int_{\vec{u}}\int_{\vec{v}}\Psi^*\left(\frac{\vec{u}-\vec{b}}{a}\right)\Psi\left(\frac{\vec{v}-\vec{b}}{a}\right)|\vec{u}-\vec{v}|^{2H}d\vec{u}d\vec{v}. \tag{14}$$

The substitutions $\vec{p}=(\vec{u}-\vec{v})/a$ and $\vec{q}=(\vec{v}-\vec{b})/a$ lead to

$$\mathrm{E}\left[|CWT^B_\Psi(a,\vec{b})|^2\right] = C_2a^{2H+2}\int_{\vec{p}}\int_{\vec{q}}\Psi^*(\vec{p}+\vec{q})\Psi(\vec{q})|\vec{p}|^{2H}d\vec{p}d\vec{q} \tag{15}$$

which can be conveniently rewritten as

$$\mathrm{E}\left[|CWT^B_\Psi(a,\vec{b})|^2\right] = C_2a^{2H+2}\int_{\vec{p}}CWT^\Psi_\Psi(1,\vec{p})|\vec{p}|^{2H}d\vec{p}. \tag{16}$$

where CWT^Ψ_Ψ represents the wavelet transform of the wavelet itself. Since the integrand in Eq. (16) is independent of a, the variance of $|CWT^B_\Psi(a,\vec{b})|$ varies as a power-law in a, and can be used to estimate the self-similarity parameter of the process. This approach therefore provides an alternative to the power spectral density for estimating H. As shown in [4], it does so reliably.

IV. SYNTHESIS OF 2D FRACTIONAL BROWNIAN MOTION USING WAVELETS

It is apparent from the foregoing that wavelet-based techniques can be readily used to analyze 2D FBM. The wavelet framework also provides an alternative to existing methods for constructing such processes. We proceed to show that wavelet functions can in fact be used to generate processes with nearly $1/\omega^\gamma$ spectra in two dimensions. The relevant theory for orthonormal wavelets in two dimensions is outlined in [19]. We follow the approach outlined by Wornell for the one-dimensional case [20]; implementations of this approach have already been successfully used to synthesize 1D FBM [11, 21]. We call a two-dimensional power spectral density "nearly $1/\omega^\gamma$" if it satisfies

$$\frac{K_1}{|\omega|^\gamma} \le S(\omega_x,\omega_y) \le \frac{K_2}{|\omega|^\gamma} \tag{17}$$

with $0 < K_1 \le K_2$ Consider the case with separable scaling and wavelet functions in two dimensions,

$$\begin{aligned} \Phi(u_x,u_y) &= \phi(u_x)\phi(u_y) \\ \Psi^1(u_x,u_y) &= \phi(u_x)\psi(u_y) \\ \Psi^2(u_x,u_y) &= \psi(u_x)\phi(u_y) \\ \Psi^3(u_x,u_y) &= \psi(u_x)\psi(u_y) \end{aligned} \tag{18}$$

where $\psi(\cdot)$ and $\phi(\cdot)$ are admissible one-dimensional wavelet and scaling functions, respectively. Furthermore, let us define $\Psi^{i,m}_{n,p}(u_x,u_y)$ as

$$\Psi^{i,m}_{n,p}(u_x,u_y) = 2^{m/2}\Psi^i(2^mu_x-n,2^mu_y-p) \tag{19}$$

for $i=1,2,3$.

Consider now the construction of a random process $X_M(u_x,u_y)$ from the sequence

$$X_M(u_x,u_y) = \sum_{m\ge M}\sum_{n,p}d^m_{n,p}\left[\Psi^{1,m}_{n,p}(u_x,u_y)+\Psi^{2,m}_{n,p}(u_x,u_y)+\Psi^{3,m}_{n,p}(u_x,u_y)\right], \tag{20}$$

where $d^m_{n,p}$ and $d^{m'}_{n,p}$ are wide-sense stationary 2D sequences that are uncorrelated for $m \ne m'$. The power spectrum of $d^m_{n,p}$ is defined in the usual way for discrete sequences, as $P_m(\omega_x,\omega_y)$. Let our candidate $1/\omega^\gamma$ process equal the limit of $X_M(u_x,u_y)$ as M goes to negative infinity,

$$X(u_x,u_y) = \lim_{M\to-\infty}X_M(u_x,u_y). \tag{21}$$

Since the $d^m_{n,p}$ are uncorrelated across m, we can define the power spectrum of $X(u_x,u_y)$ in the limit as:

$$S(\omega_x,\omega_y) = \lim_{M\to-\infty}\sum_m P_m(\omega_x,\omega_y)\left\{|\hat{\Psi}^1(2^{-m}\omega_x,2^{-m}\omega_y)|^2+|\hat{\Psi}^2(2^{-m}\omega_x,2^{-m}\omega_y)|^2+|\hat{\Psi}^3(2^{-m}\omega_x,2^{-m}\omega_y)|^2\right\} \tag{22}$$

where $\hat{\Psi}^i(\omega_x,\omega_y)$ is the two-dimensional Fourier transform of $\Psi^i(u_x,u_y)$. Let us choose sequences $d^m_{n,p}$ to have spectra $P_m(\omega_x,\omega_y)$ which vary as a power-law function of m:

$$P_m(\omega_x,\omega_y) = 2^{-\gamma m}\sigma^2, \tag{23}$$

where we set the variance $\sigma^2=1$ with no loss of generality.

We now exploit the relations that exist among the two-dimensional wavelet and scaling functions, building on the fact that in one dimension [16]

$$\begin{aligned} \hat{\phi}(\omega) &= H(\omega/2)\hat{\phi}(\omega/2) \\ \hat{\psi}(\omega) &= G(\omega/2)\hat{\phi}(\omega/2) \end{aligned} \tag{24}$$

where $\hat{\phi}(\omega)$ and $\hat{\psi}(\omega)$ are the one-dimensional Fourier transforms of $\phi(x)$ and $\psi(x)$, respectively, and $H(\omega)$ and $G(\omega)$ are the filter functions relating $\hat{\phi}(\omega)$ and $\hat{\psi}(\omega)$ to $\hat{\phi}(\omega/2)$. Note that $H(\omega)$ and $G(\omega)$ satisfy [16, 20]

$$\begin{array}{l} G(\omega) = \exp(-j\omega)H^*(\omega+\pi) \\ |H(\omega)|^2 + |H(\omega+\pi)|^2 = 1. \end{array} \tag{25}$$

Applying these relations to the two-dimensional case gives the relations

$$\begin{array}{rcl} \hat{\Phi}(\omega_x,\omega_y) &=& H(\omega_x/2)H(\omega_y/2)\hat{\Phi}(\omega_x/2,\omega_y/2) \\ \hat{\Psi}^1(\omega_x,\omega_y) &=& H(\omega_x/2)G(\omega_y/2)\hat{\Phi}(\omega_x/2,\omega_y/2) \\ \hat{\Psi}^2(\omega_x,\omega_y) &=& G(\omega_x/2)H(\omega_y/2)\hat{\Phi}(\omega_x/2,\omega_y/2) \\ \hat{\Psi}^3(\omega_x,\omega_y) &=& G(\omega_x/2)G(\omega_y/2)\hat{\Phi}(\omega_x/2,\omega_y/2). \end{array} \tag{26}$$

Substituting Eqs. (25) and (26) into Eq. (22) yields, after some algebra:

$$S(\omega_x,\omega_y) = (2^\gamma - 1)\sum_m 2^{-\gamma m}|\hat{\Phi}(2^{-m}\omega_x, 2^{-m}\omega_y|^2. \tag{27}$$

The problem is now to show that there exists K_1 and K_2 such that Eq. (17) is satisfied. We use the identity

$$S(\omega_x,\omega_y) = 2^{-n\gamma}S(2^{-n}\omega_x, 2^{-n}\omega_y) \tag{28}$$

for all values of n, which can be easily verified. Assuming a specific ω_x and ω_y, we can always choose n_0, $\omega_{x,0}$, and $\omega_{y,0}$ such that the following conditions are satisfied:

$$\begin{array}{rcl} \omega_x &=& 2^{n_0}\omega_{x,0} \\ \omega_y &=& 2^{n_0}\omega_{y,0} \\ 1 &\le& \sqrt{\omega_{x,0}^2+\omega_{y,0}^2} \le 2. \end{array} \tag{29}$$

Using Eq. (28), we obtain

$$S(\omega_x,\omega_y) = S(2^{n_0}\omega_{x,0}, 2^{n_0}\omega_{y,0}) = 2^{-n_0\gamma}S(\omega_{x,0},\omega_{y,0}) \tag{30}$$

with $1 \le \sqrt{\omega_{x,0}^2+\omega_{y,0}^2} \le 2$, whereupon our problem is reduced to establishing that

$$\left[\inf_{1\le|\omega_0|\le 2} S(\omega_{x,0},\omega_{y,0})\right]|\omega|^{-\gamma} \le S(\omega_x,\omega_y)$$

and

$$S(\omega_x,\omega_y) \le 2^\gamma|\omega|^{-\gamma}\left[\sup_{1\le|\omega_0|\le 2} S(\omega_{x,0},\omega_{y,0})\right] \tag{31}$$

with $|\omega| = \sqrt{\omega_x^2+\omega_y^2} = 2^{n_0}|\omega_0|$.

To prove Eq. (31), therefore, it is sufficient to find upper and lower bounds for $S(\omega_{x,0},\omega_{y,0})$ on $1 \le |\omega_0| \le 2$. First we consider the upper bound of Eq. (31). Assume that $|\hat{\Phi}(0,0)| = 1$ so that $|\hat{\Phi}(\omega_x,\omega_y)| \le 1$. From the properties of the one-dimensional scaling function, it is not hard to show that there exists a number $K_3 > 1$ such that

$$|\hat{\Phi}(\omega_x,\omega_y)| \le \frac{K_3}{1+|\omega|}. \tag{32}$$

Using this, we can rewrite Eq. (27) as

$$\begin{array}{l} S(\omega_{x,0},\omega_{y,0})/(2^\gamma-1) \\ = \sum_m 2^{-\gamma m}|\hat{\Phi}(2^{-m}\omega_{x,0}, 2^{-m}\omega_{y,0})|^2 \\ = \left[\sum_{m=0}^{\infty}|\hat{\Phi}(2^{-m}\omega_{x,0}, 2^{-m}\omega_{y,0})|^2 \right. \\ \quad \left. + \sum_{m=-\infty}^{-1}|\hat{\Phi}(2^{-m}\omega_{x,0}, 2^{-m}\omega_{y,0})|^2\right] \\ \le \left[\sum_{m=0}^{\infty} 2^{-\gamma m} \right. \\ \quad \left. + \sum_{m=1}^{\infty} 2^{\gamma m}|\hat{\Phi}(2^{m}\omega_{x,0}, 2^{m}\omega_{y,0})|^2\right] \\ \le \left[\sum_{m=0}^{\infty} 2^{-\gamma m} + \sum_{m=1}^{\infty} 2^{\gamma m}K_3^2/(1+2^m|\omega_0|)^2\right] \\ \le \left[\sum_{m=0}^{\infty} 2^{-\gamma m} + \sum_{m=1}^{\infty} K_3^2 2^{(\gamma-2)m}\right] \le \infty \end{array} \tag{33}$$

for $0 < \gamma < 2$, where we have used the fact that $|\omega_0| \le 2$ in the defined range. Thus, the supremum of $S(\omega_{x,0},\omega_{y,0})$ exists, proving the upper bound of Eq. (31).

To show that the lower bound also holds, we invoke the continuity of $\hat{\Phi}(\omega_x,\omega_y)$ at $(0,0)$. Since $\hat{\Phi}(0,0) = 1$, there must exist an n_0 such that

$$\begin{array}{c} |\hat{\Phi}(\omega_x,\omega_y)| \ge 1/2 \\ \text{for } \omega_x < 2^{-n_0} \text{ and } \omega_y < 2^{-n_0}. \end{array} \tag{34}$$

Hence

$$|\hat{\Phi}(2^{-n_0-1}\omega_{x,0}, 2^{-n_0-1}\omega_{y,0})| \ge 1/2, \tag{35}$$

since the largest values of $\omega_{x,0}$ and $\omega_{y,0} = 2$. Therefore

$$\begin{array}{l} S(\omega_{x,0},\omega_{y,0}) = (2^\gamma-1)\sum_m 2^{-\gamma m}|\hat{\Phi}(2^{-m}\omega_x, 2^{-m}\omega_y)|^2 \\ \quad \ge (2^\gamma-1)2^{-\gamma(n_0-1)}|\hat{\Phi}(2^{-n_0-1}\omega_{x,0}, 2^{-n_0-1}\omega_{y,0})|^2 \\ \quad \ge (2^\gamma-1)2^{-\gamma(n_0-1)}2^{-2} > 0, \end{array} \tag{36}$$

so that the lower bound in Eq. (31) exists. Accordingly, two-dimensional sequences $d_{n,p}^m$ satisfying Eq. (23) can be used to synthesize processes with power-law spectral densities such as 2D FBM.

V. CONCLUSION

We have shown how one-dimensional fractional Brownian motion is readily extended to two dimensions (it can, in fact, be generally extended to n dimensions). Even though this is a non-stationary process, we can calculate a meaningful power spectral density, and we have shown that isotropic 2D FBM has an isotropic power-law spectral density. Processes with such densities arise in a variety of settings; examples include models of fractal landscapes, texture recognition, and spatial noise in infrared focal plane arrays.

Wavelet theory has already been shown to provide an appropriate tool both for the analysis and synthesis of long-range dependent processes in one dimension. The results here demonstrate that it is similarly useful in two dimensions. From an analysis point of view, the wavelet transform of 2D FBM retains the self-similar properties of the original signal, and can be used to estimate the self-similarity parameter of the process. This parameter is important in texture recognition, and as a measure of the roughness of a fractal surface. We have also shown how wavelets can be used in the synthesis of 2D processes with power-law spectral densities.

This work provides an appropriate framework for the development of detection and estimation algorithms to combat the effects of power-law fixed-pattern noise. Such techniques have already been considered in one dimension [9, 10].

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