Fractal Point Events in Physics, Biology, and Communication Networks

Malvin Carl Teich
Boston University
http://people.bu.edu/teich

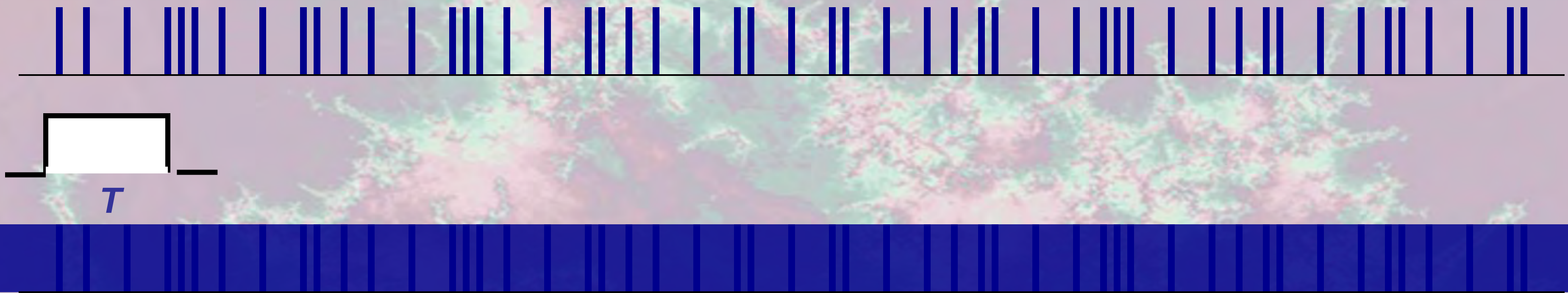
1. OUTLINE

1. OUTLINE
2. POINT EVENTS
3. FRACTALS AND FRACTAL POINT EVENTS
4. FRACTAL PHOTONS
5. HEART RATE VARIABILITY
6. NETWORKS
7. SENSORY DETECTION

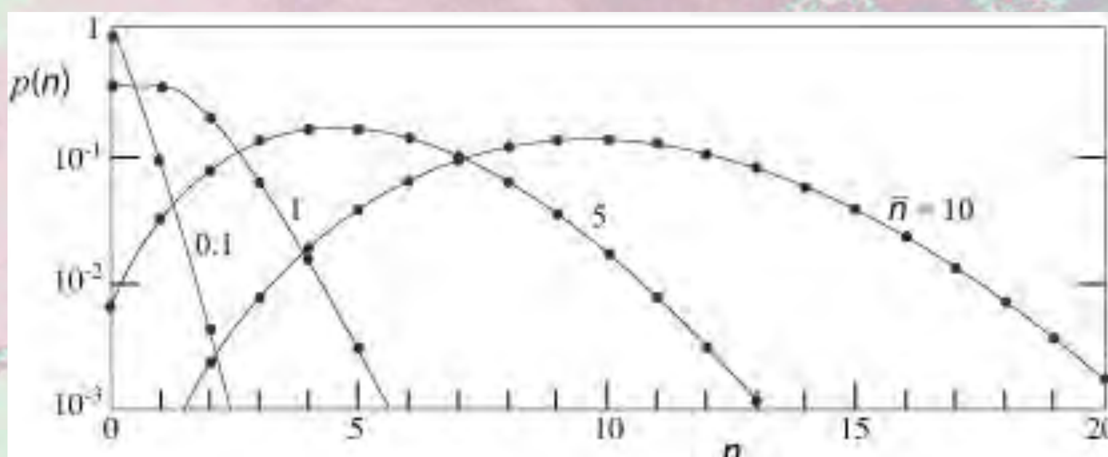
2. POINT EVENTS

COUNTING POINT EVENTS IN A FIXED TIME WINDOW *T*

Poisson process (zero-memory):



For this process, the counting distribution $p(n)$ (i.e., the relative frequency or probability mass function) for the number of events *n* is characterized by the Poisson distribution, $p(n)=(\lambda T)^n e^{-\lambda T}/n!$, whose variance-to-mean ratio *F* is unity, so $\mathrm{var}(n)\equiv\sigma^2=\bar{n}=\lambda T$:

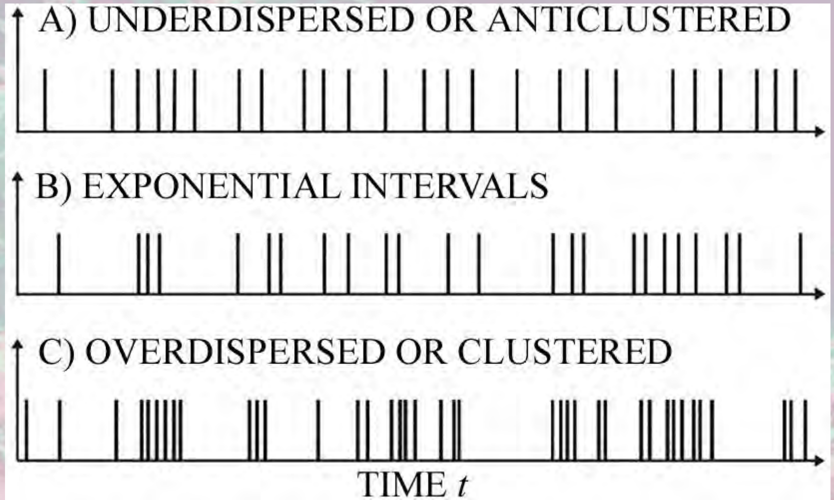


Poisson

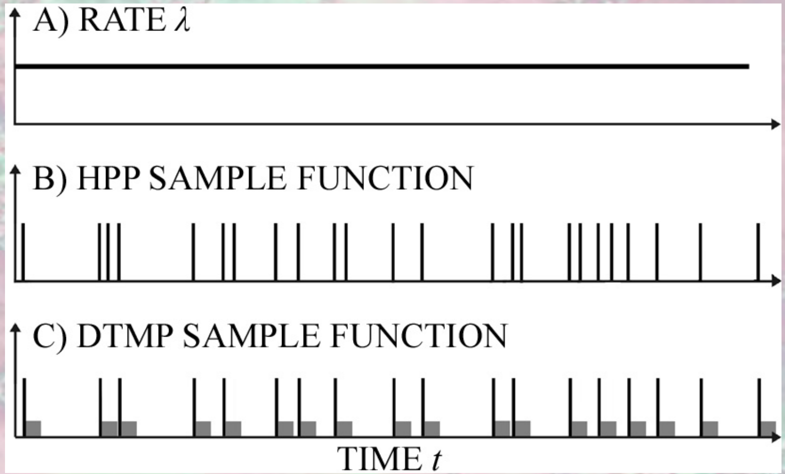
After Saleh and Teich, *Fundamentals of Photonics*, 2nd Ed. (Wiley, 2007), Chap. 12.

ANTICLUSTERED AND CLUSTERED POINT EVENTS

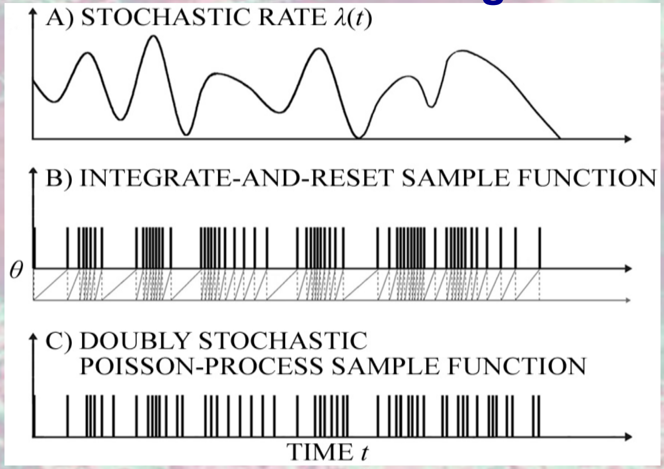
HPP:



Dead-time anticlustering:



Rate-variation clustering:

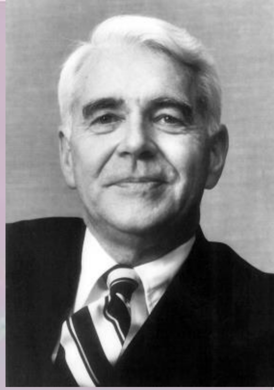


Cox

IMPORTANT EXAMPLES OF ANTICLUSTERED EVENTS VIA DEAD-TIME-MODIFICATION.
IMPORTANT EXAMPLES OF CLUSTERED EVENTS: NEGATIVE-BINOMIAL (NB) AND NEYMAN TYPE-A (NTA) DISTRIBUTIONS, AND VARIATIONS THEREOF

BOSTON UNIVERSITY

NEURAL COUNTING AND PHOTON COUNTING



McGill

W. J. McGill, “Signal Detection Theory,” presented at the *University Seminar on Mathematical Methods in the Social Sciences*, Columbia University, March 1974.

VOLUME 36, NUMBER 13 PHYSICAL REVIEW LETTERS 29 MARCH 1976

Neural Counting and Photon Counting in the Presence of Dead Time*

Malvin Carl Teich† and William J. McGill
Columbia University, New York, New York 10027
(Received 10 November 1975)

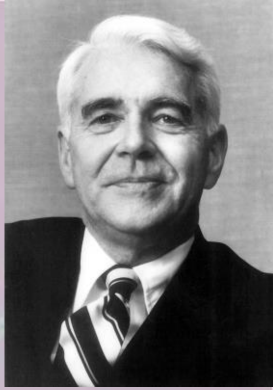
The usual stimulus-based neural counting model for audition is found to be mathematically identical to the well-known semiclassical formalism for photon counting. In particular, we explicitly demonstrate the equivalence of McGill's noncentral negative binomial distribution and Peřina's multimode confluent hypergeometric distribution for a coherent signal imbedded in chaotic noise. Dead-time corrections, important both in neural counting and in photon counting, are incorporated in a generalized form of this distribution. Some specific implications of these results are discussed.

NTA DISTRIBUTION – NEURAL EVENTS

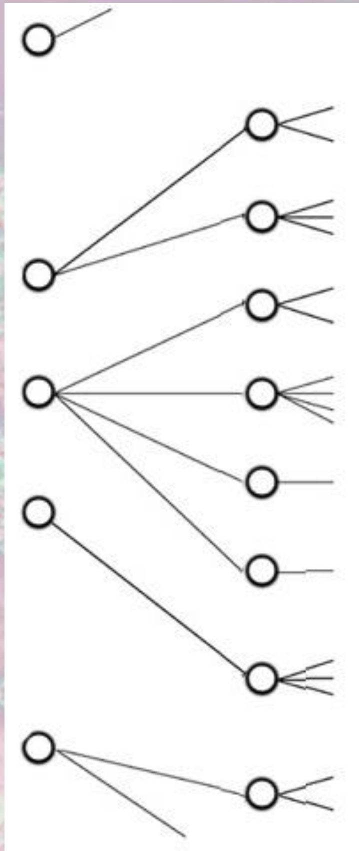
Reprinted from JOURNAL OF MATHEMATICAL PSYCHOLOGY
Vol. 4 No 3 ,October 1967
Printed in Belgium

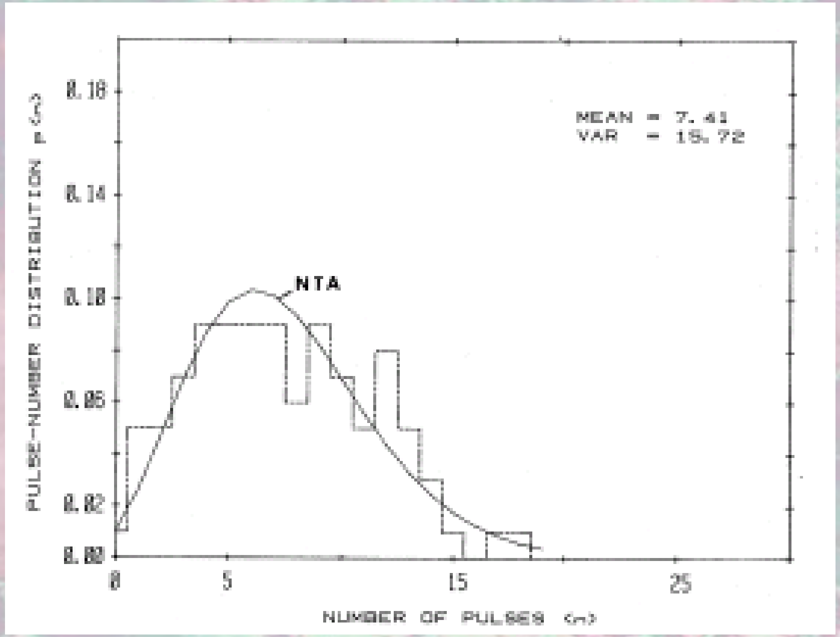
Neural Counting Mechanisms and Energy Detection in Audition

WILLIAM J. McGILL
University of California San Diego, California 92037



McGill





After Saleh and Teich, "Multiplication and Refractoriness in the Cat's Retinal-Ganglion-Cell Discharge at Low Light Levels," *Biol. Cybern.* **52**, 101-107 (1985).

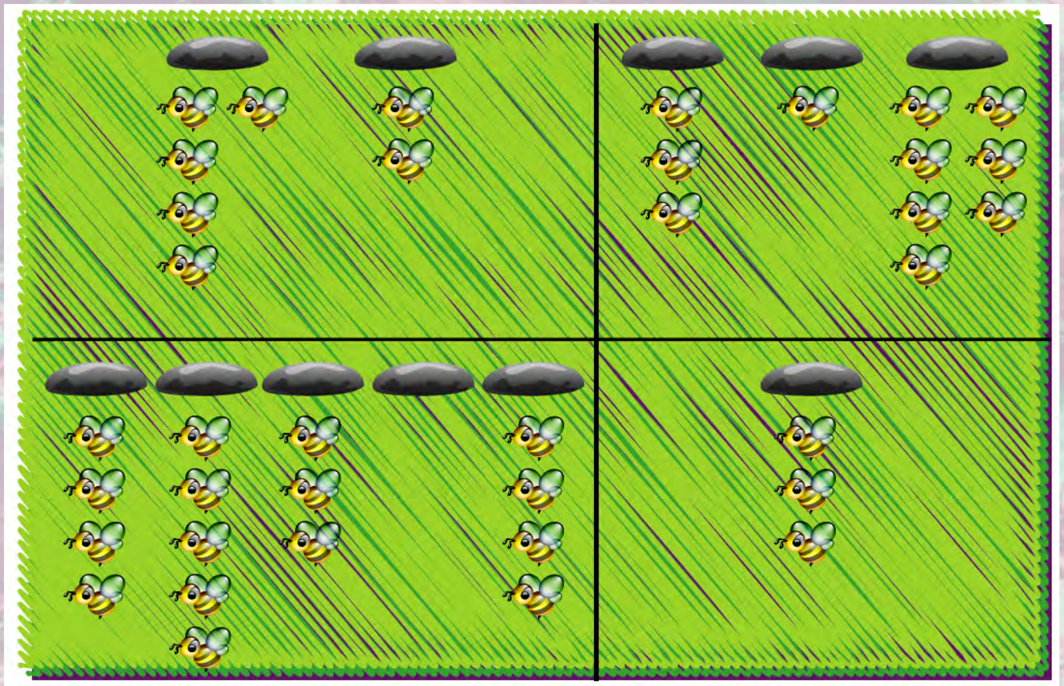
NEYMAN TYPE-A DISTRIBUTION – BUGS

On a New Class of "Contagious" Distributions, Applicable in Entomology and Bacteriology

J. Neyman

The Annals of Mathematical Statistics, Vol. 10, No. 1 (Mar., 1939), 35-57.

Neyman



$$p(n) = \sum_{m=0}^{\infty} p\left(n \middle| m\right) p(m)$$

$$= \sum_{m=0}^{\infty} \frac{(\alpha m)^n e^{-\alpha m}}{n!} \frac{\langle m \rangle^m e^{-\langle m \rangle}}{m!}$$

$$p(0) = \exp\left[-\langle m \rangle \left(1 - e^{-\alpha}\right)\right]$$

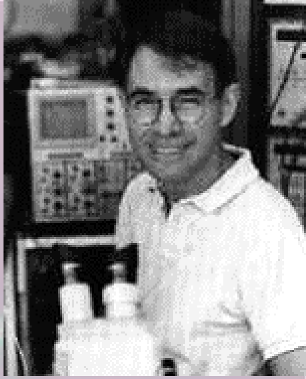
BOSTON UNIVERSITY

NTA DISTRIBUTION – VISUAL PERCEPTION

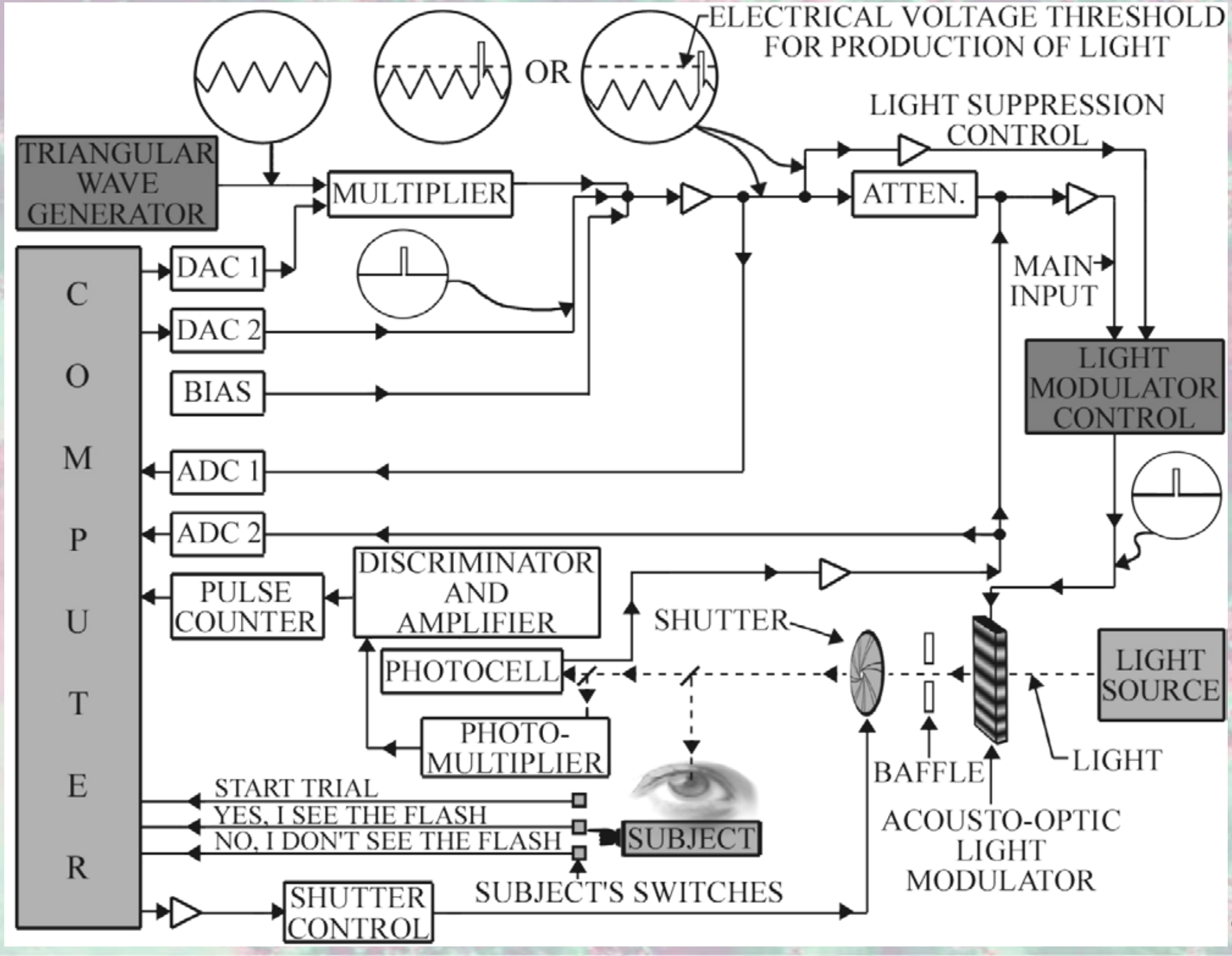
Photon Counting and Energy Detection in Vision

Malvin Carl Teich and Paul R. Prucnal
Department of Electrical Engineering and Computer Science, Columbia University, New York, New York 10027

J. Opt. Soc. Am., Vol. 67, No. 10, October 1977



Prucnal



NTA DISTRIBUTION – PHOTONS

IEEE JOURNAL OF QUANTUM ELECTRONICS, VOL. QE-17, NO. 12, DECEMBER 1981

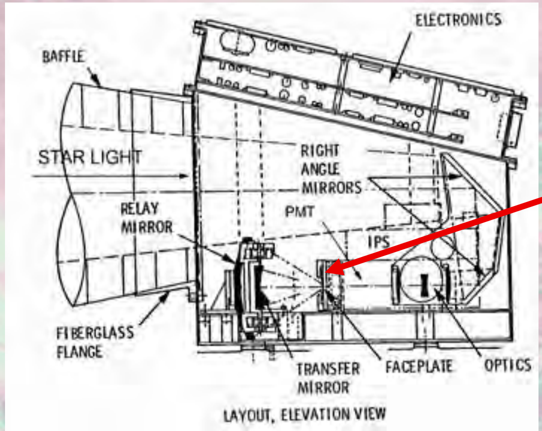
Discrimination of Shot-Noise-Driven Poisson Processes by External Dead Time: Application to Radioluminescence from Glass

BAHAA E. A. SALEH, MEMBER, IEEE, JOSEPH T. TAVOLACCI, MEMBER, IEEE, AND MALVIN CARL TEICH, SENIOR MEMBER, IEEE

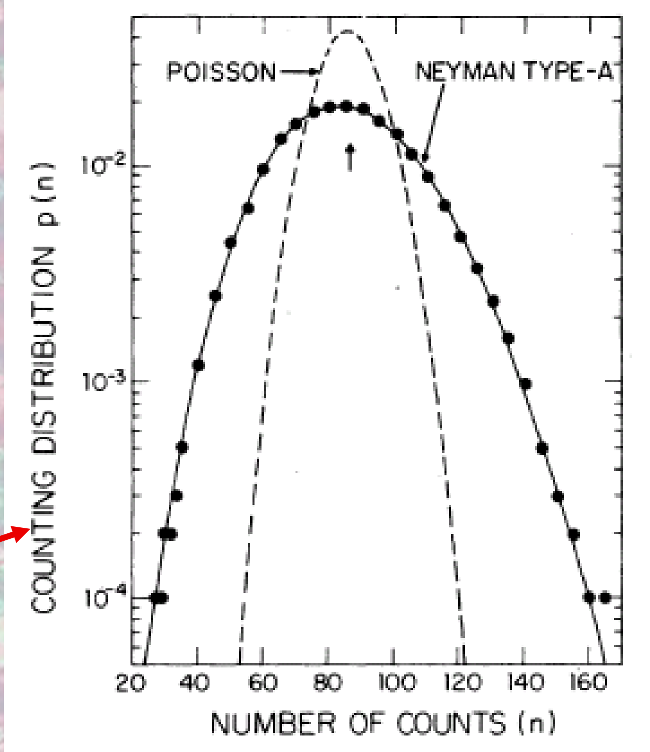
Galileo

GALILEO MISSION TIMELINE:

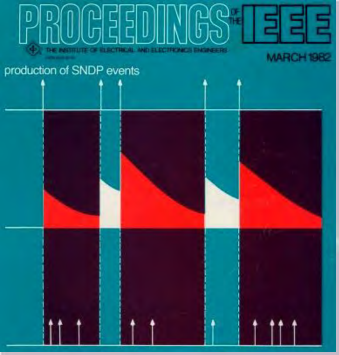
IDEA:	Oct. 1977
LAUNCH:	Oct. 1989
ARRIVAL:	Dec. 1995
DEATH:	Sep. 2003
PHOTOS:	14,000
DATA:	30 Gbytes



PMT FACEPLATE
BETALUMINESCENCE



After Teich and Saleh, "Fluctuation Properties of Multiplied-Poisson Light: Measurement of the Photon-Counting Distribution for Radioluminescence Radiation from Glass," *Phys. Rev. A* **24**, 1651-1654 (1981).

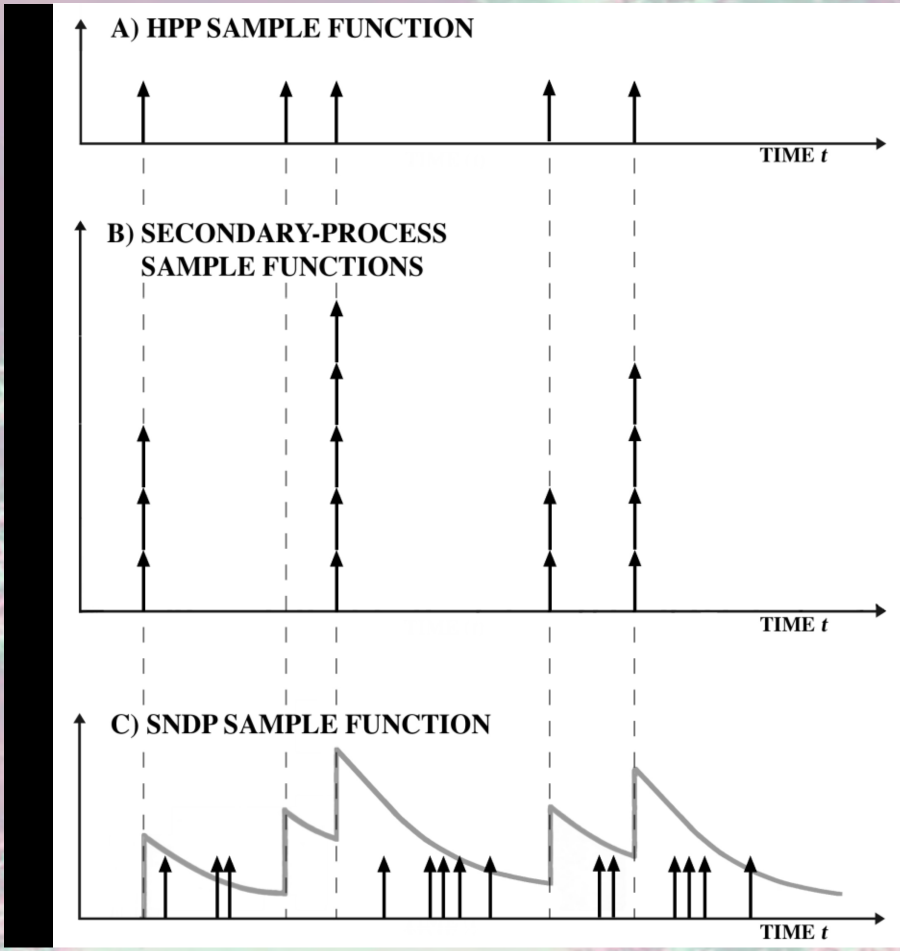


NTA DISTRIBUTION – SHOT-NOISE-DRIVEN POISSON

PROCEEDINGS OF THE IEEE, VOL. 70, NO. 3, MARCH 1982

Multiplied-Poisson Noise in Pulse, Particle, and Photon Detection

BAHAA E. A. SALEH, MEMBER, IEEE, AND MALVIN CARL TEICH, SENIOR MEMBER, IEEE



CASCADE VARIANCE THEOREM

$$\langle n\rangle = \langle \alpha\rangle\langle m\rangle$$

$$\mathrm{var}(n) = \langle\alpha\rangle^2\,\mathrm{var}(m) + \langle m\rangle\,\mathrm{var}(\alpha)$$

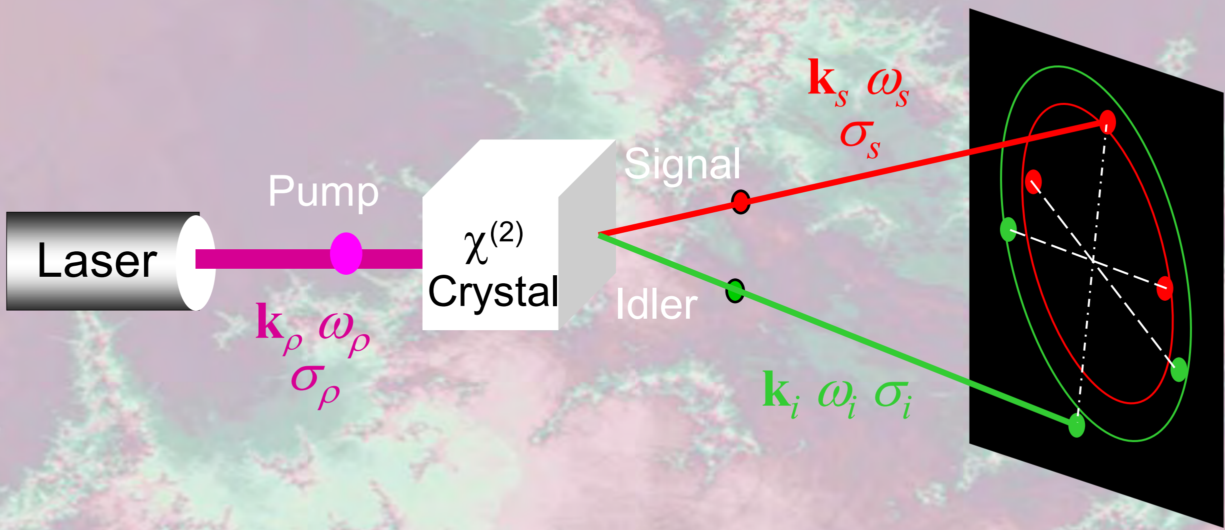
$$F_n \equiv \frac{\mathrm{var}(n)}{\langle n\rangle} = \langle\alpha\rangle F_m + F_\alpha$$

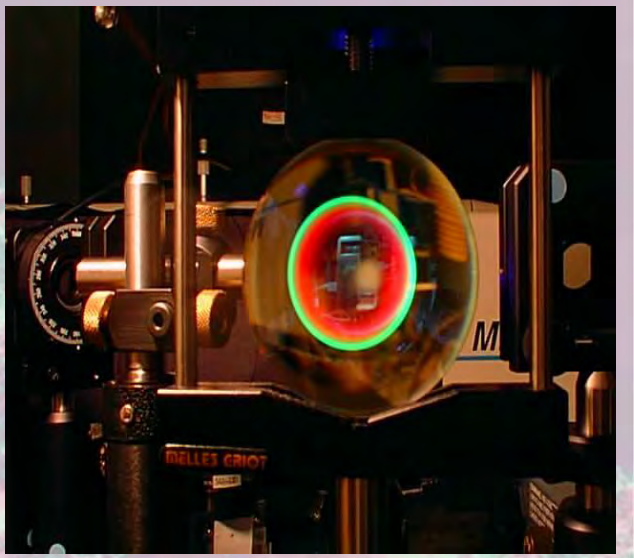
for $F_m = F_\alpha = 1:$

$$F_n = 1 + \langle\alpha\rangle \qquad \text{(NTA)}$$

BOSTON UNIVERSITY

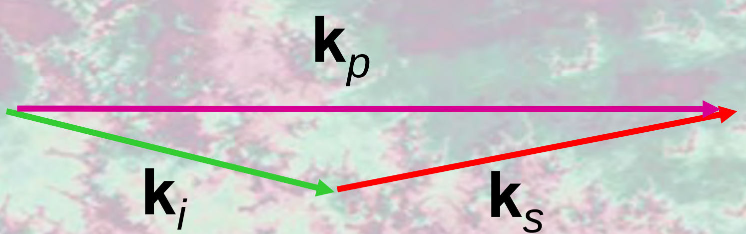
GENERATING PAIRS OF POINT EVENTS VIA SPONTANEOUS PARAMETRIC DOWN-CONVERSION: ENTANGLED PHOTONS





Conservation of energy $\omega_\rho = \omega_s + \omega_i$

Conservation of momentum $\mathbf{k}_\rho = \mathbf{k}_s + \mathbf{k}_i$

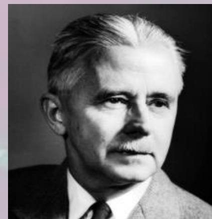


Adapted from Joobeur, Saleh, and Teich, "Spatiotemporal Coherence Properties of Entangled Light Beams Generated by Parametric Down-Conversion," *Phys. Rev. A* **50**, 3349-3361 (1994).

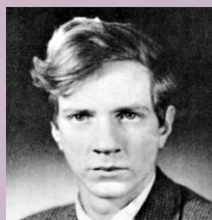
BOSTON UNIVERSITY

RANDOM POINT EVENTS IN NEUROPHYSIOLOGY

Hartline



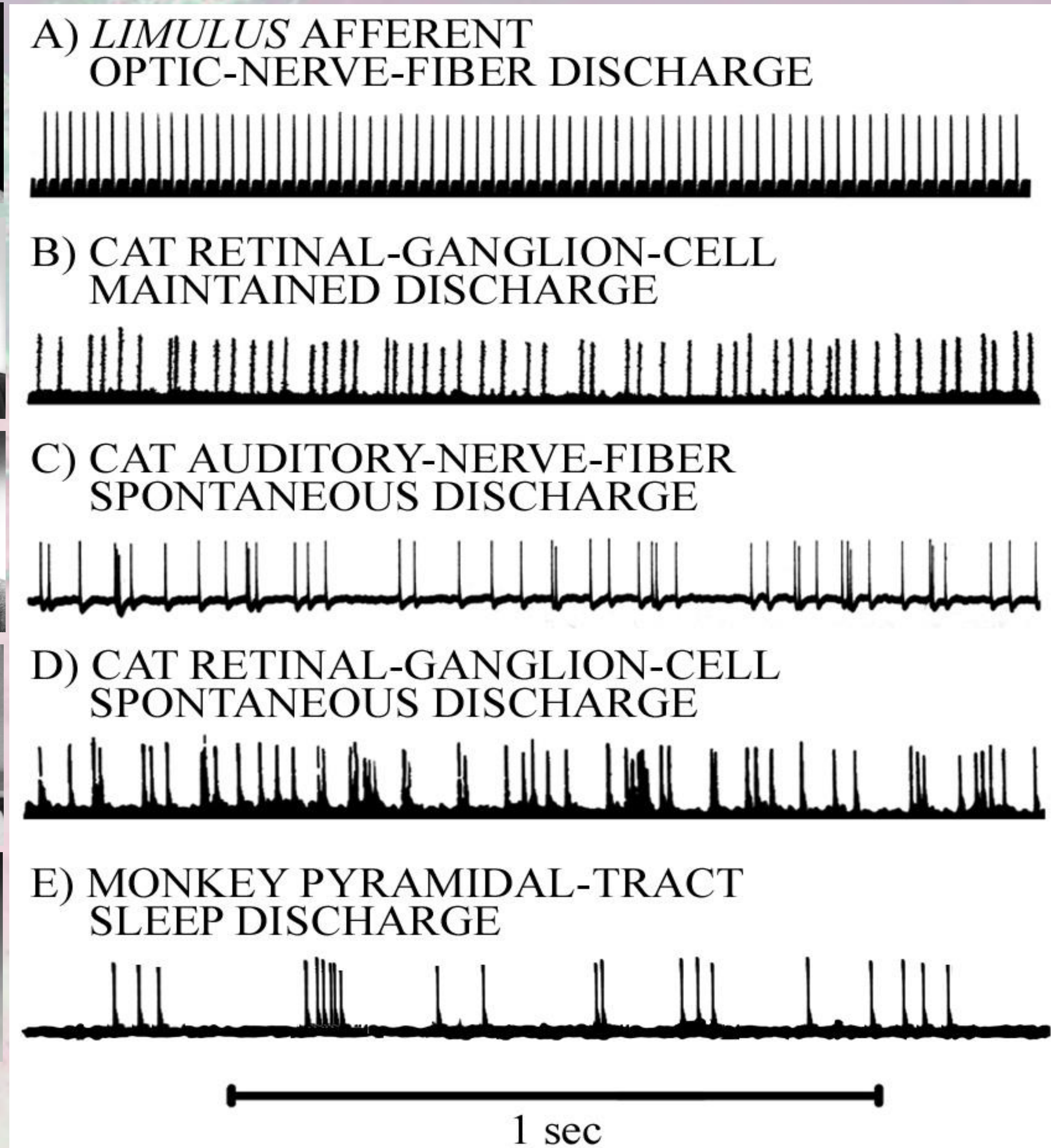
Barlow
Levick



Davis
Tasaki

Kuffler

Evarts



BOSTON UNIVERSITY

WHAT ARE THE COUNTING STATISTICS OF THESE POINT EVENTS?

COUNTING STATISTICS FOR POINT EVENTS IN THE MAMMALIAN AUDITORY SYSTEM

1110 J. Acoust. Soc. Am. 77 (3), March 1985 0001-4966/85/031110-19$00.80 © 1985 Acoustical Society of America 1110

Pulse-number distribution for the neural spike train in the cat's auditory nerve

Malvin C. Teich and Shyam M. Khanna
Department of Electrical Engineering, Columbia University, New York, New York 10027 and Fowler Memorial Laboratory, Department of Otolaryngology, Columbia College of Physicians and Surgeons, New York, New York 10032

Khanna

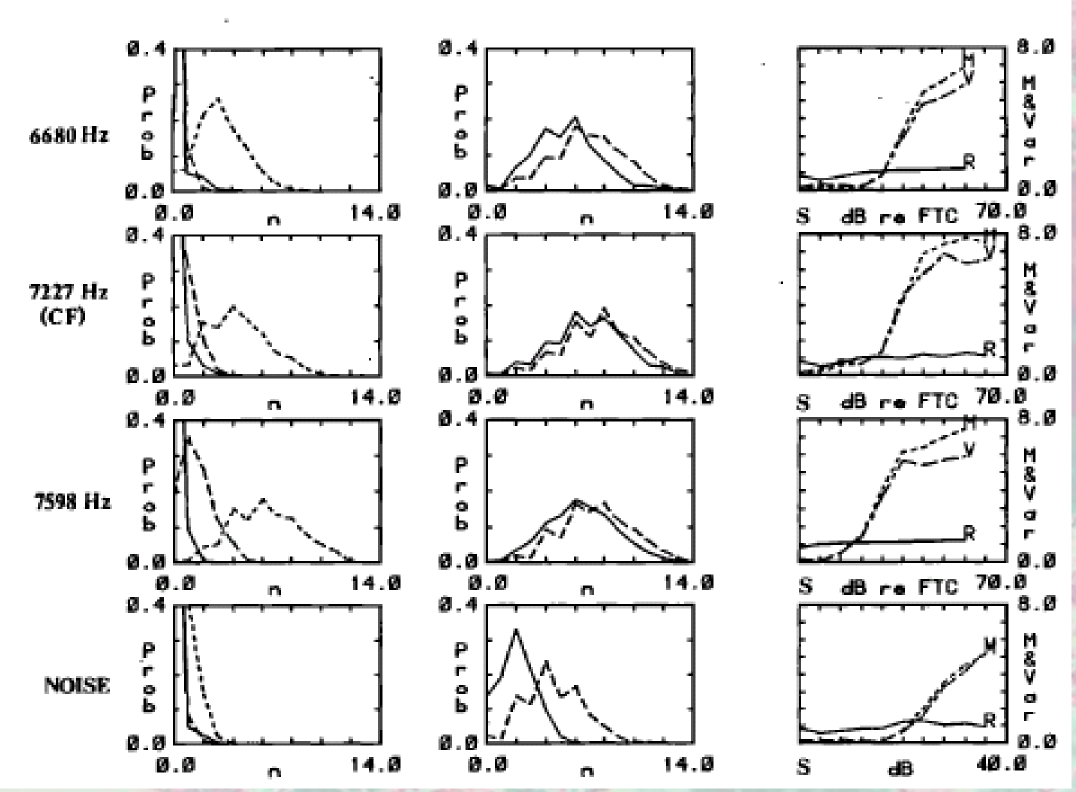
Eighth-nerve-fiber neural-counting experiments carried out at different sound-pressure levels using windows of

T = 50 msec

and

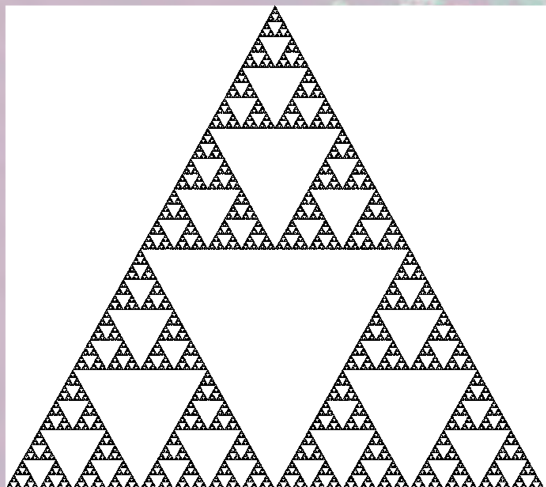
T = 200 msec

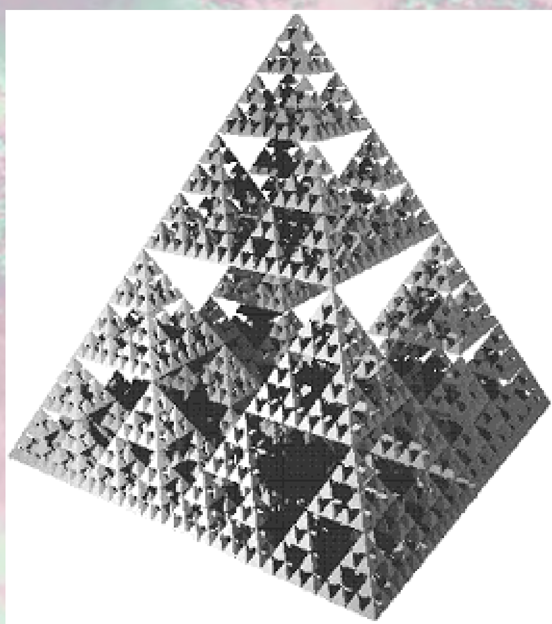
duration

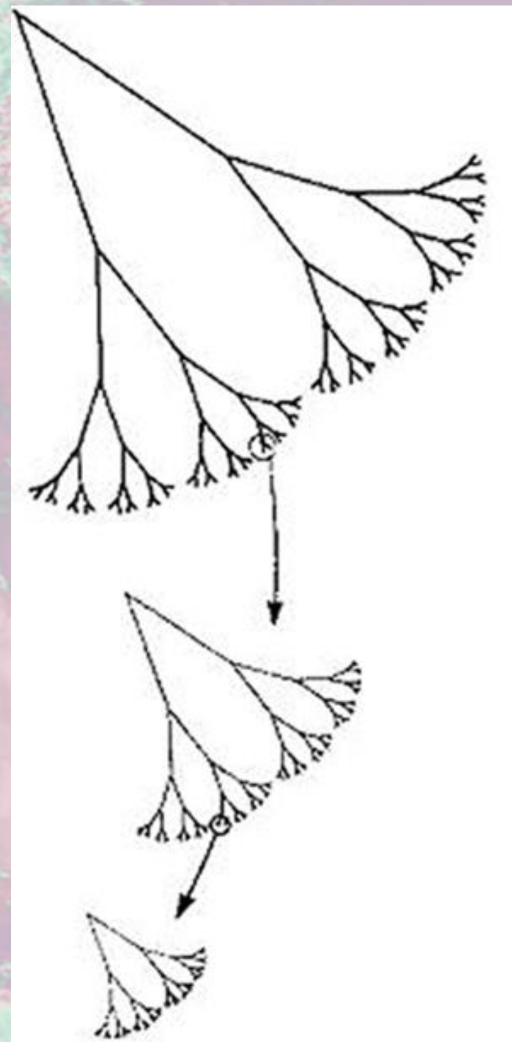


SOME EVIDENCE OF EVENT PAIRS: MIGHT A MULTINOMIAL MODEL REVEAL A HIDDEN NEURAL CODE?

3. FRACTALS AND FRACTAL POINT EVENTS







FORMS OF FRACTALS
- Deterministic
- Random
- Static
- Dynamical process

CUTOFFS
- Inner
- Outer

DEFINITIONS
- Scaling (both cutoffs)
- Fractal (no inner cutoff)
- Long-Range Dependence (no outer cutoff)

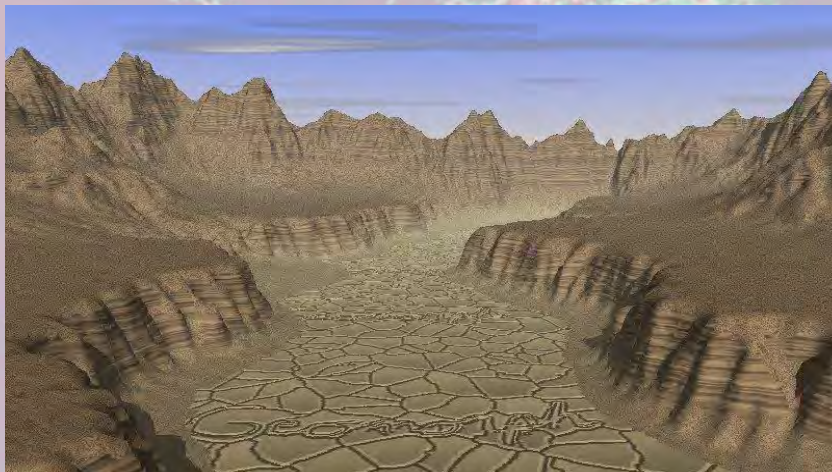
Mandelbrot

Adapted from B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, 1983).

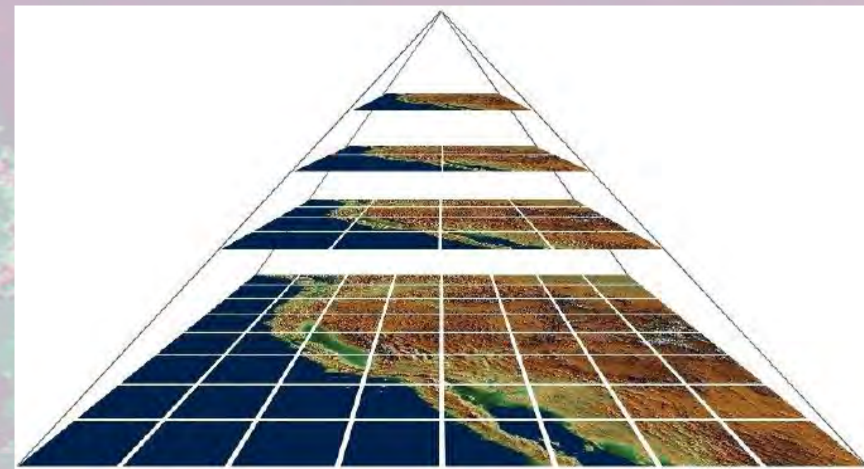
BOSTON UNIVERSITY

VIEWING THE WORLD AT MULTIPLE SCALES

Volume:



Area:



Line:

Coastline Lengths at Different Scales:

MEASUREMENT SCALE (km)	MEASURED LENGTH (km)
0.694	534
6.94	314
69.4	133

Richardson

Penck (1894); Richardson (1961): $\boxed{d \propto s^{c}}$

The dependence of the measurement outcome on the scale chosen to make that measurement is the hallmark of a FRACTAL OBJECT

BOSTON UNIVERSITY

UBIQUITY OF FRACTAL BEHAVIOR

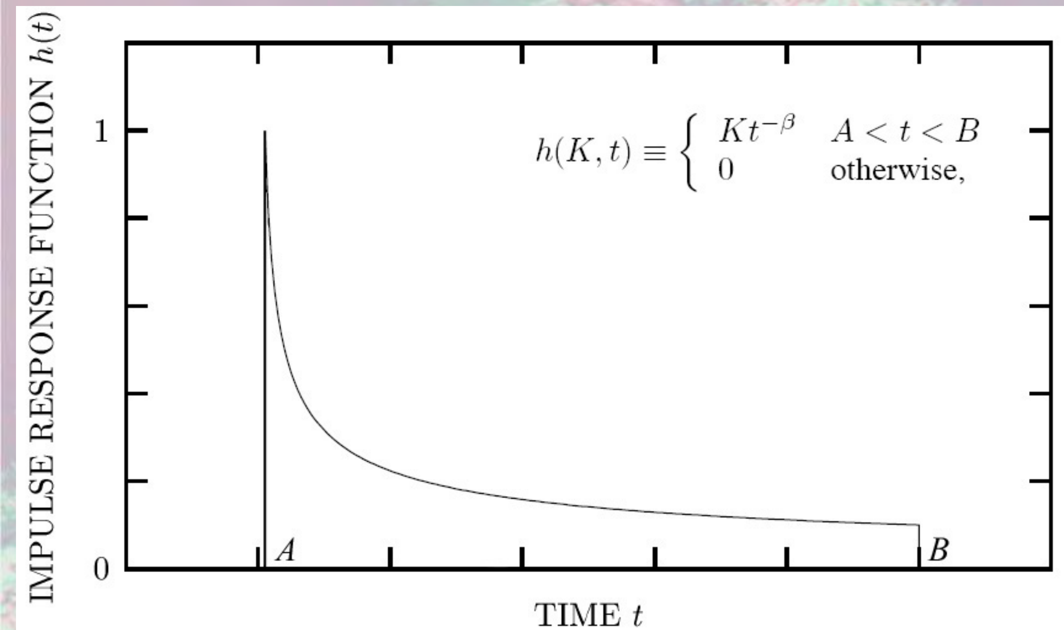
- Mathematics and physical sciences
  - Fractal geometry of nature
  - Noise in electronic components
  - Fabricated nonperiodic layered structures
  - Errors in telephone networks
  - Photon statistics of Čerenkov radiation
  - Earthquake patterns
  - Computer network traffic
- Neurosciences
  - Ion channels
  - Membrane voltages
  - Vesicular exocytosis and MEPCs
  - Action-potential sequences
  - Networks of cortical neurons
  - Loudness and brightness functions
  - Natural course of forgetting
- Medicine and human behavior
  - Human standing and human gait
  - Mood
  - Human heartbeat patterns
  - Movement patterns influenced by drugs

FRACTAL SHOT NOISE

IEEE TRANSACTIONS ON INFORMATION THEORY, VOL. 36, NO. 6, NOVEMBER 1990

Power-Law Shot Noise

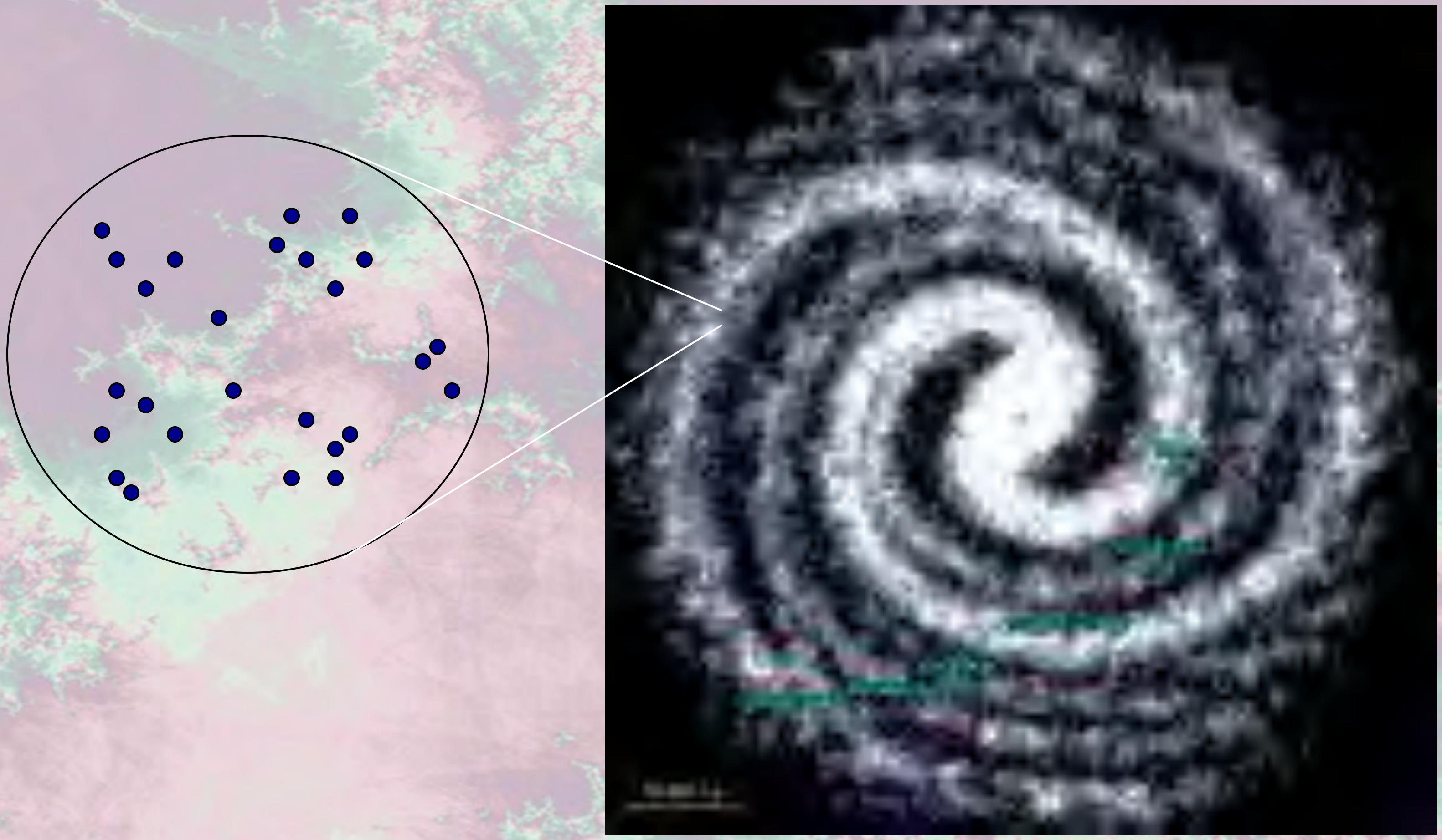
STEVEN B. LOWEN, STUDENT MEMBER, IEEE, AND MALVIN C. TEICH, FELLOW, IEEE



	$A > 0$, $B = \infty$	$A = 0$, $B = \infty$	$A = 0$, $B < \infty$	$A > 0$, $B < \infty$
$0 < \beta < \frac{1}{2}$ $(2 > \alpha > 1)$	$\Pr\{X = \infty\} = 1$ no $S(f)$			$X \to$ Gaussian $S \sim 1/f^\alpha$
$\frac{1}{2} \leq \beta < 1$ $(1 \geq \alpha > 0)$			$E[X^2] = \infty$ $S \sim 1/f^\alpha$	
$\beta = 1$			$E[X] = \infty$ no $S(f)$	$X \to$ Gaussian $S(f)$ not $1/f^\alpha$
$\beta > 1$ $(0 < \zeta < 1)$	$X \to$ Gaussian $S(f)$ not $1/f^\alpha$	$X =$ stable no $S(f)$	$X \to$ stable no $S(f)$	

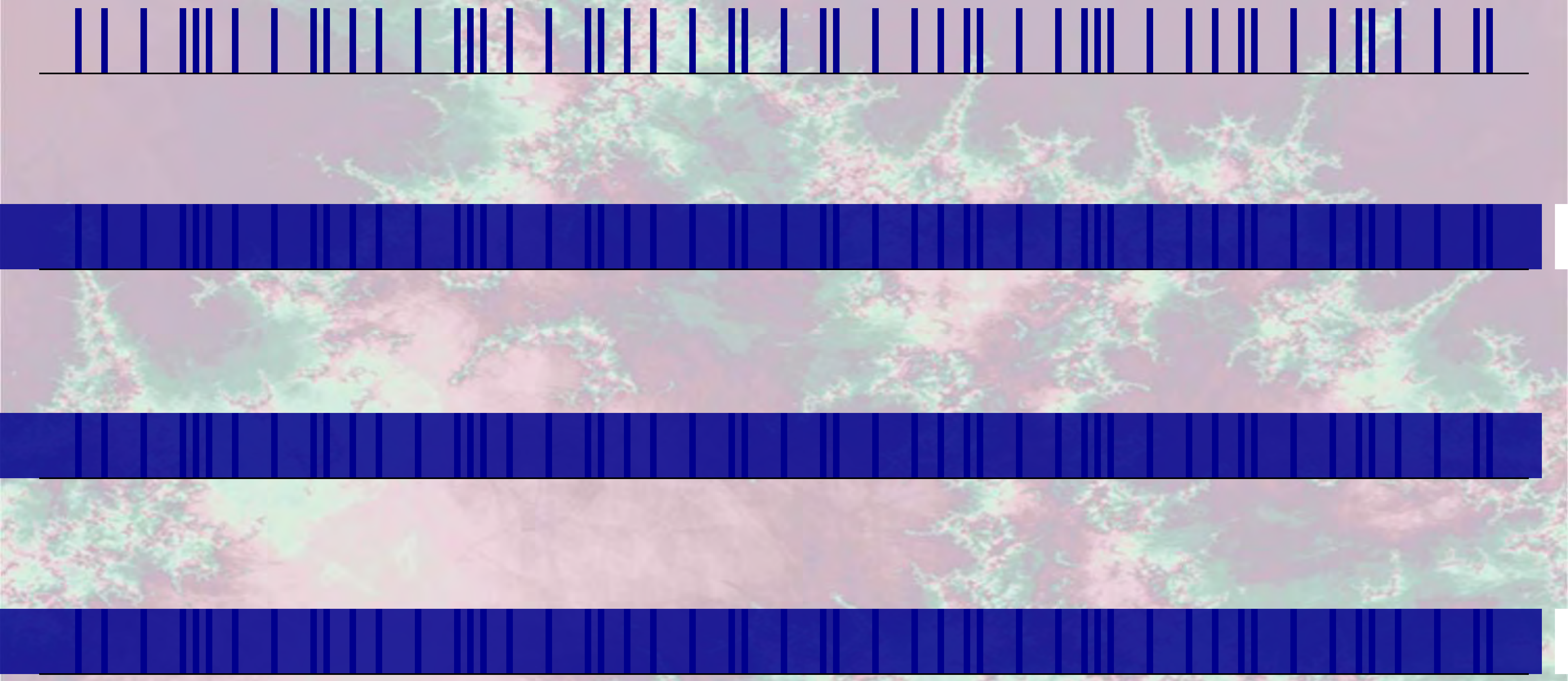
BOSTON UNIVERSITY

RANDOM POINT EVENTS IN SPACE AT MULTIPLE SCALES



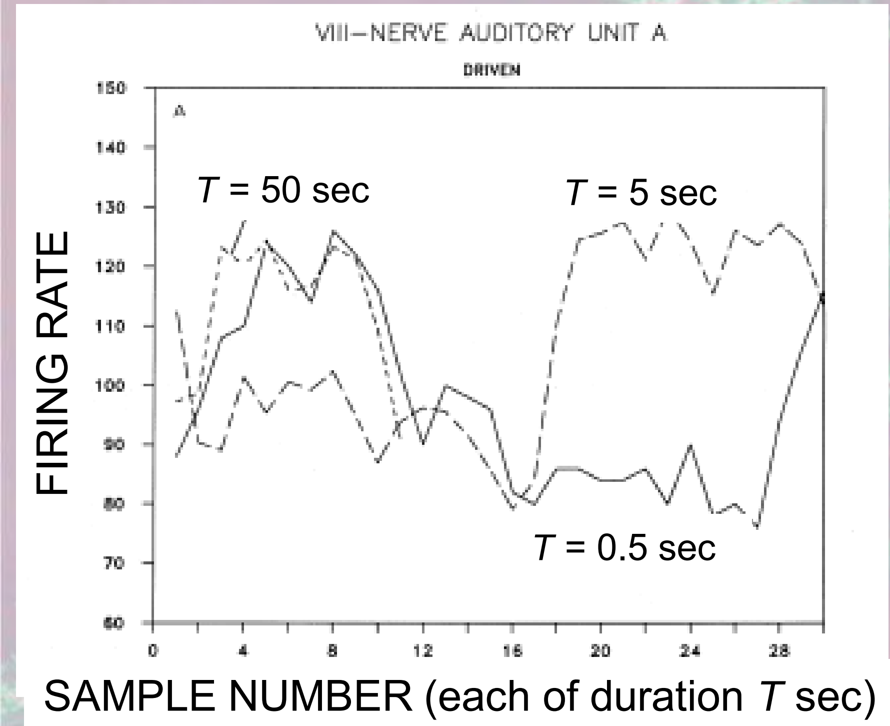
STARS, LIKE GALAXIES, TEND TO OCCUR IN CLUSTERS.

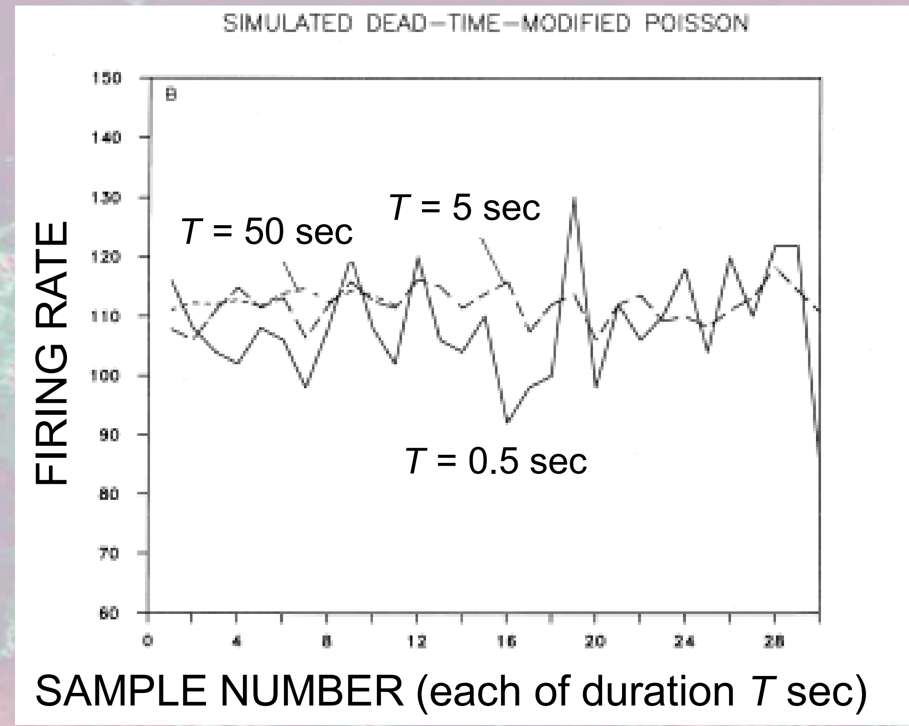
RANDOM POINT EVENTS IN TIME AT MULTIPLE SCALES

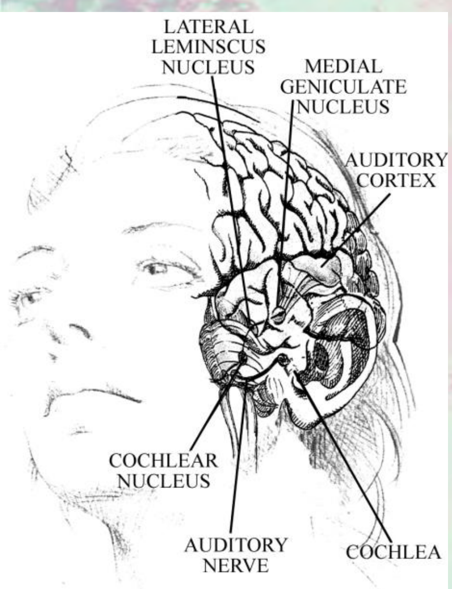


HOW DO THE RATE FLUCTUATIONS OF A SEQUENCE OF RANDOM POINT EVENTS BEHAVE AS THE COUNTING WINDOW DURATION INCREASES?

RATE FLUCTUATIONS IN AN AUDITORY NERVE FIBER







SNR_λ is observed to be essentially independent of T (counting window duration)

For Poisson: $\mathrm{var}(n) \equiv \sigma_n^2 = \bar{n}$

$\mathrm{SNR} = \bar{n}/\sigma_n = \sqrt{n}$

Rate $\lambda \equiv n/T$ so $\bar{\lambda} = \bar{n}/T$ and $\sigma_\lambda = \sigma_n/T$

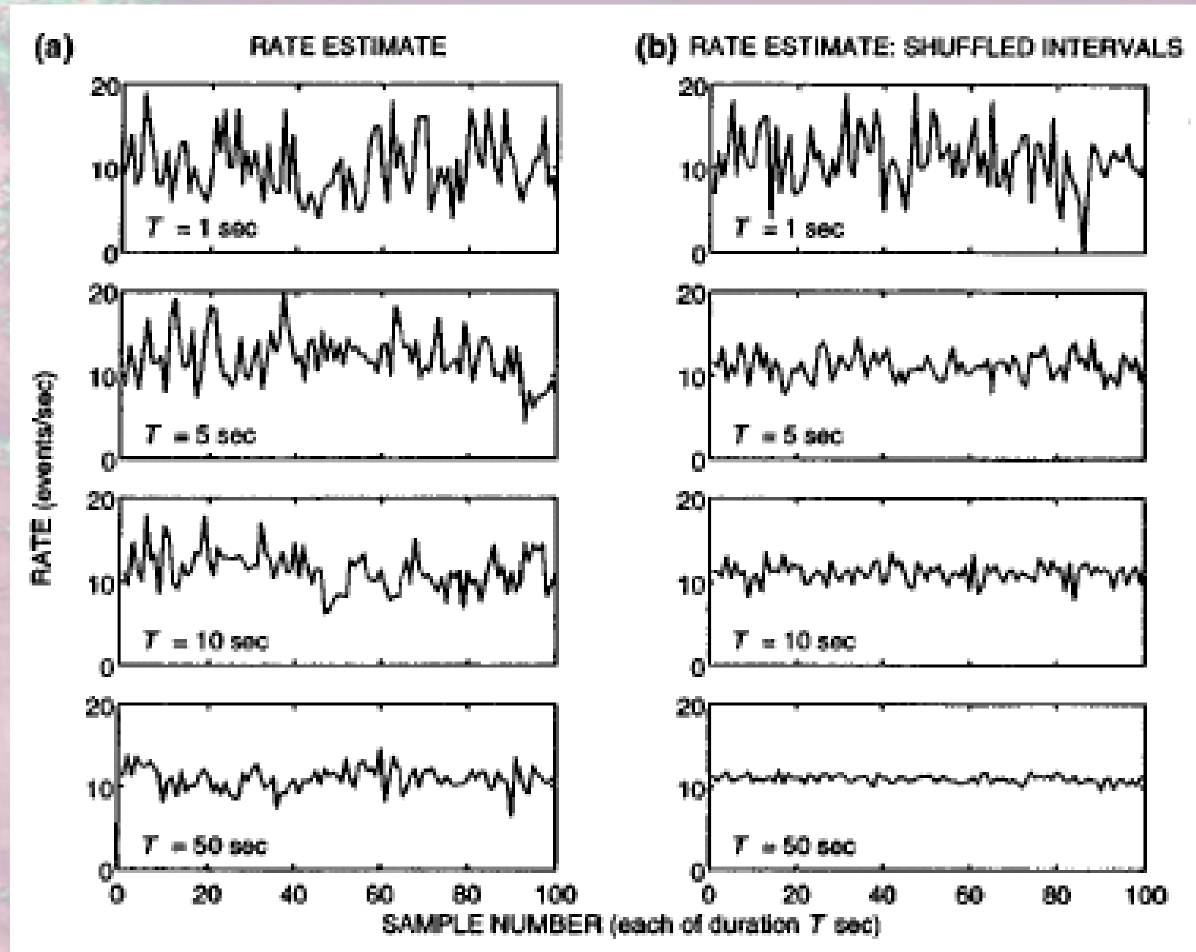
Thus, $\mathrm{SNR}_\lambda \equiv \bar{\lambda}/\sigma_\lambda = \mathrm{SNR}_n \propto T^{1/2}$

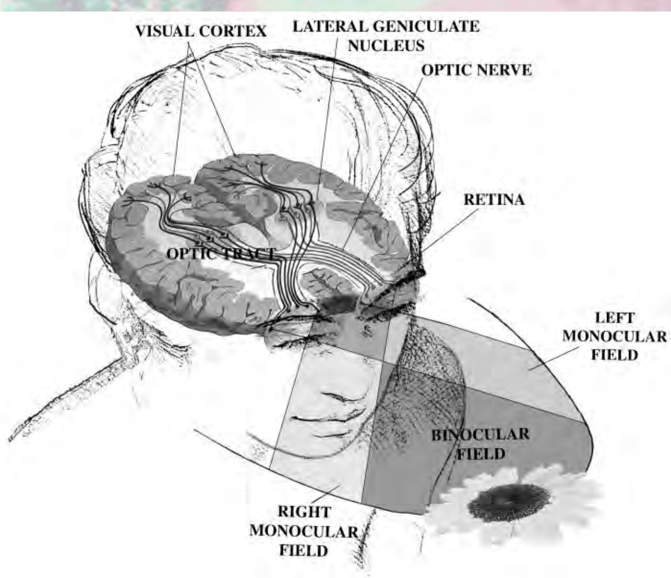
Hence, $\mathrm{SNR}_\lambda \uparrow$ as $T \uparrow$

After Teich, "Fractal Character of the Auditory Neural Spike Train," *IEEE Trans. Biomed. Eng.* **36**, 150-160 (1989); and Teich, Johnson, Kumar, and Turcott, "Rate Fluctuations and Fractional Power-Law Noise Recorded from Cells in the Lower Auditory Pathway of the Cat," *Hearing Res.* **46**, 41-52 (1990).

RATE FLUCTUATIONS IN A VISUAL-SYSTEM NERVE FIBER (RGC MAINTAINED DISCHARGE)

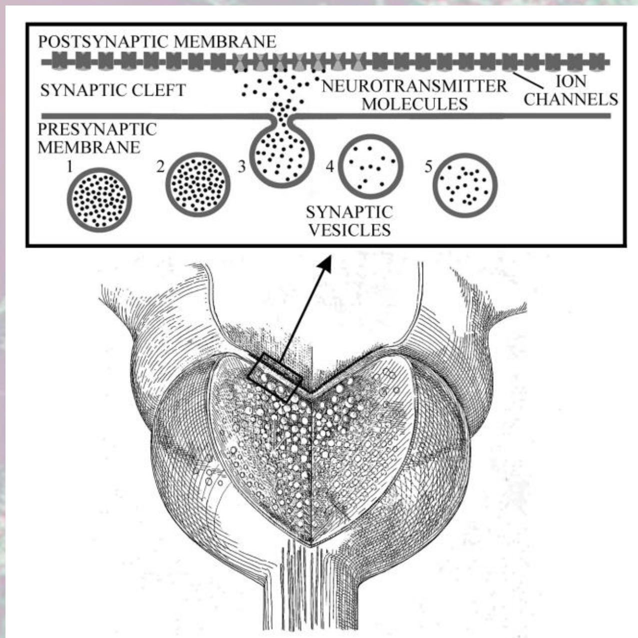
SNR_{λ} increases with T, but far more slowly than for the shuffled intervals (Poisson data)

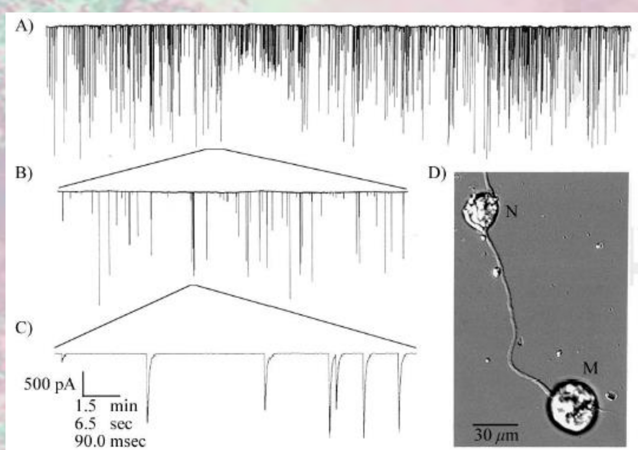


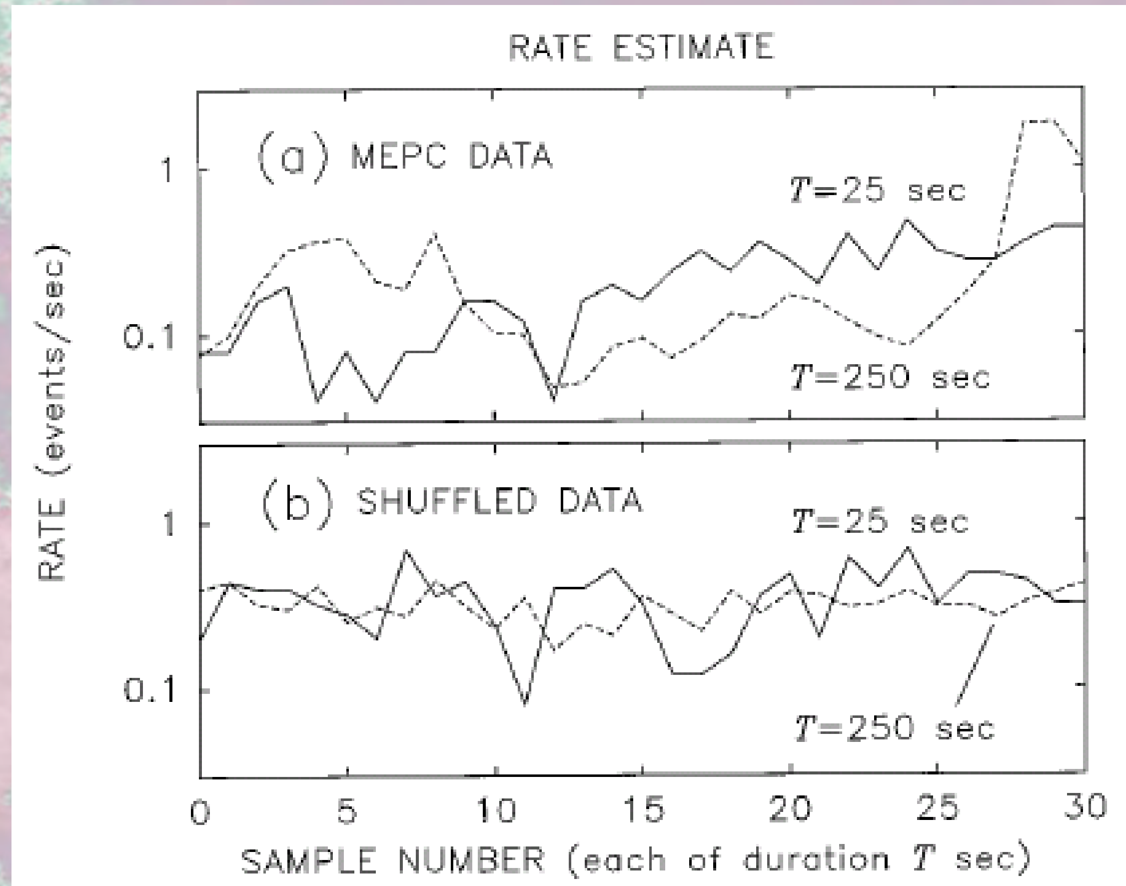


After Teich, Heneghan, Lowen, Ozaki, and Kaplan, "Fractal Character of the Neural Spike Train in the Visual System of the Cat," *J. Opt. Soc. Am. A* 14, 529-546 (1997).

RATE FLUCTUATIONS OF VESICULAR NEUROTRANSMITTER EXOCYTOSIS AT *XENOPUS* NEUROMUSCULAR JUNCTION



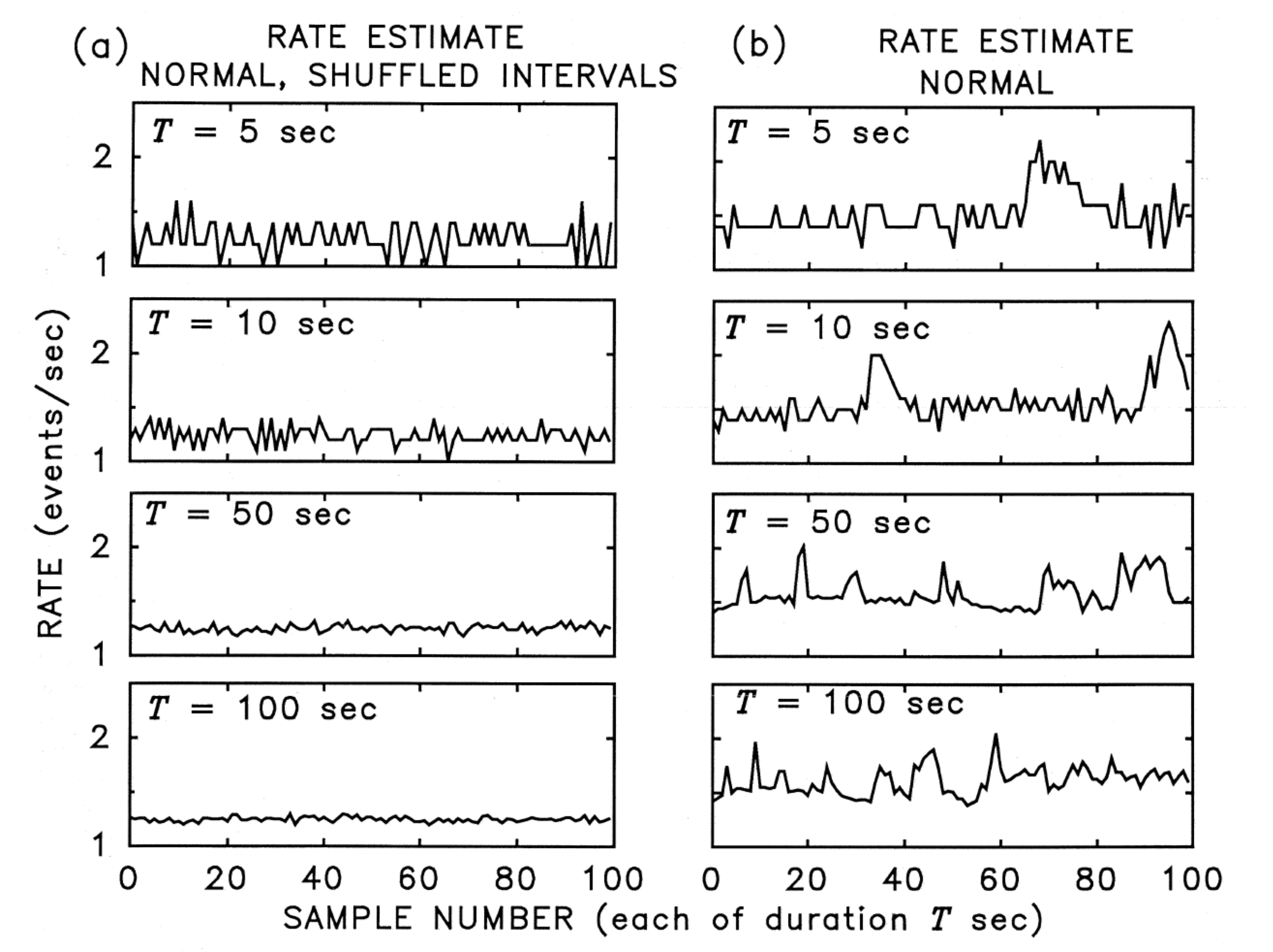




SNR_λ is essentially independent of T for the miniature end-plate current (MEPC) data

After Lowen, Cash, Poo, and Teich, "Quantal Neurotransmitter Secretion Rate Exhibits Fractal Behavior," *J. Neurosci.* **17**, 5666-5677 (1997).

RATE FLUCTUATIONS IN THE HUMAN HEARTBEAT

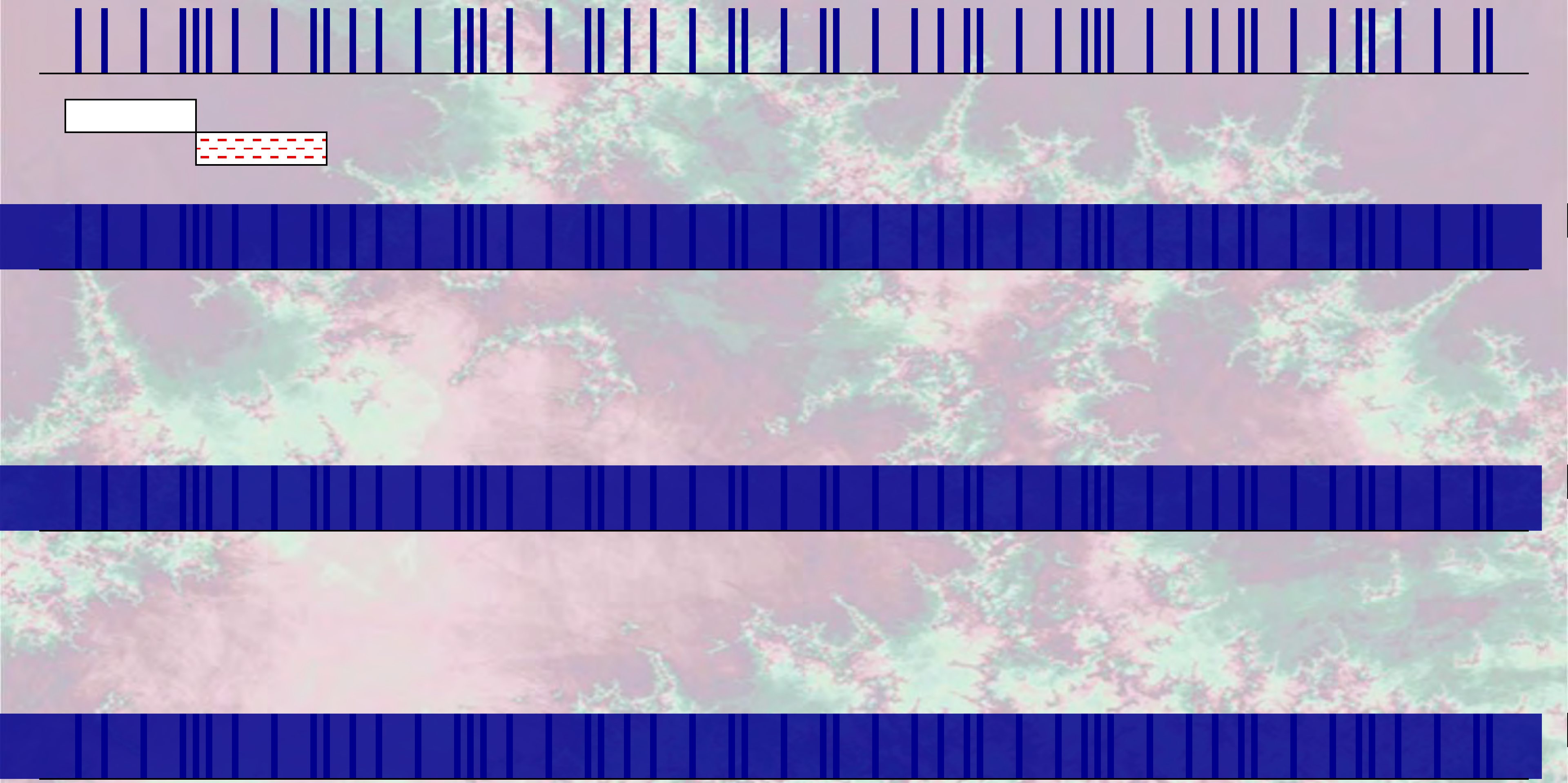


SNR$_\lambda$ is nearly independent of T for the human heart rate

After Turcott and Teich, "Fractal Character of the Electrocardiogram: Distinguishing Heart-Failure and Normal Patients," *Ann. Biomed. Eng.* **24**, 269-293 (1996).

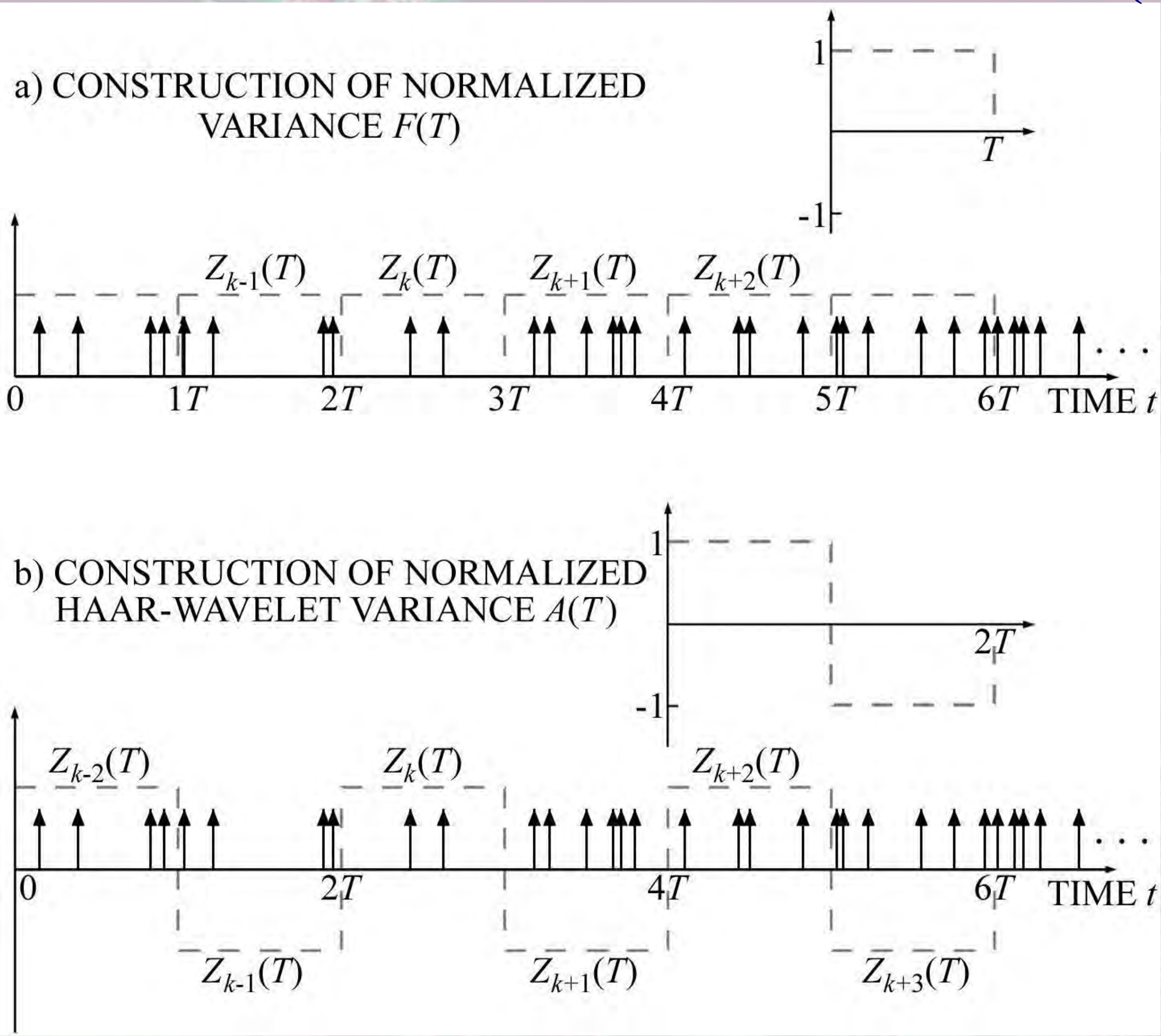
BOSTON UNIVERSITY

COUNTING POINT EVENTS IN A DIFFERENTIATING TIME WINDOW [NORMALIZED HAAR-WAVELET VARIANCE $A(T)$]

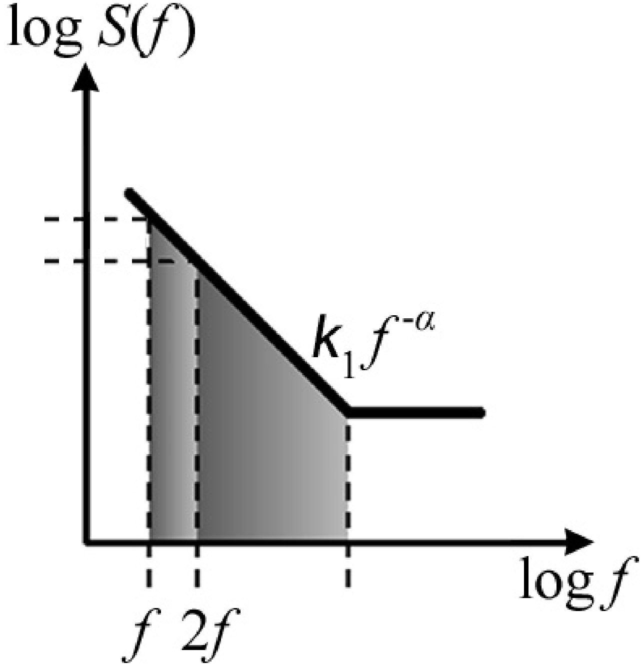


Random point events in time often exhibit self-scaling that takes the form of power-law behavior in their statistics (e.g., spectrum and count variance-to-mean ratio):
FRACTAL BEHAVIOR OF POINT EVENTS

NORMALIZED VARIANCE *F*(*T*) AND NORMALIZED HAAR-WAVELET VARIANCE *A*(*T*)



POWER-LAW SCALE INVARIANCE

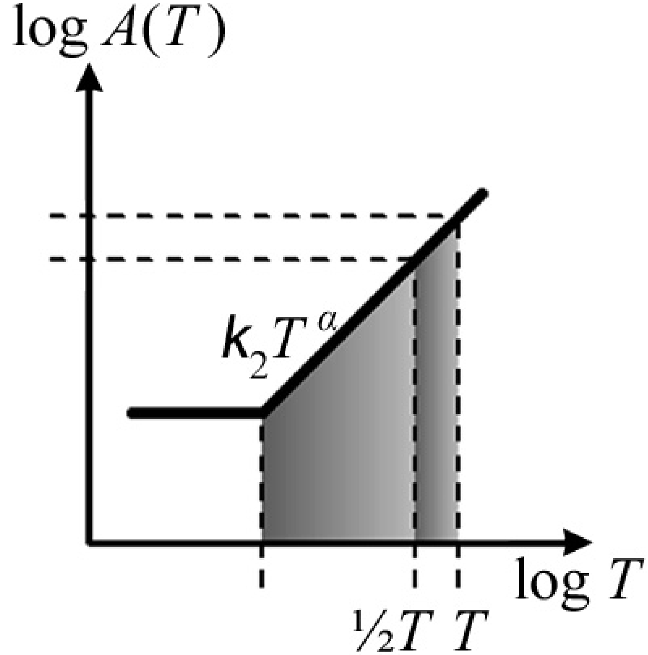


$$\frac{S(f)}{S(2f)} = \frac{k_1 f^{-\alpha}}{k_1 (2f)^{-\alpha}} = 2^{\alpha}$$

INDEPENDENT OF f

↓

SCALE INVARIANT



$$\frac{A(T)}{A(T/2)} = \frac{k_2 T^{\alpha}}{k_2 (T/2)^{\alpha}} = 2^{\alpha}$$

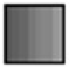
INDEPENDENT OF T

↓

SCALE INVARIANT

Pareto

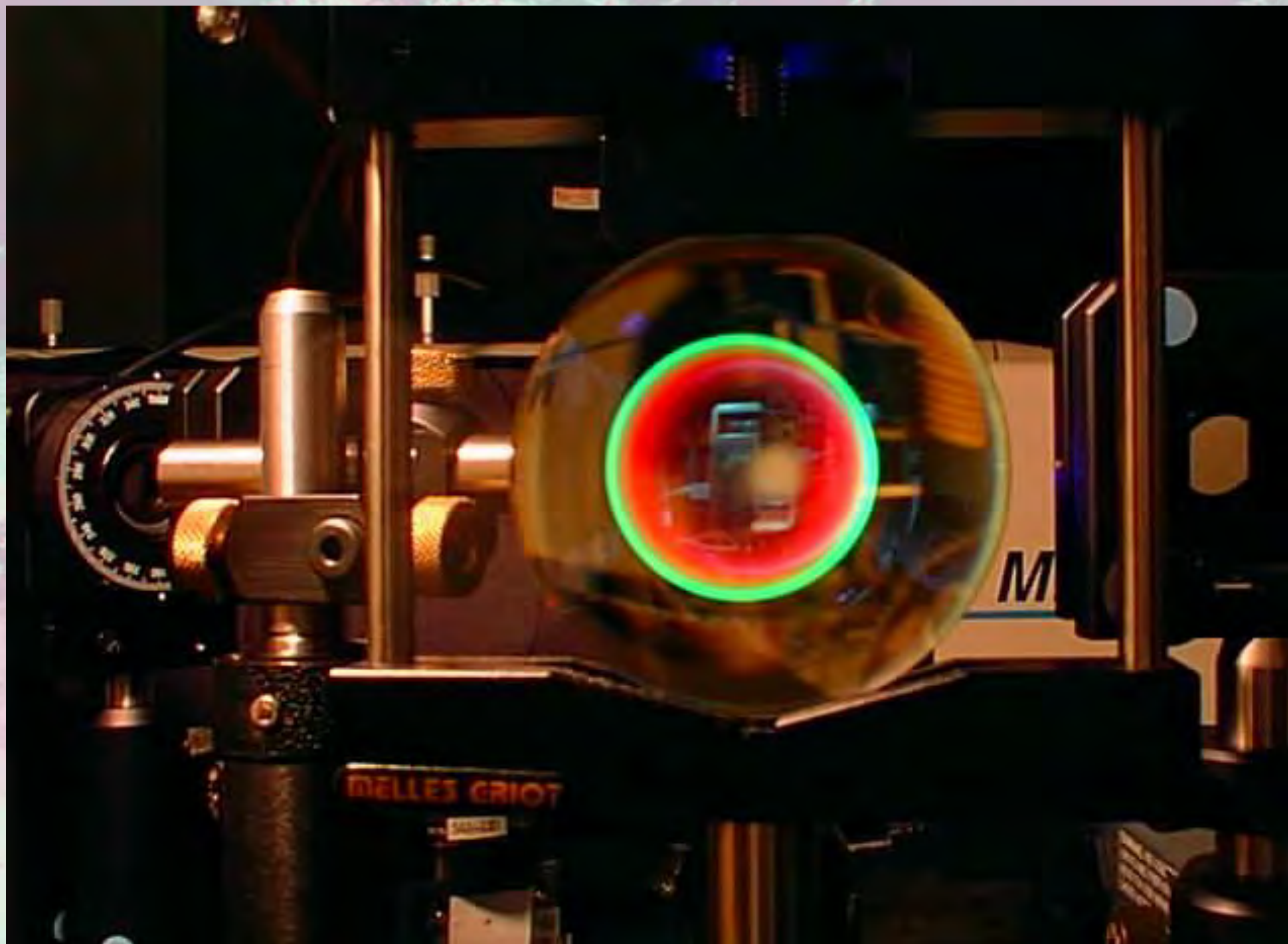
α = SCALING EXPONENT

 = SCALING RANGE

BOSTON UNIVERSITY

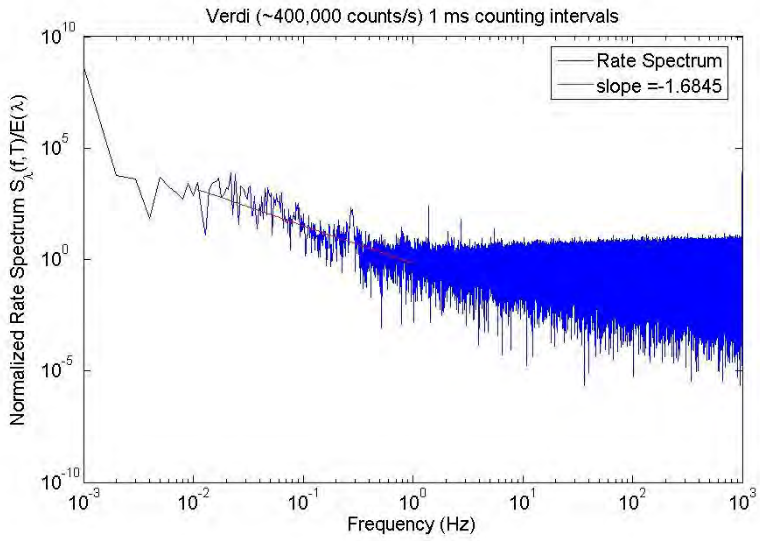
4. FRACTAL PHOTONS

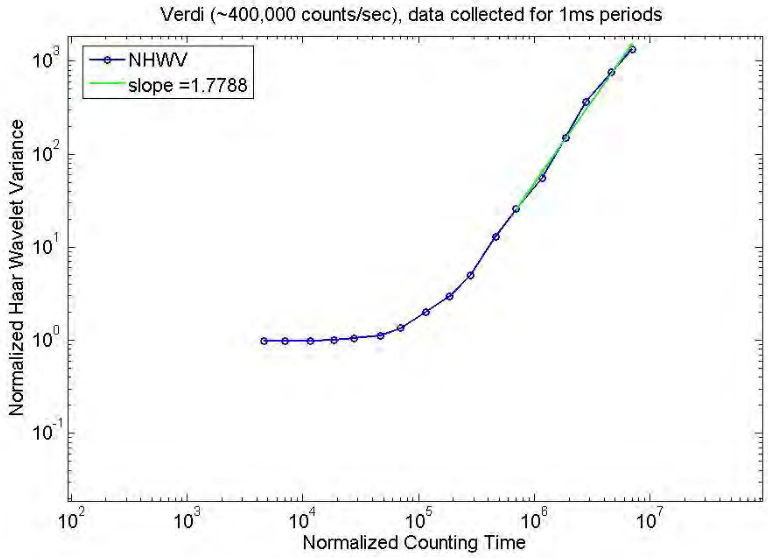
BU PHOTONICS CENTER
QUANTUM PHOTONICS LAB – ROOM 733



LASER

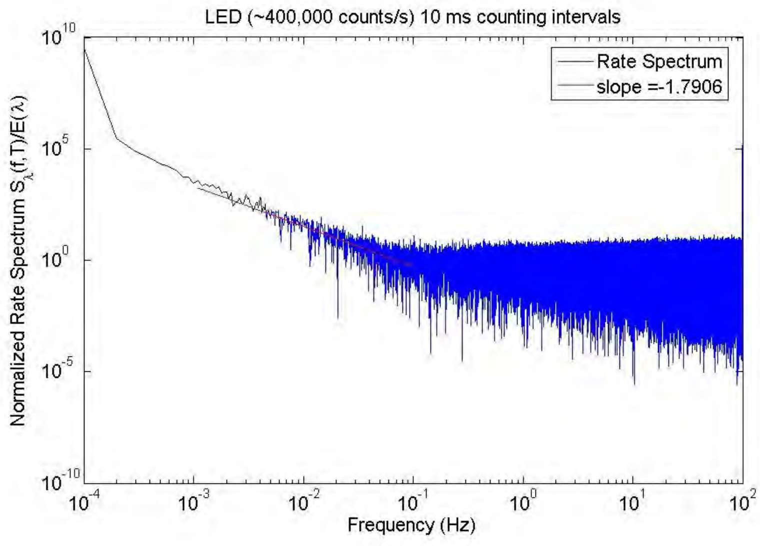
FREQUENCY-DOUBLED $Nd^{3+}:YVO_4$ LASER OPERATING AT 532 nm

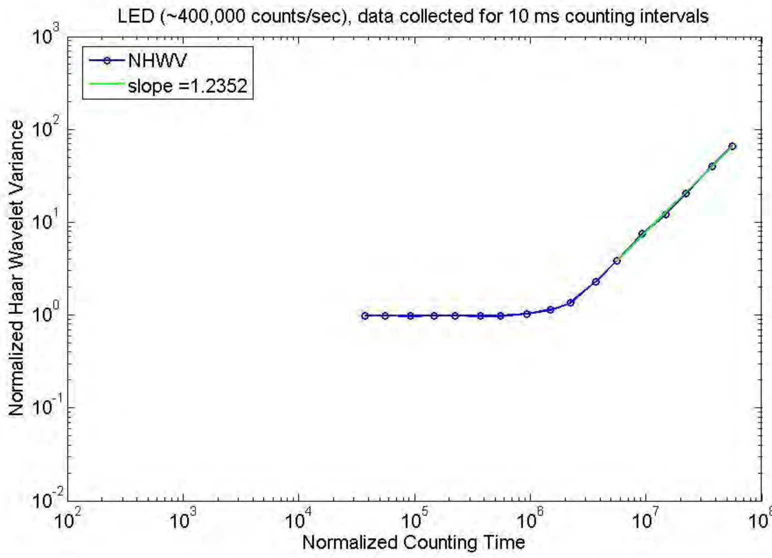




LED

COMMERCIAL BLUE LED WITH CENTRAL WAVELENGTH OF 430 nm





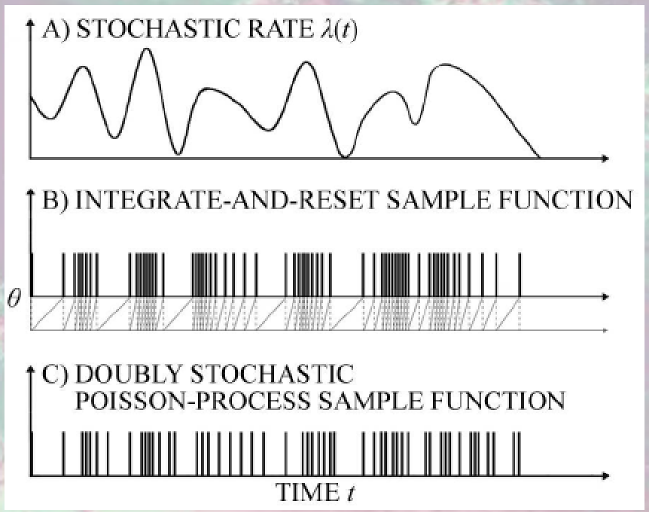
ORIGINS OF FRACTAL BEHAVIOR

- Empirical power-law behavior
- Diffusion
- Convergence to stable (Lévy) distributions
- Lognormal distribution
- Self-organized criticality
- Highly optimized tolerance
- Scale-free networks
- Superposition of relaxation processes

FRACTAL-BASED POINT-PROCESSES – MODELS

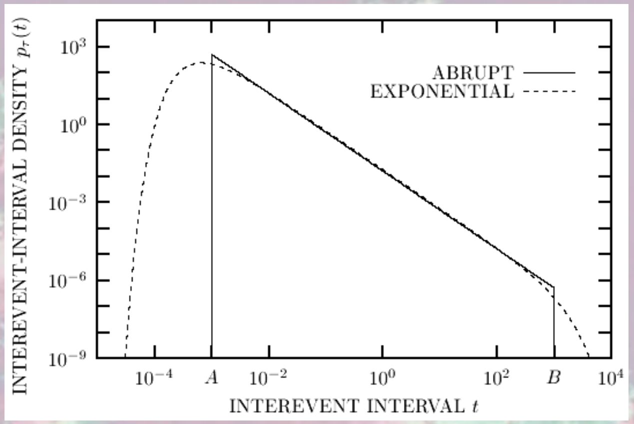
Fractal Gaussian Noise

Kolmogorov



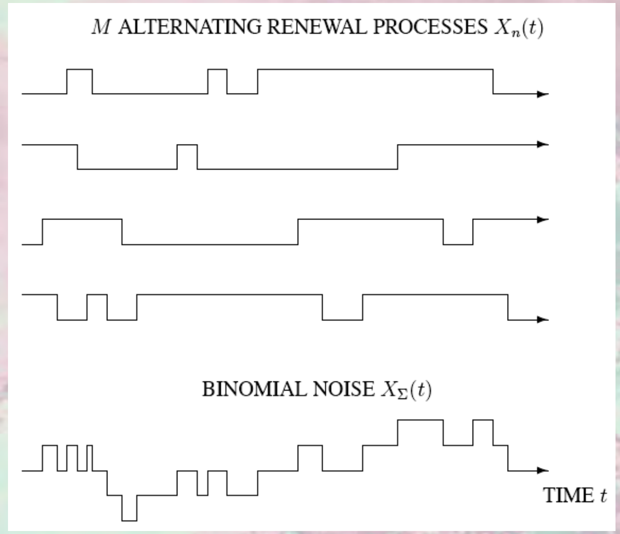
Fractal Renewal Process

Mandelbrot



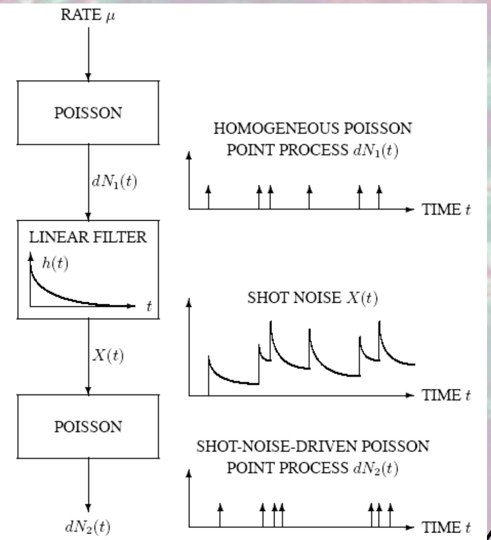
Alternating Fractal Renewal Process

Gauss



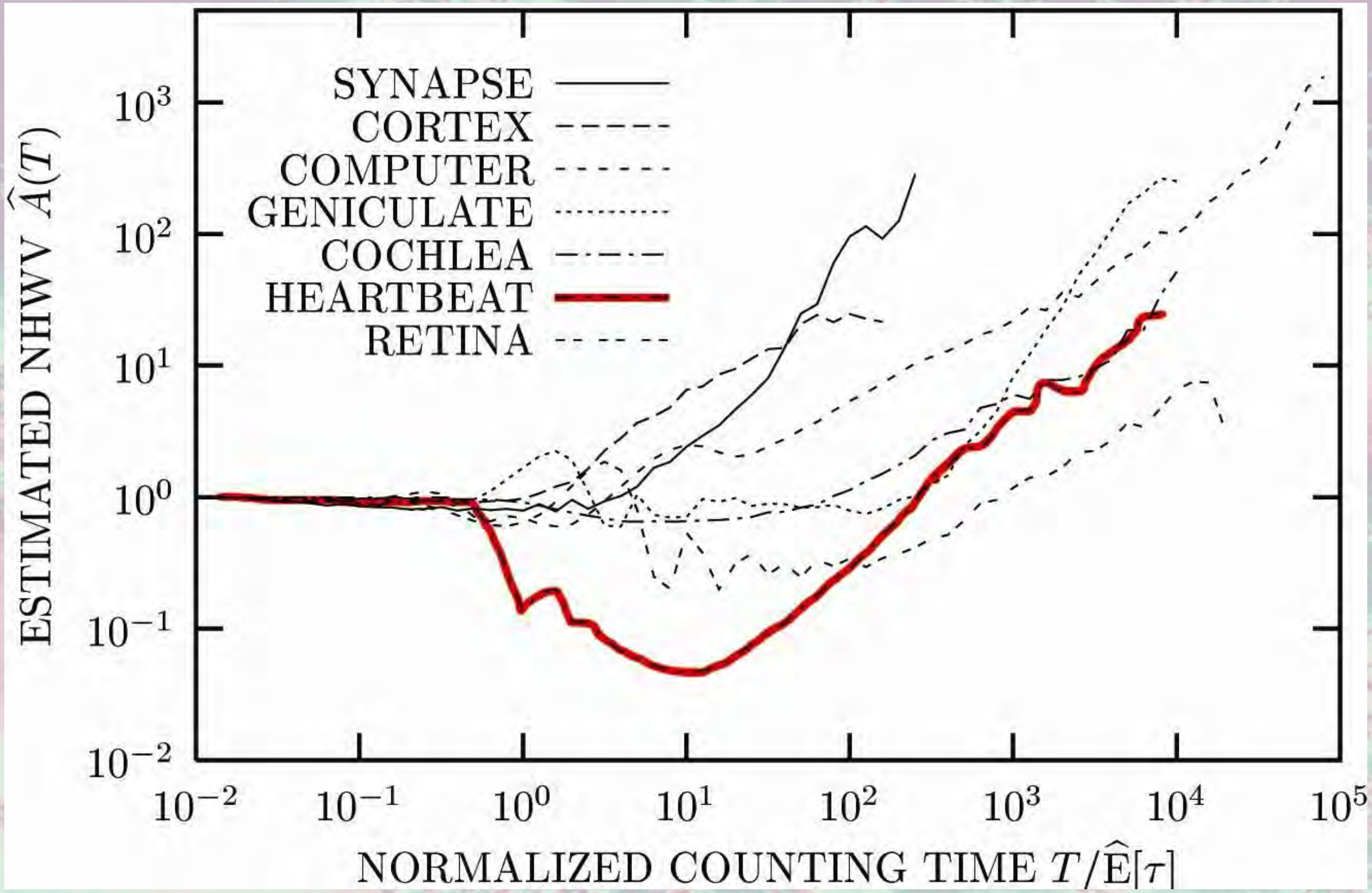
Fractal Shot-Noise-Driven Poisson

Bartlett



5. HEART RATE VARIABILITY

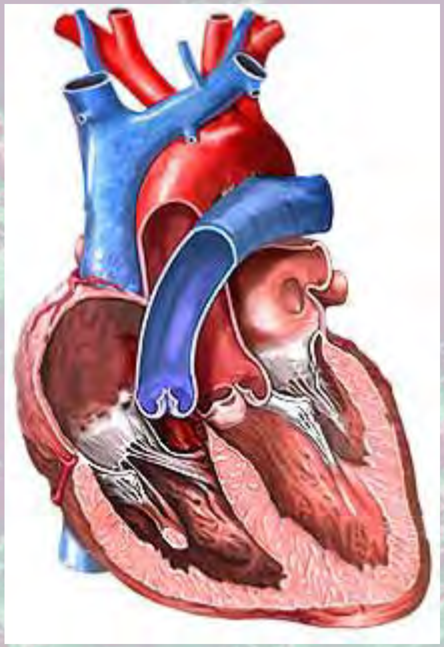
CAN BE STUDIED VIA:
COUNTS -- Duration of time window selected by experimenter affects observation
or
TIME INTERVALS: More exhaustive since all information is retained



After Lowen & Teich, *Fractal-Based Point Processes* (Wiley, 2005).

CONGESTIVE HEART FAILURE

INABILITY OF HEART TO INCREASE CARDIAC OUTPUT IN PROPORTION TO METABOLIC DEMANDS



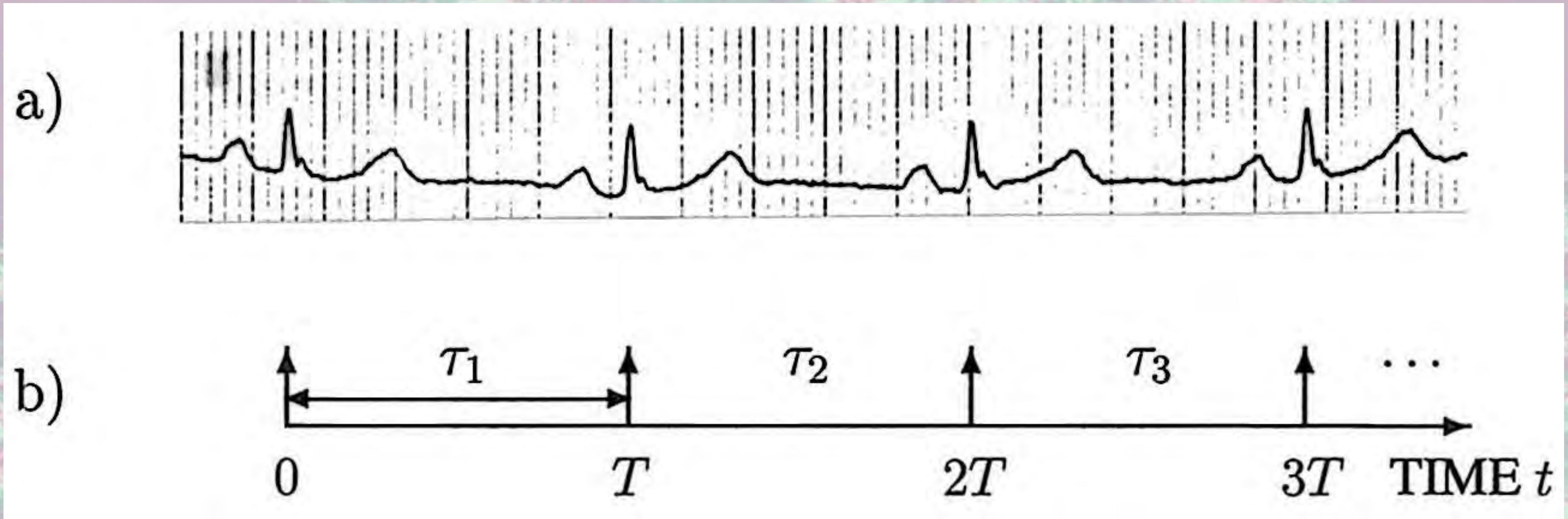
Symptom complex:
Many different presentations and etiologies

Typical symptoms:
- Shortness of breath
- Swelling in legs
- General fatigue and weakness

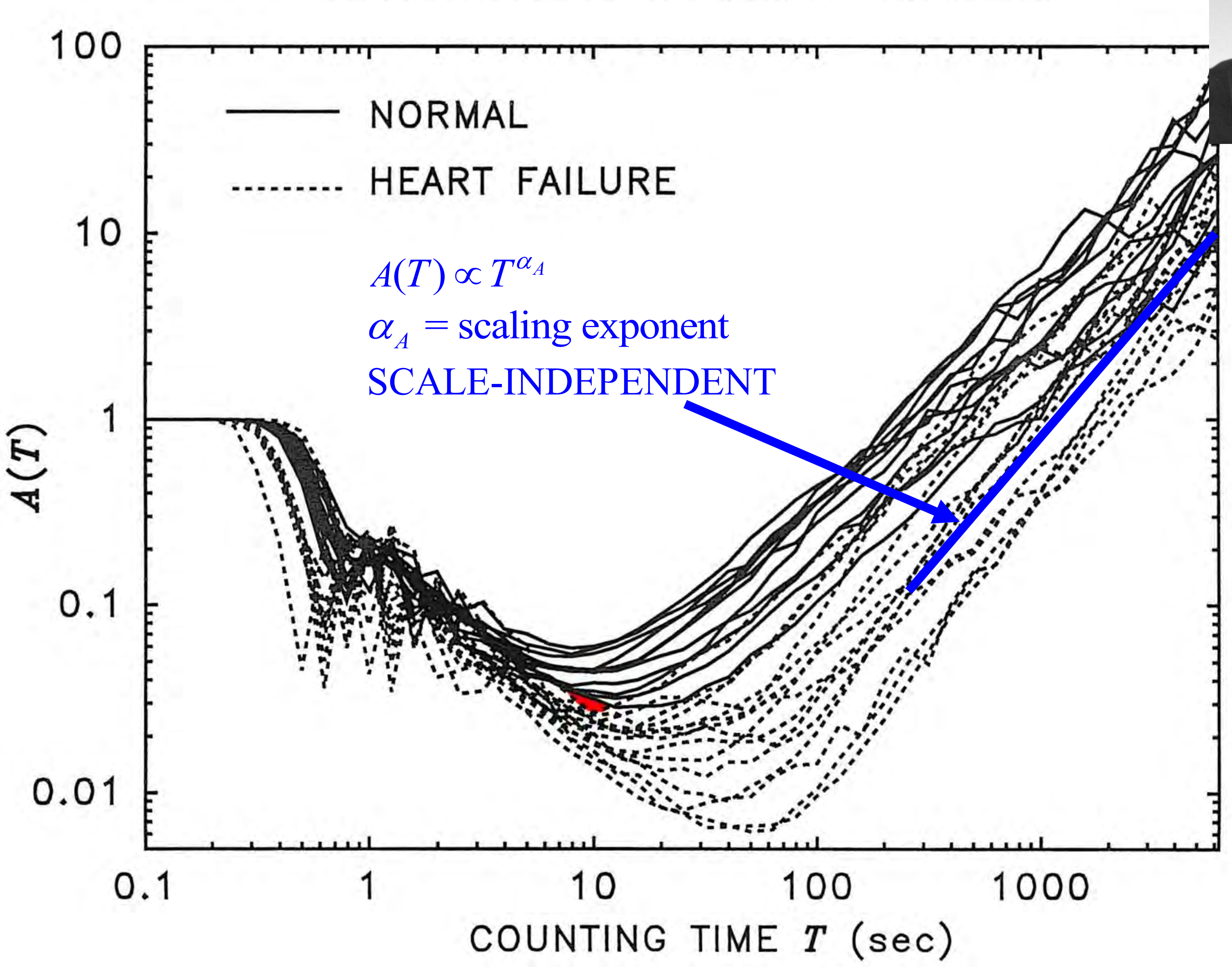
Clinical diagnostics:
- Ascultate heart
- Carotid pulse
- Electrocardiogram
- Chest radiograph

Collaborators:
- **Steven Lowen, Harvard Medical School**
- **Conor Heneghan, University College Dublin**
- **Robert Turcott, Stanford Medical School**
- **Markus Feurstein, Wirtschaftsuniversität Wien**
- **Stefan Thurner, Allgemeines Krankenhaus Wien**

ELECTROCARDIOGRAM



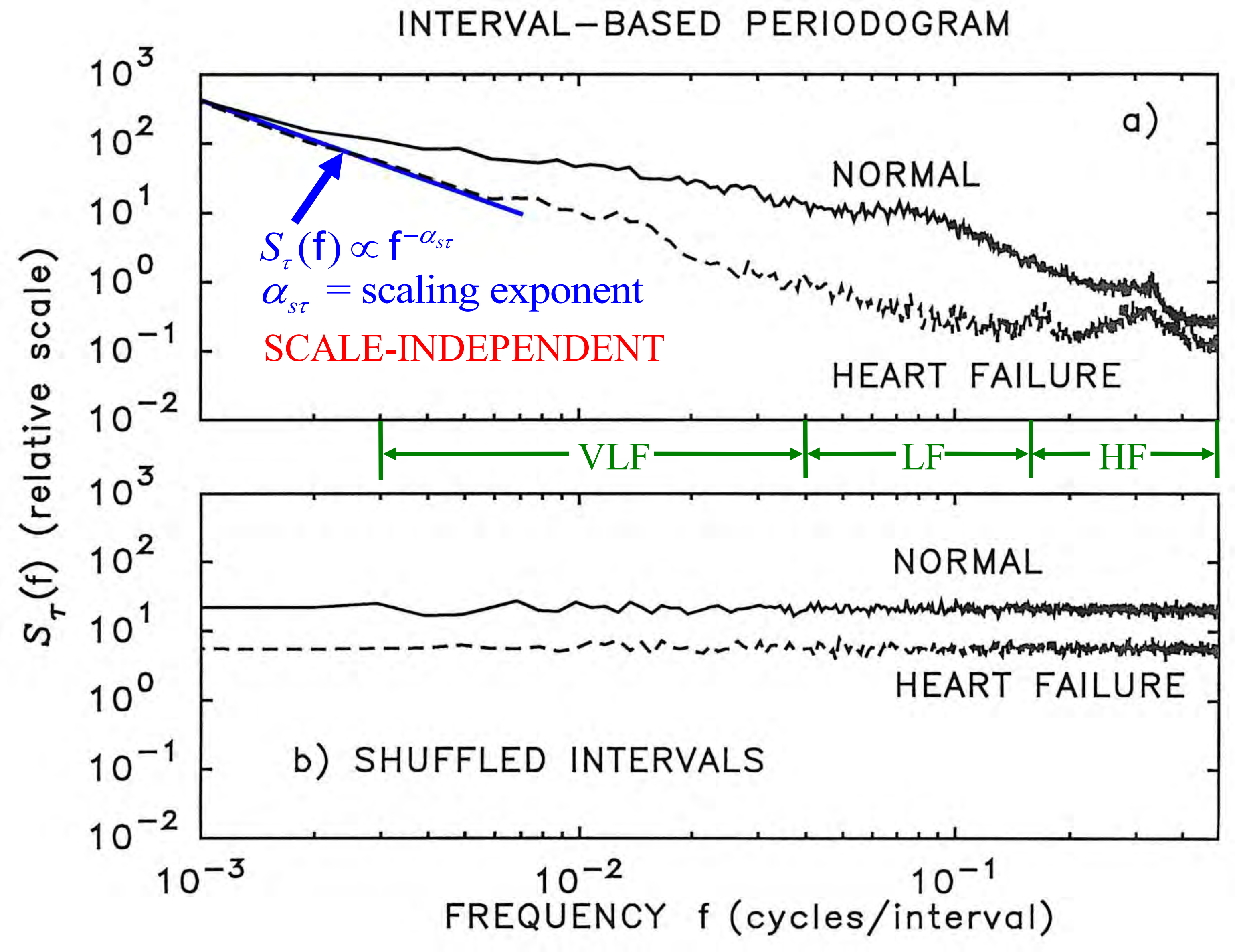
NORMALIZED HAAR-WAVELET VARIANCE



Allan

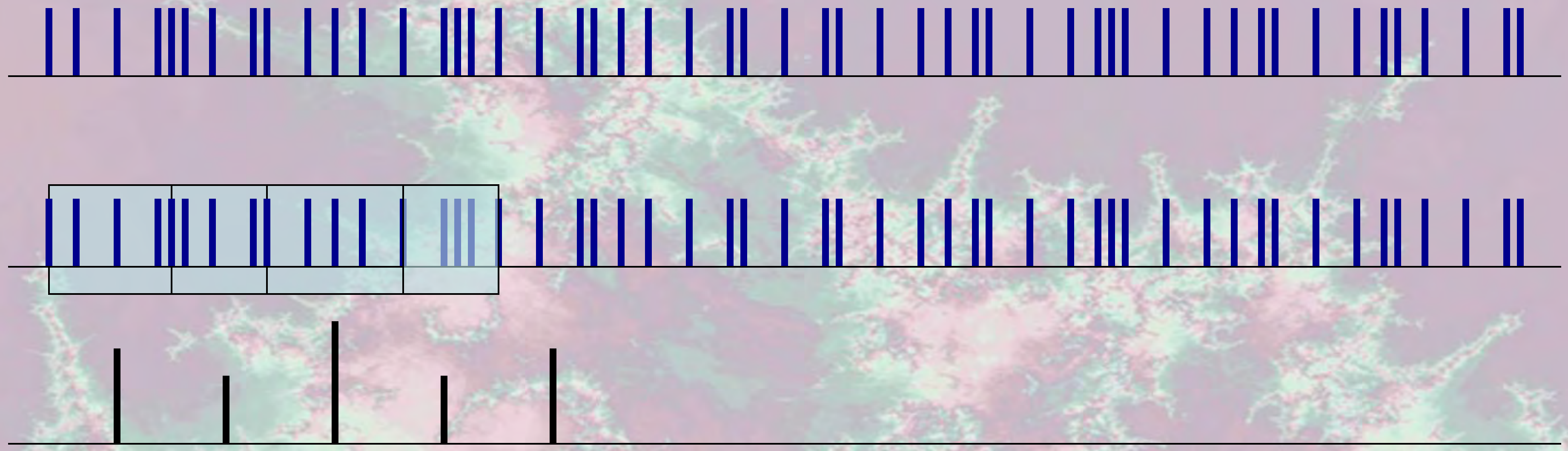
After Turcott & Teich, *Ann. Biomed. Eng*. **24**, 269-293 (1996).

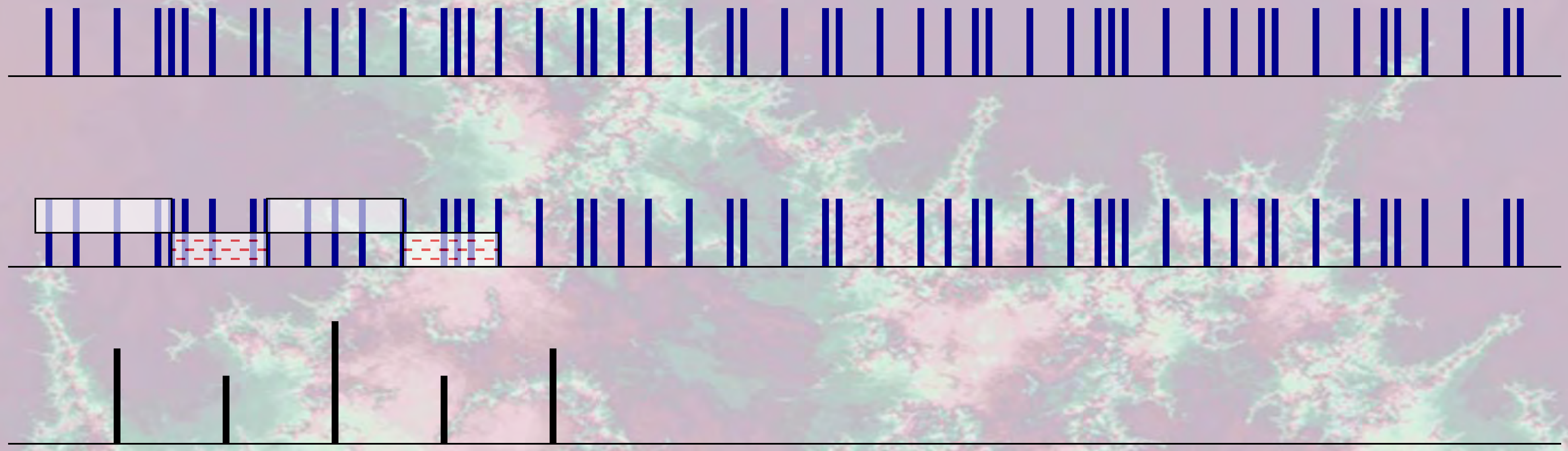
INTERVAL-BASED MEASURES: SPECTRUM



Fourier

INTERVAL-BASED HAAR WAVELET





INTERVAL-BASED TIME-SCALE ANALYSIS

Haar

DISCRETE WAVELET TRANSFORM

EXAMINES ALL SCALES

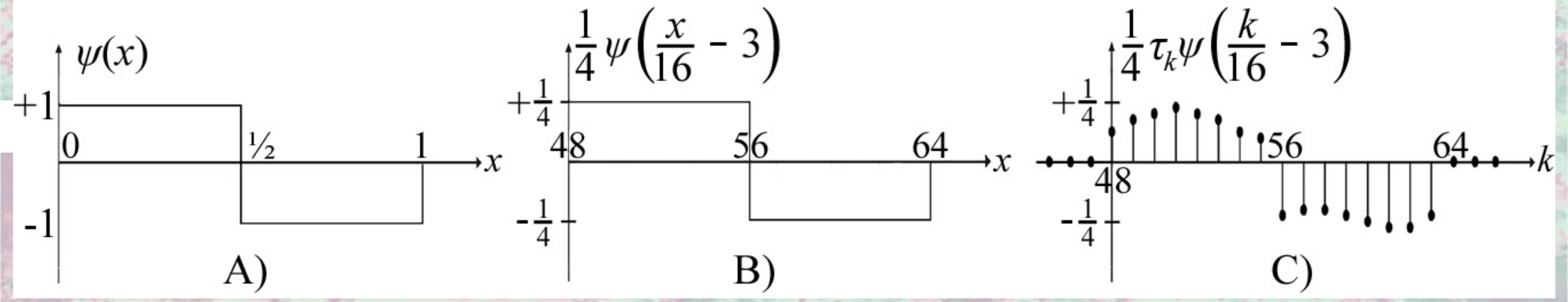
MITIGATES AGAINST NONSTATIONARITIES

m = scale index; 2^m = scale

$$W_{\psi,\tau}^{\mathrm{wav}}(m,i) = \sum_k 2^{-m/2}\,\psi(2^{-m}k - i)\,\tau_k$$

$$\sigma_{\mathrm{wav}}^2 \equiv \mathrm{Var}\left[W_{\psi,\tau}^{\mathrm{wav}}(m,i)\right] = 2^{-m}\sum_k\sum_l \psi(2^{-m}k - i)\,\psi(2^{-m}l - i)\,R_\tau(l-k)$$

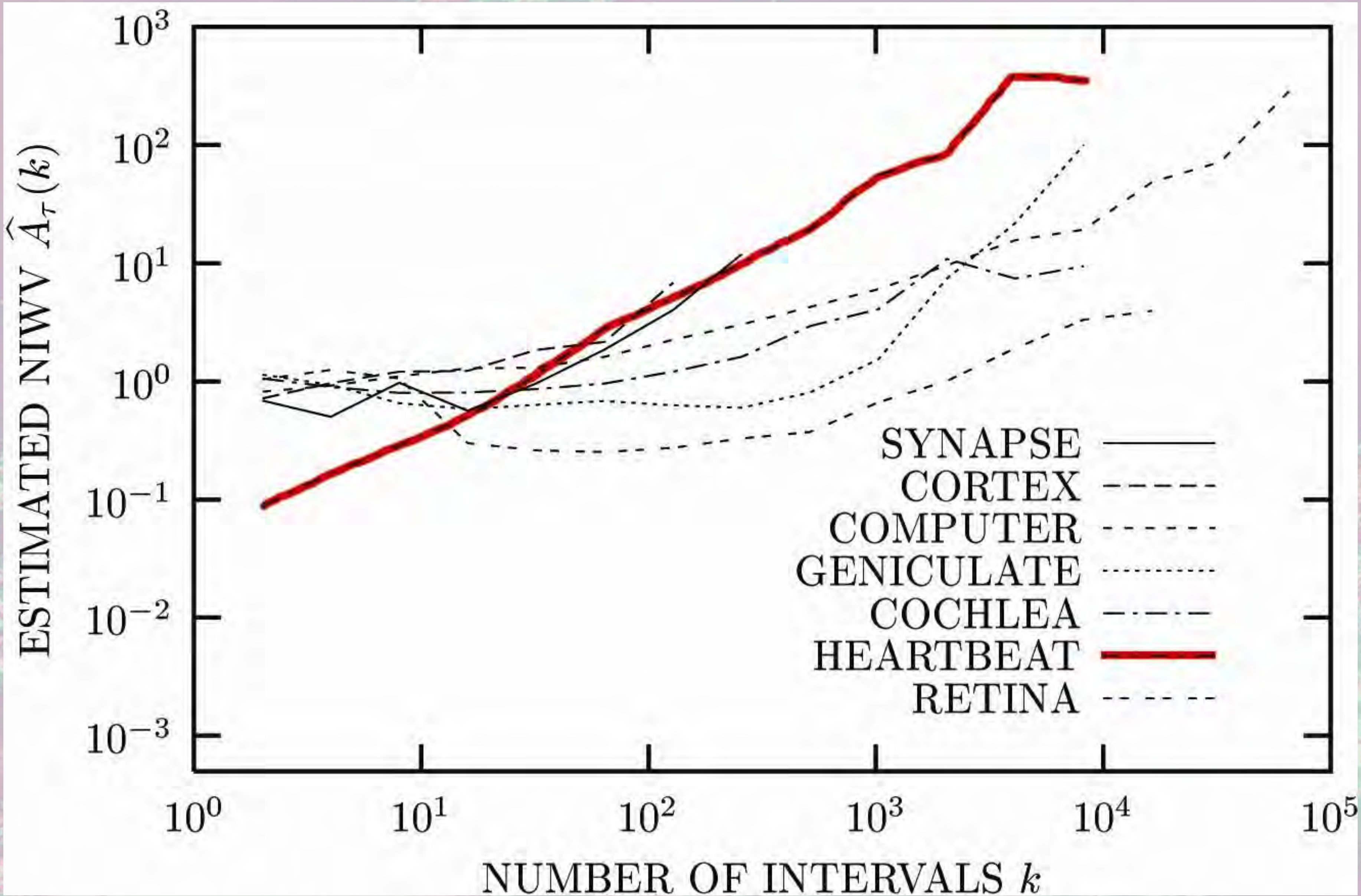
$$A_\tau(k) \equiv \mathrm{Var}\left[W_{\psi,\tau}^{\mathrm{wav}}(m,i)\right] \Big/ \mathrm{Var}\,[\tau]$$



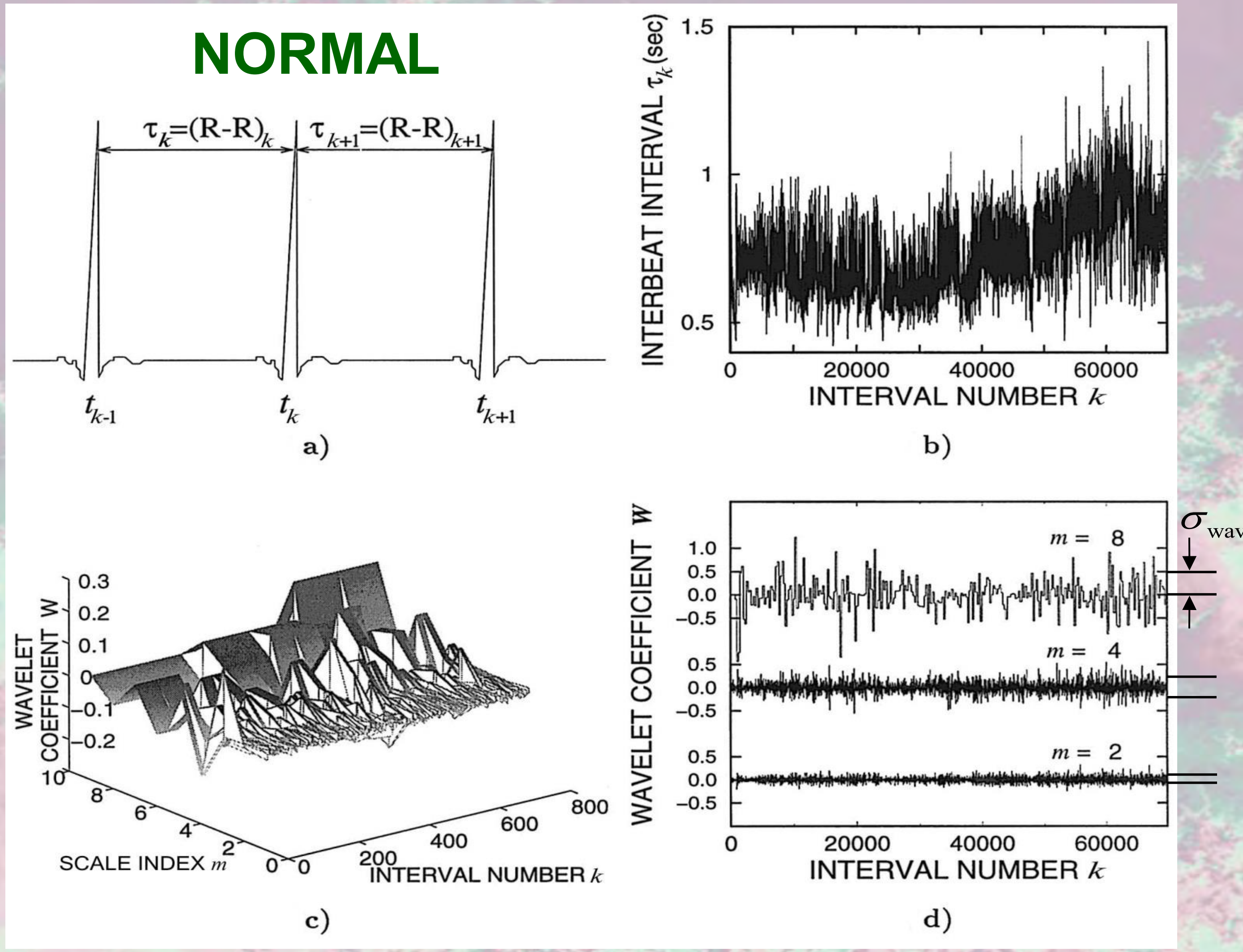
After Teich, Lowen, Jost, Vibe-Rheymer, and Heneghan, "Heart-Rate Variability: Measures and Models," in *Nonlinear Biomedical Signal Processing*, Vol. **II**, *Dynamic Analysis and Modeling*, edited by M. Akay (IEEE Press, New York, 2001), ch. 6, pp. 159-213.

BOSTON UNIVERSITY

INTERVAL-BASED MEASURES: NIWV



After Lowen & Teich, *Fractal-Based Point Processes* (Wiley, 2005).

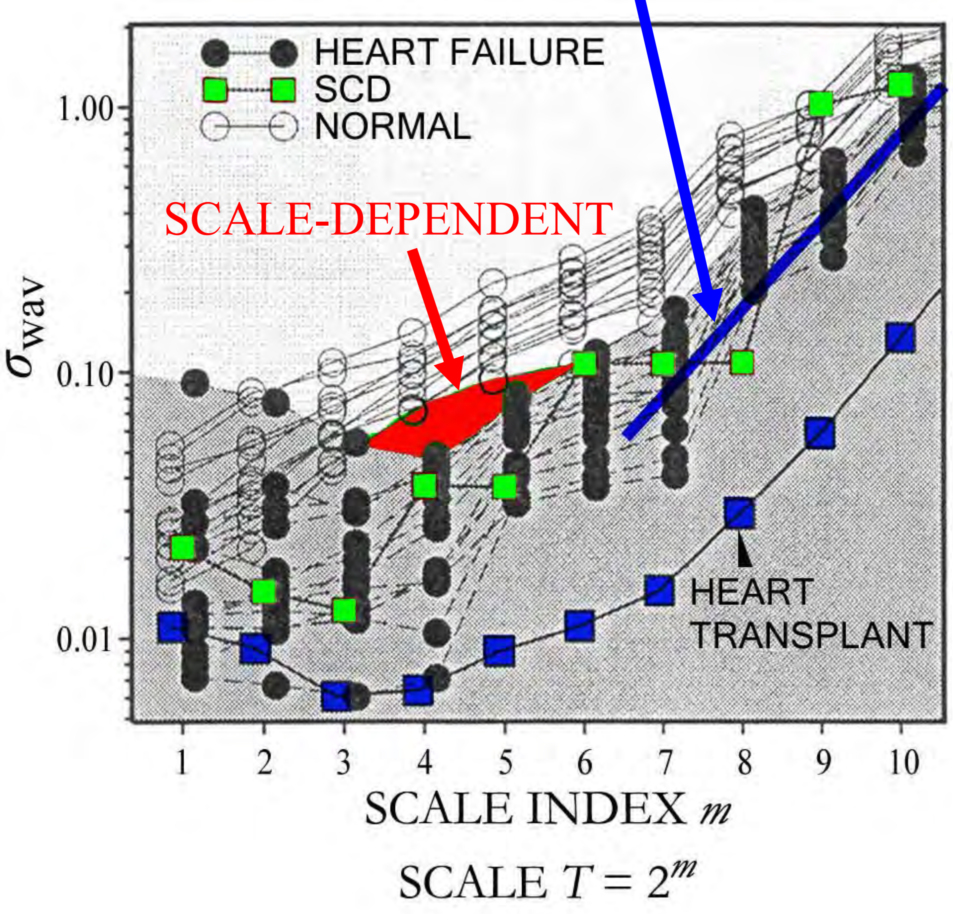


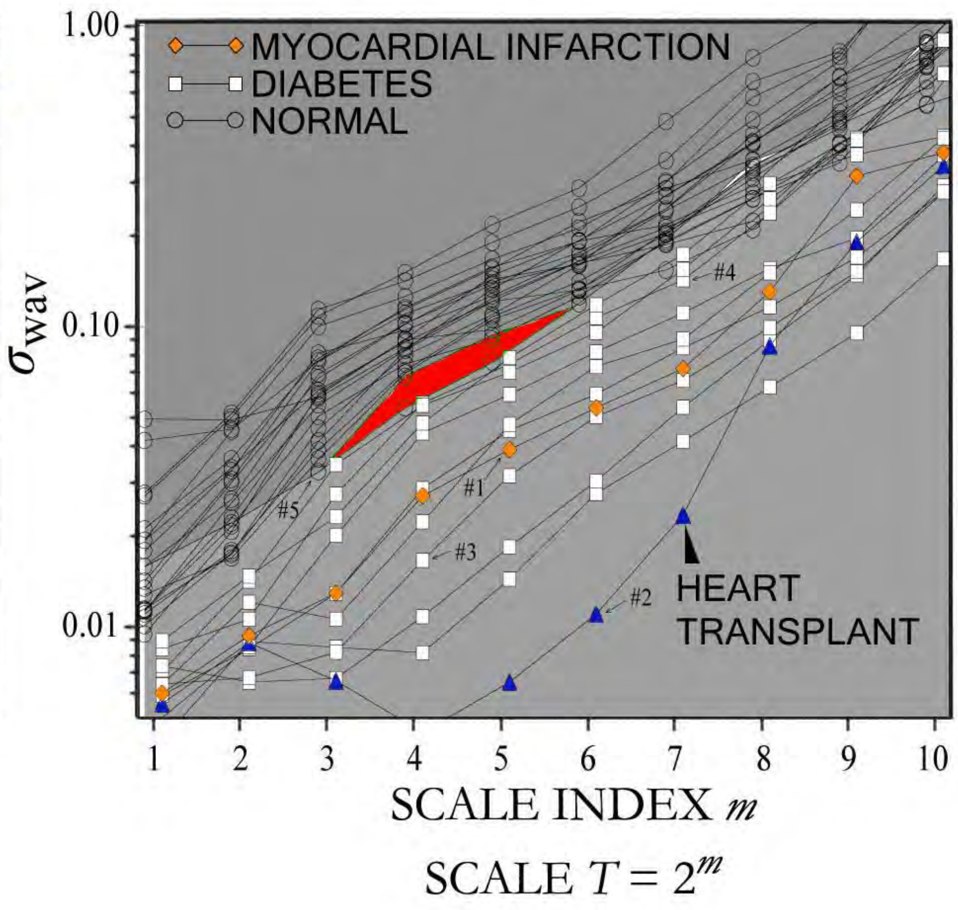
After Teich, Lowen, Jost, Vibe-Rheymer, and Heneghan, "Heart-Rate Variability: Measures and Models," in *Nonlinear Biomedical Signal Processing*, Vol. **II**, *Dynamic Analysis and Modeling*, edited by M. Akay (IEEE Press, New York, 2001), ch. 6, pp. 159-213.

$\sigma^2_{\mathrm{wav}}(T) \propto T^{\alpha_{A\tau}}$

$\alpha_{A\tau}$ = scaling exponent

SCALE-INDEPENDENT





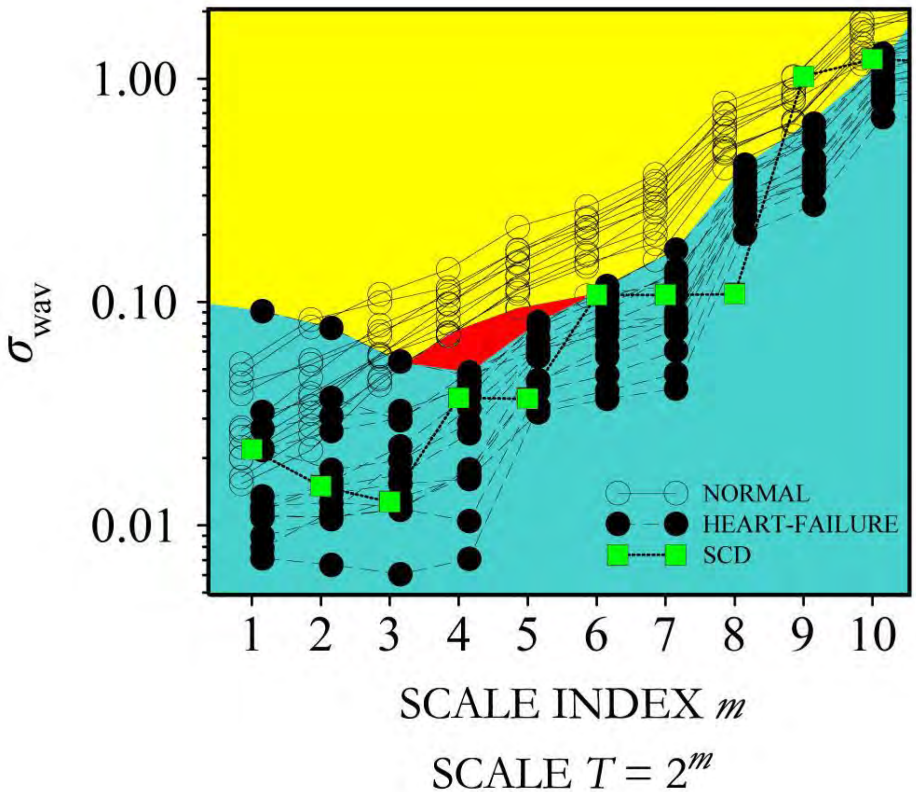
After Teich, *Proc. Int. Conf. IEEE Eng. Med. Biol. Soc.* **20**, 1136-1141 (1998).

After Ashkenazy *et al.*, *Fractals* **6**, 197-203 (1998).

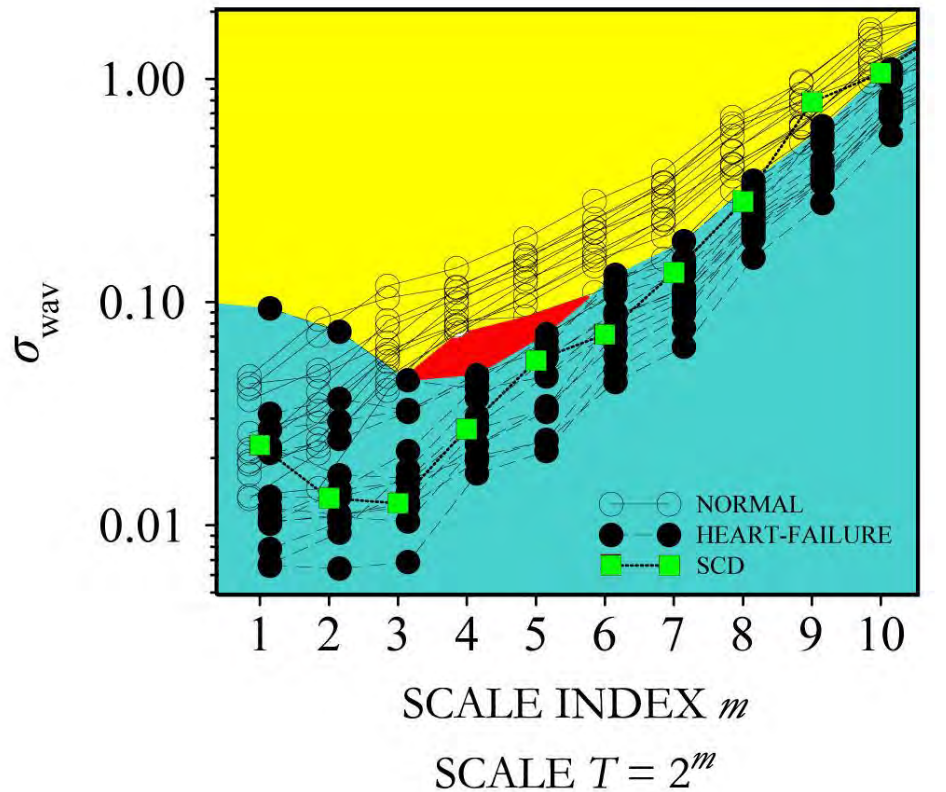
ROBUSTNESS WITH FORM OF WAVELET


Daubechies

Haar wavelet

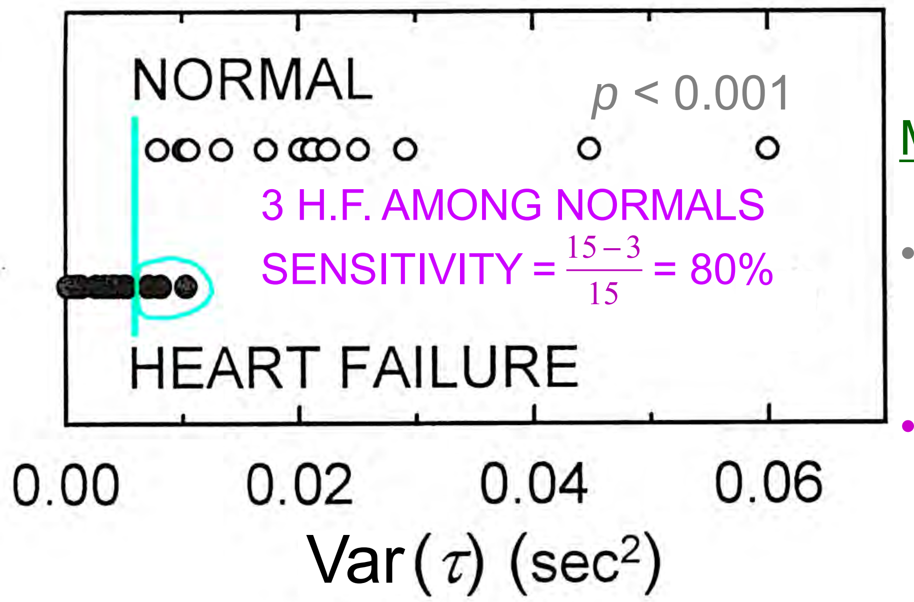


Daubechies 10-tap wavelet

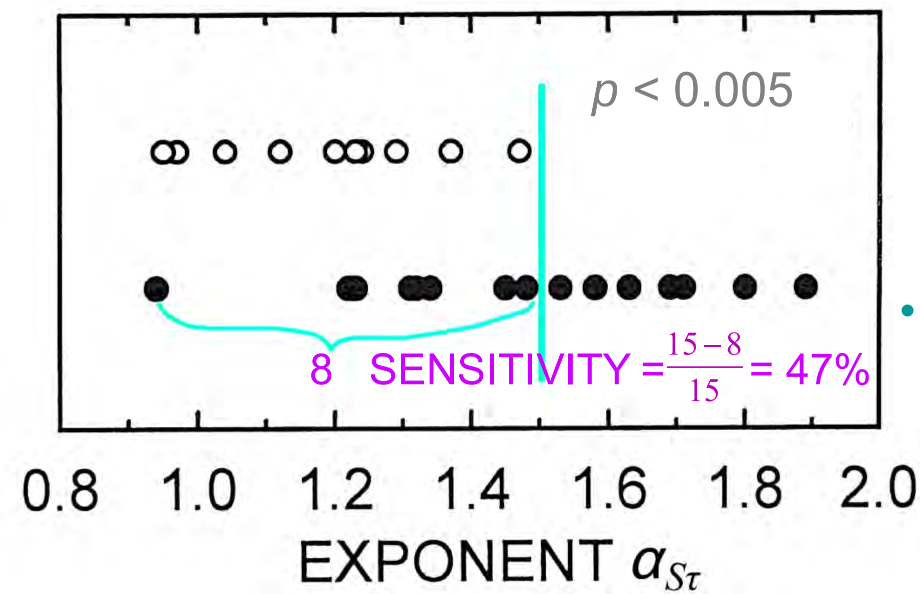


S. Thurner, M. C. Feurstein, and M. C. Teich, "Multiresolution Wavelet Analysis of Heartbeat Intervals Discriminates Healthy Patients from Those with Cardiac Pathology," *Phys. Rev. Lett.* **80**, 1544-1547 (1998).

IDENTIFYING PATIENTS WITH CARDIAC DYSFUNCTION



SCALE-DEPENDENT



SCALE-INDEPENDENT

MEASURES OF STATISTICAL SIGNIFICANCE

- *p* VALUE, d', AND VARIANTS (rely on Gaussian assumption)
- SENSITIVITY/SPECIFICITY MEASURES OF CLINICAL SIGNIFICANCE (distribution free)

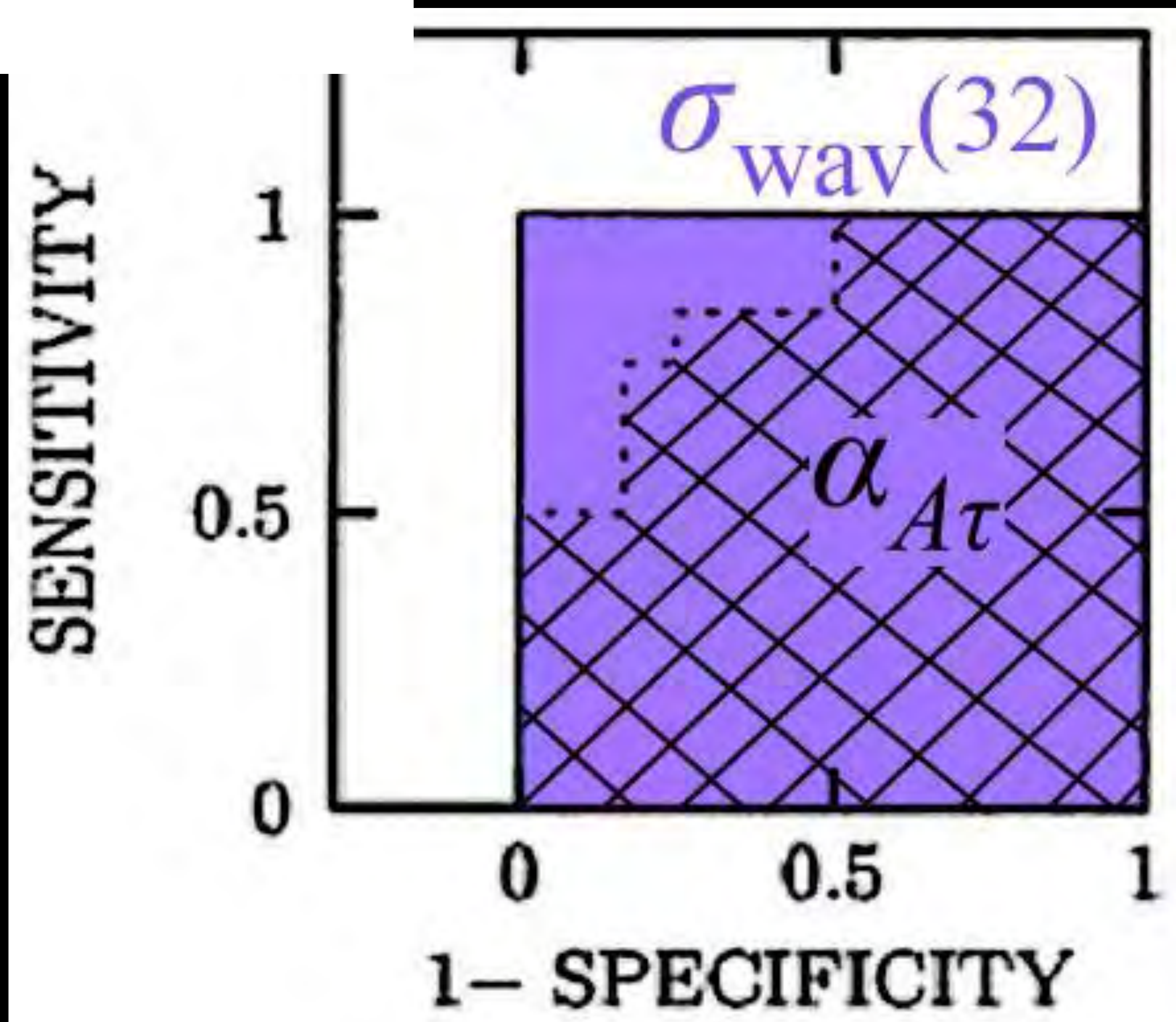
  SENSITIVITY ≡ proportion of heart-failure patients that are properly identified

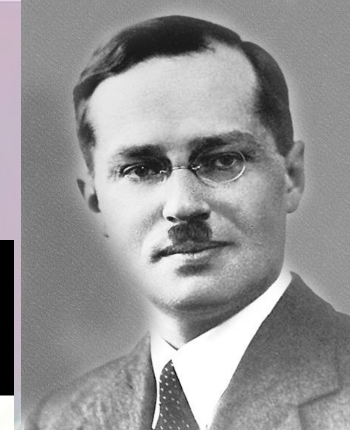
  e.g., Hypothesis that all normal patients are so identified ≡ 100% SPECIFICITY
- ROC CURVES & AREA UNDER ROC

ROC CURVES & AREA UNDER ROC

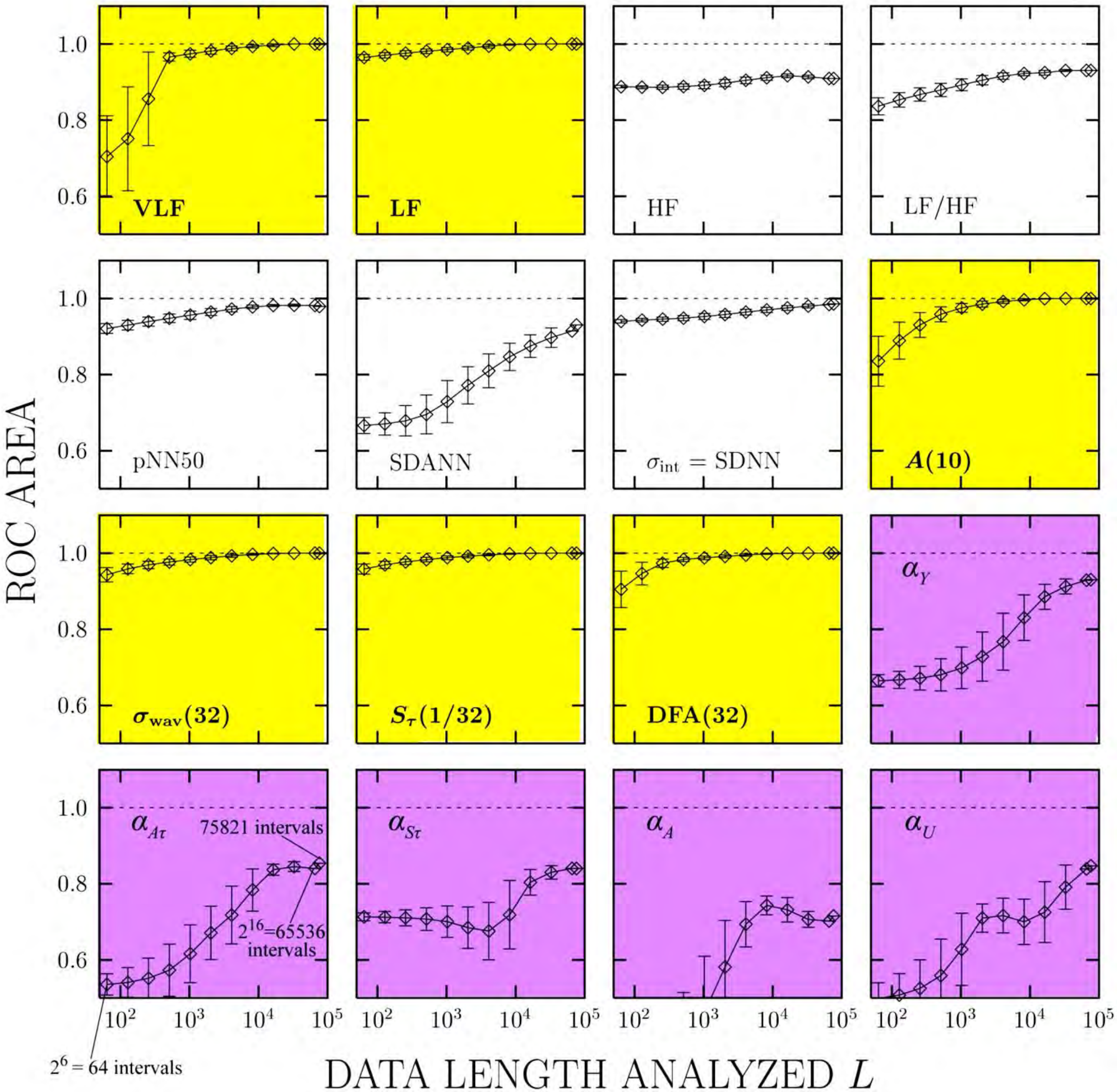
SCALE-DEPENDENT σ_{wav} (32)

SCALE-INDEPENDENT $\alpha_{A\tau}$

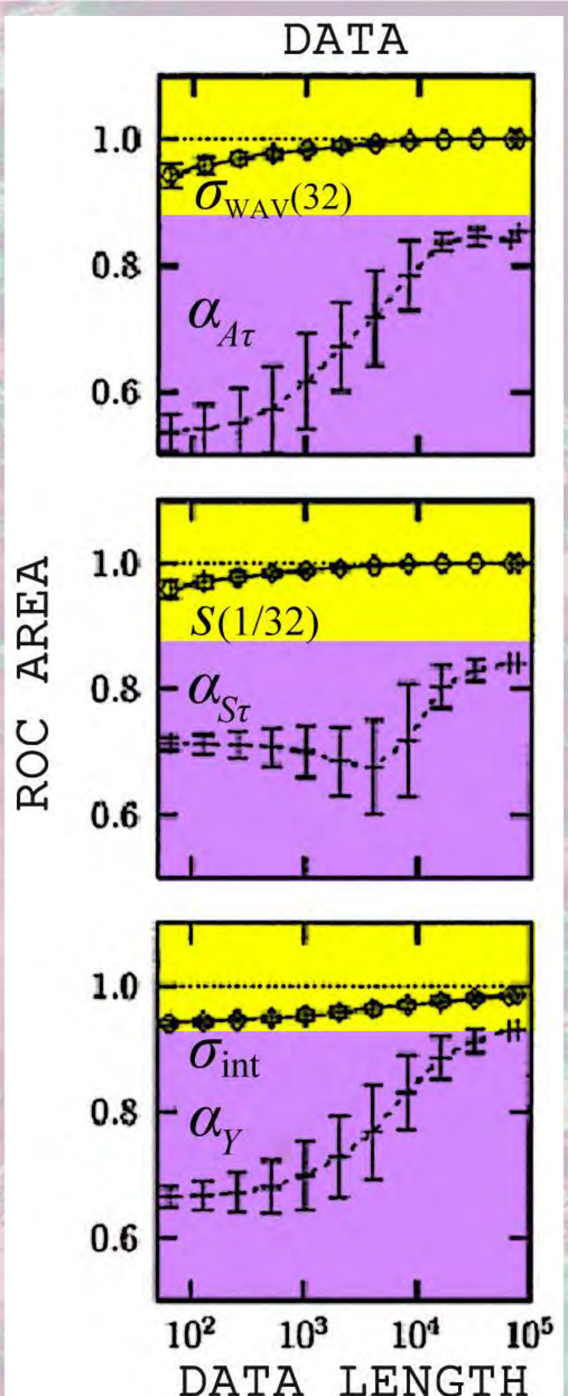




ROC-AREA CURVES: NORMAL & CHF DATA

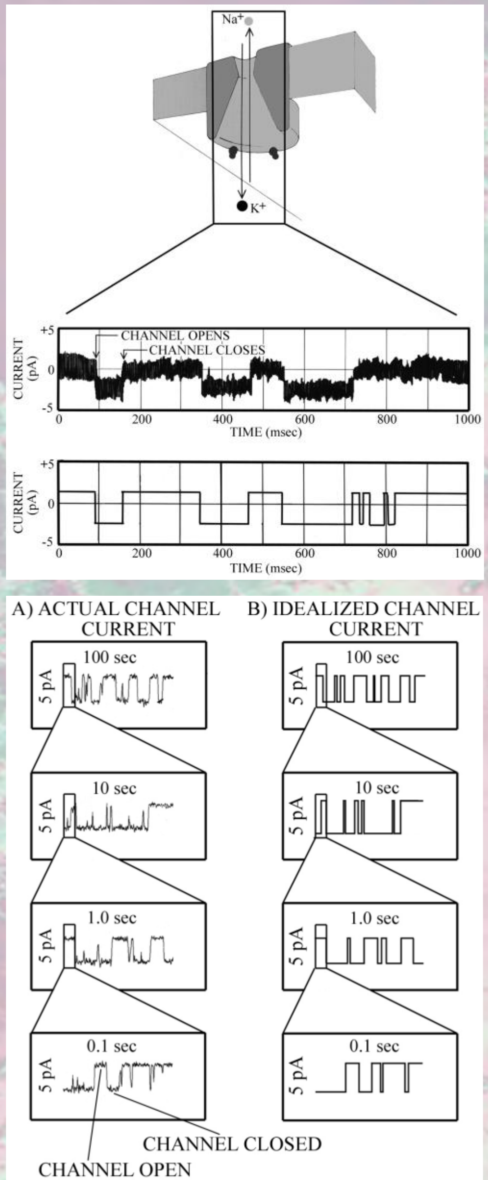


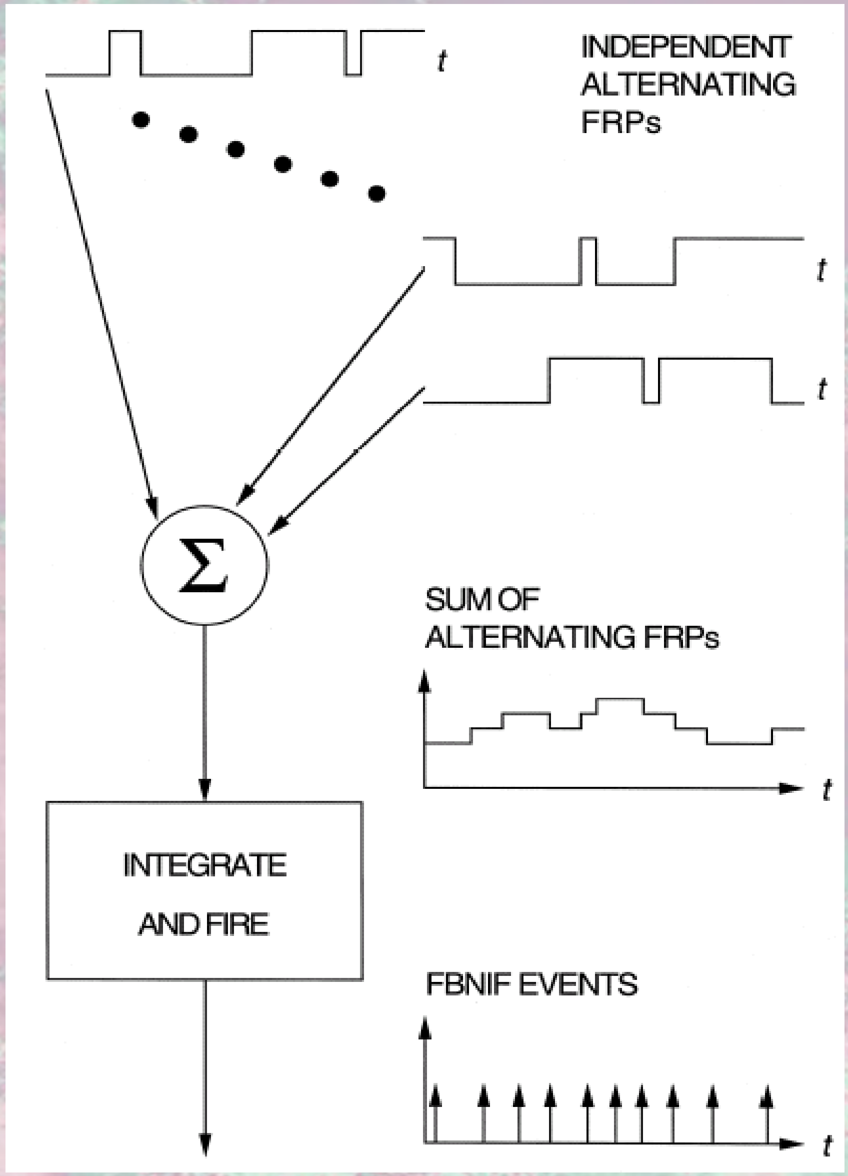
BOSTON UNIVERSITY



BOSTON UNIVERSITY

PHYSIOLOGICAL ORIGIN OF FRACTAL BEHAVIOR





Bernoulli

Gauss

Kolmogorov

Lapicque

Adapted from S. Thurner, S. B. Lowen, M. Feurstein, C. Heneghan, H. G. Feichtinger, and M. C. Teich, "Analysis, Synthesis, and Estimation of Fractal-Rate Stochastic Point Processes," *Fractals* **5**, 565-595 (1997).

6. NETWORKS

EXAMPLES OF SCALE-FREE NETWORKS

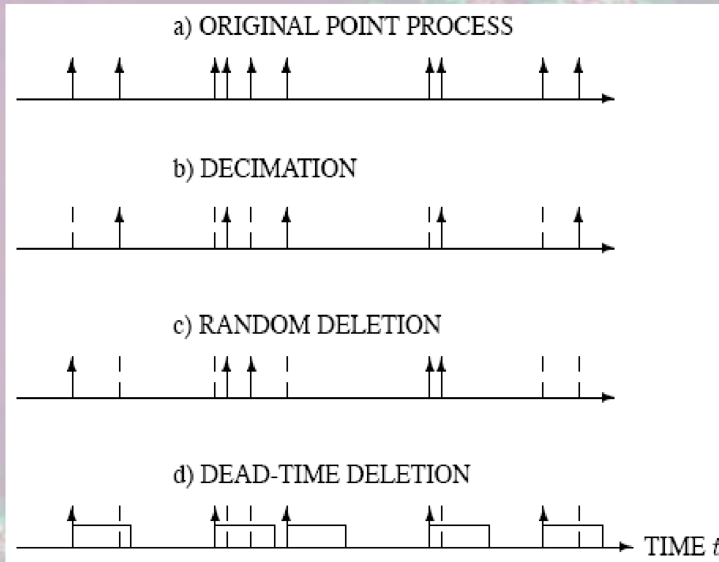
- Cellular metabolic networks
- Air transportation
- Internet

  (as of 2005, >100,000 separate networks, >100 million hosts, millions of routers, billions of web locations, tens of billions of catalogued documents)
- Web
- Scientific collaborations (linked by joint publications)
- Scientific papers (linked by citations)
- People (connected by professional associations or friendships)
- Businesses (linked by joint ventures)

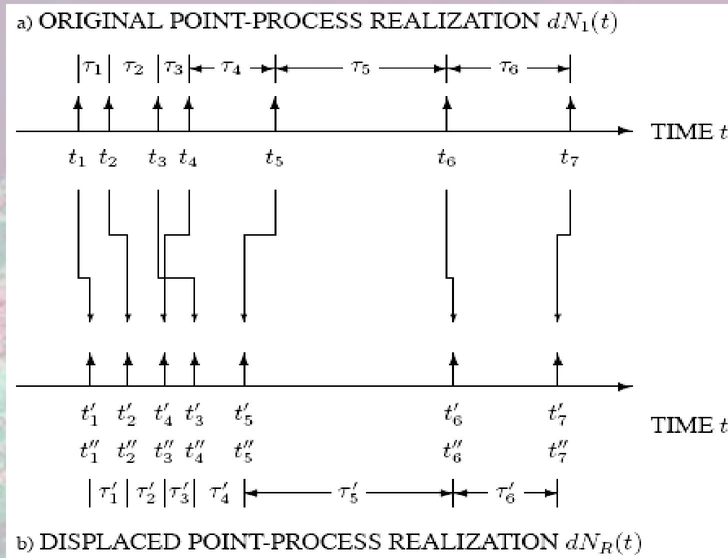
SUCH NETWORKS ARE ROBUST ABAINST ACCIDENTAL FAILURES BECAUSE RANDOM BREAKDOWNS SELECTIVELY AFFECT THE MOST PLENTIFUL NODES, WHICH ARE THE LEAST CONNECTED

SURROGATE DATA ANALYSIS

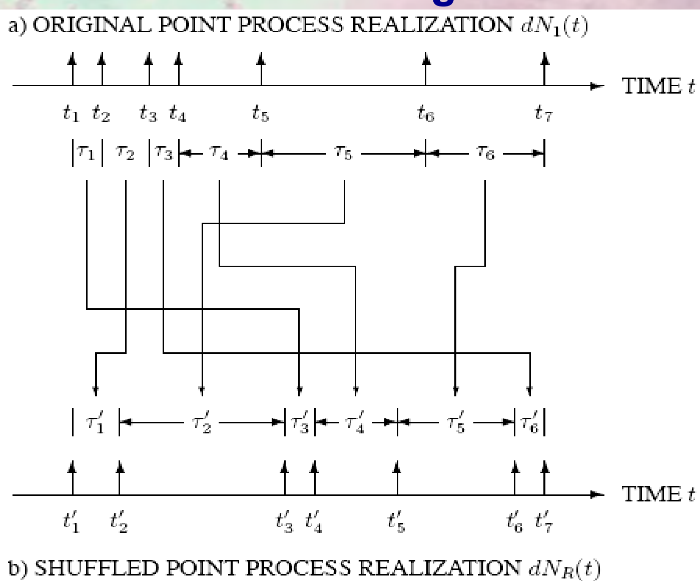
Deletion



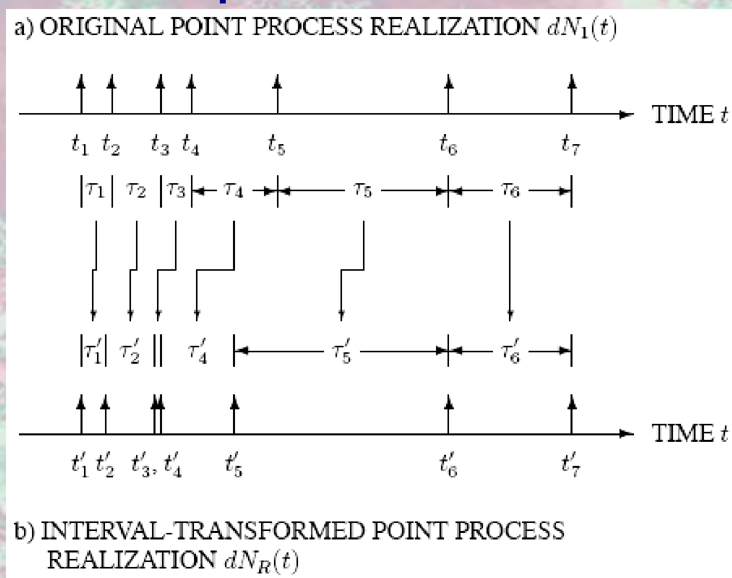
Displacement



Shuffling

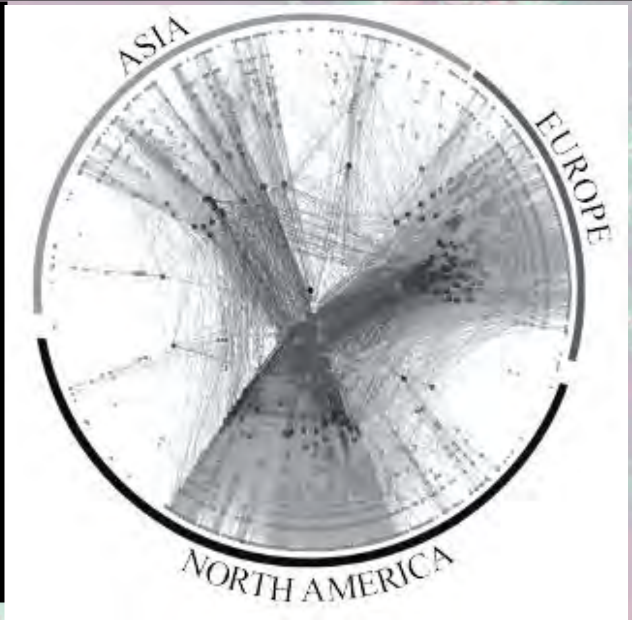


Exponentialization

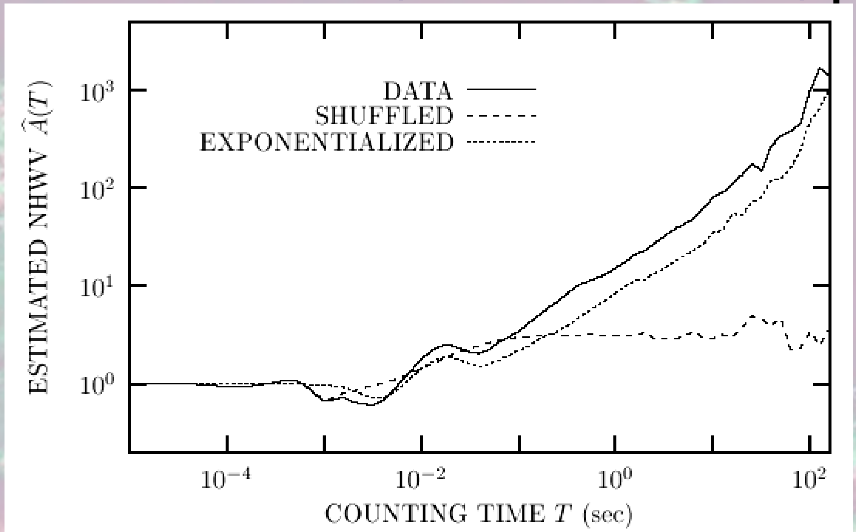


APPLICATIONS: COMPUTER NETWORK TRAFFIC

Snapshot of 11,000 ISPs from CAIDA (April/May 2003)

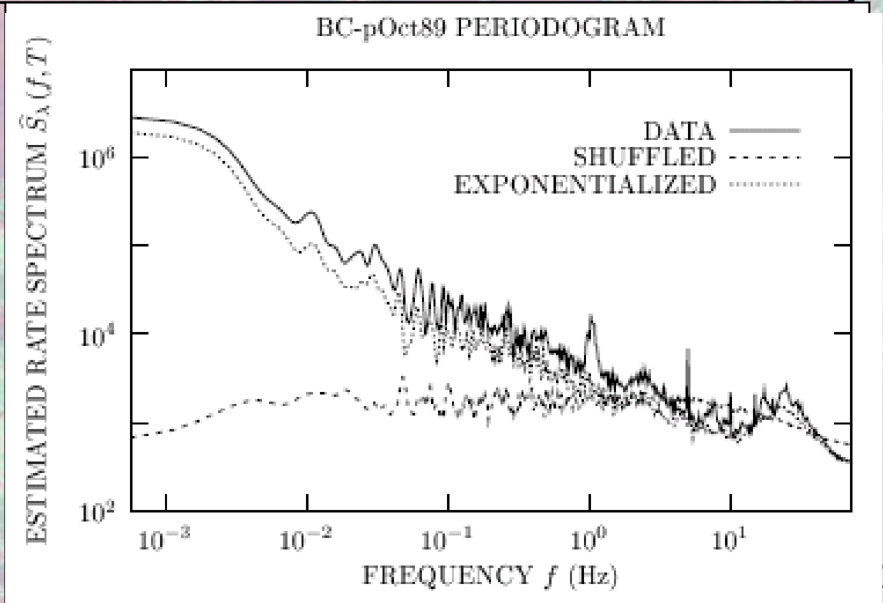


1 million consecutive Ethernet packet arrivals over 29 min (data set BC-pOct89)



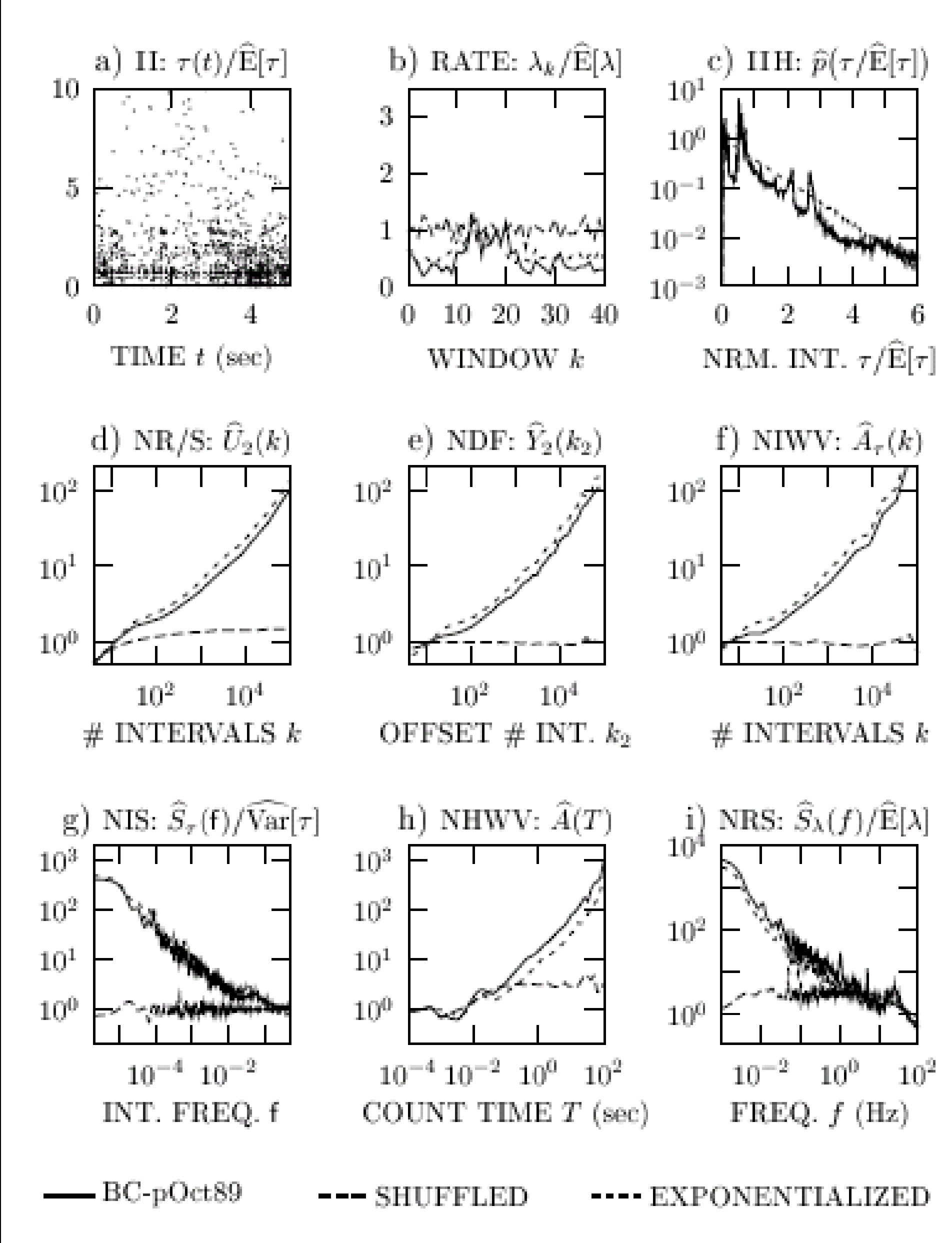
Erlang

Palm



IDENTIFYING THE NETWORK-TRAFFIC POINT PROCESS
TABLEAU OF NINE STATISTICAL MEASURES

CLASSIC BC-pOct89 DATA SET
1-million Consecutive Ethernet Packet Arrivals at Main Ethernet Cable



SIMULATED FRACTAL NEYMAN-SCOTT POINT PROCESS
(Rectangular Fractal-Shot-Noise-Driven Point Process: RFSNDP)

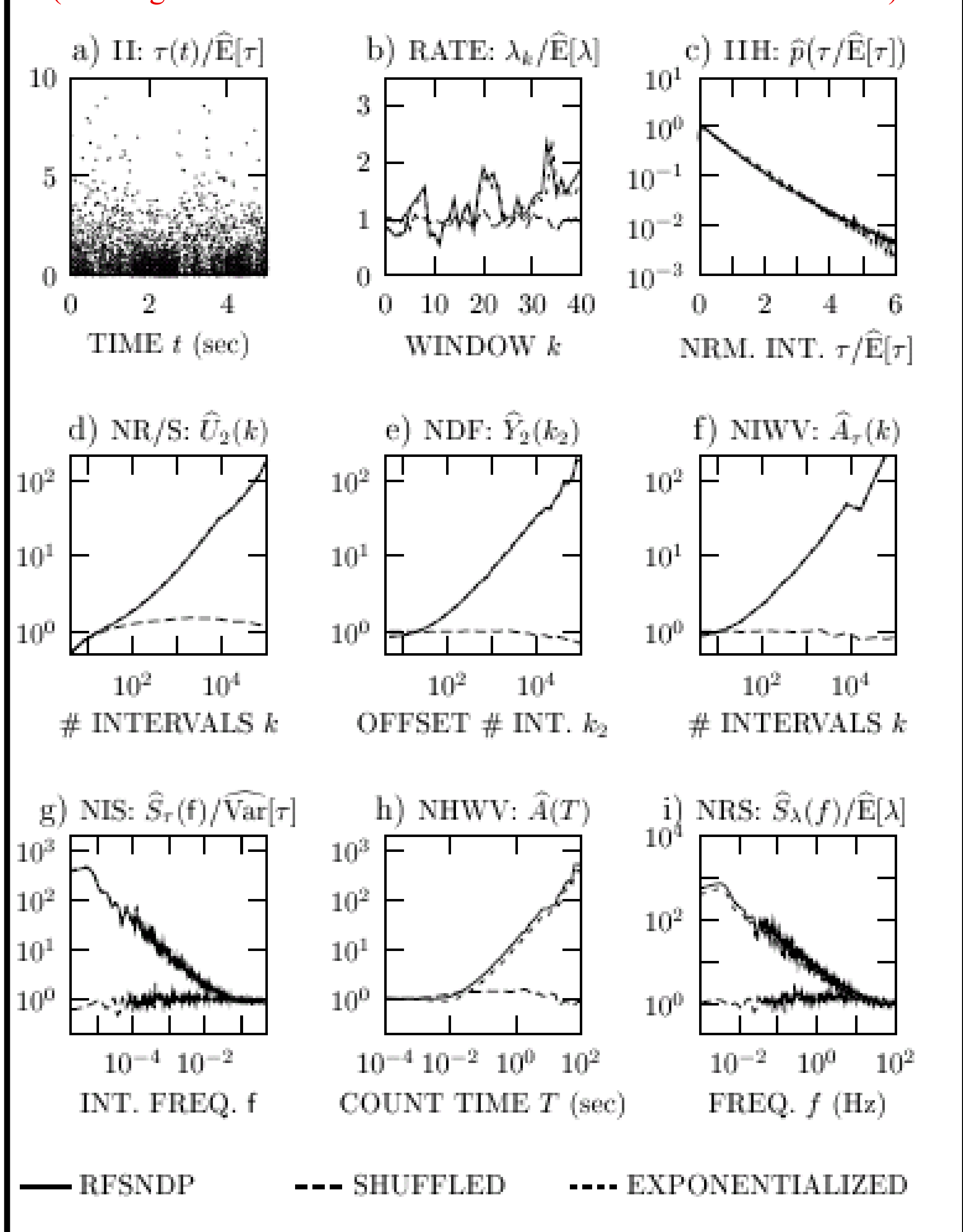
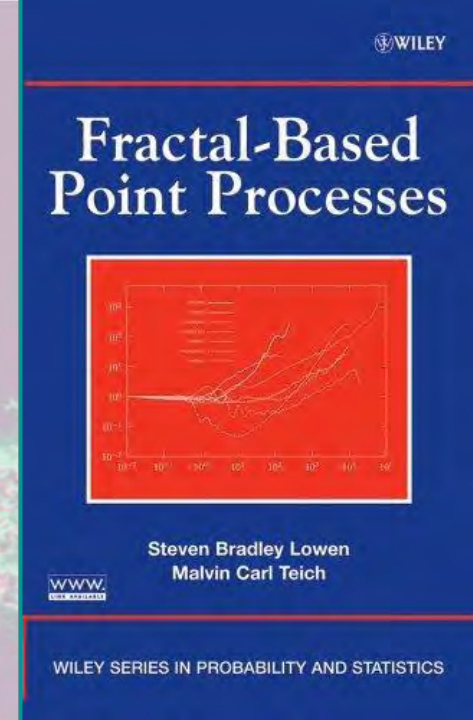


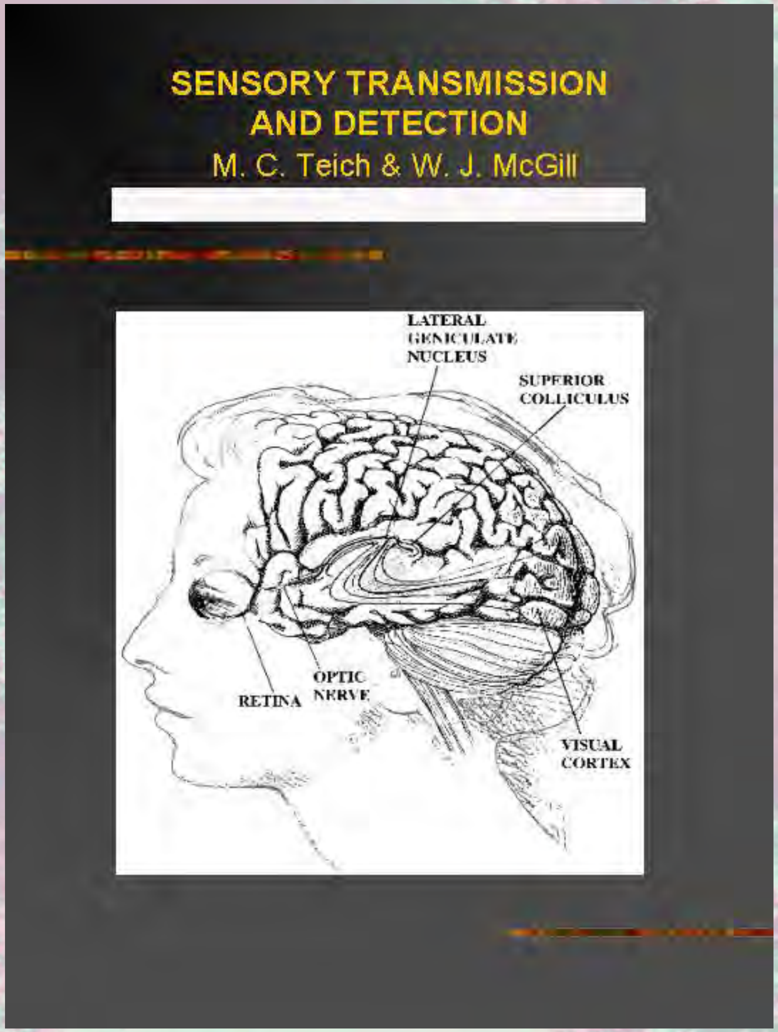
Table of Contents:

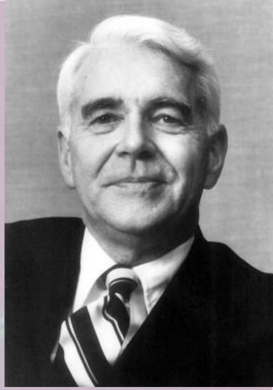
Preface.
List of Figures.
List of Tables.
Authors.
1. Introduction.
2. Scaling, Fractals, and Chaos.
3. Point Processes: Definition and Measures.
4. Point Processes: Examples.
5. Fractal and Fractal-Rate Point Processes.
6. Processes Based on Fractional Brownian Motion.
7. Fractal Renewal Processes.
8. Processes Based on the Alternating Fractal Renewal Process.
9. Fractal Shot Noise.
10. Fractal-Shot-Noise-Driven Point Processes.
11. Operations.
12. Analysis and Estimation.
13. Computer Network Traffic.
Appendix A: Derivations.
Appendix B: Problem Solutions.
Appendix C: List of Symbols.
Bibliography.
Author Index.
Subject Index.



BOSTON UNIVERSITY

7. SENSORY DETECTION





McGill

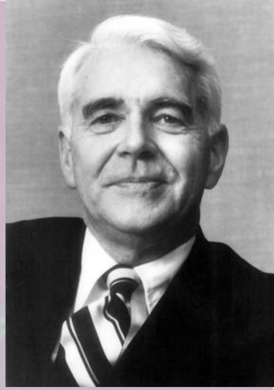
http://people.bu.edu/teich

JOURNAL OF MATHEMATICAL PSYCHOLOGY 39, 146–163 (1995)

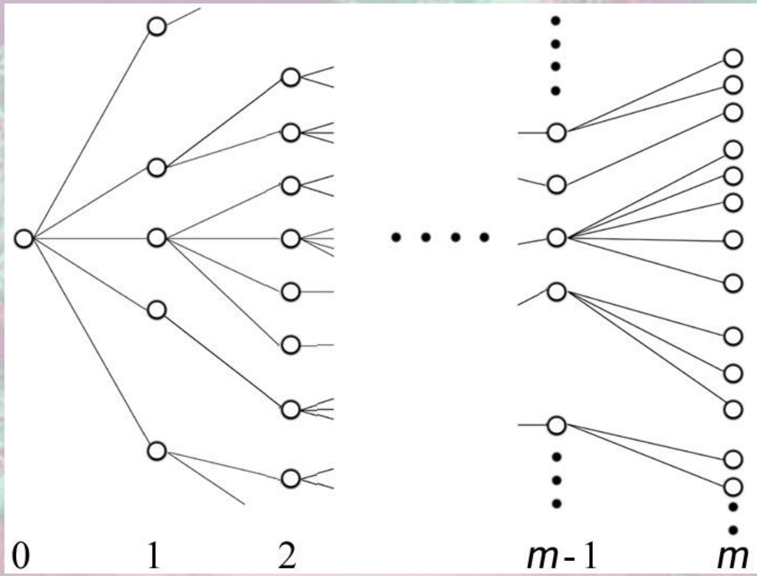
Alerting Signals and Detection in a Sensory Network

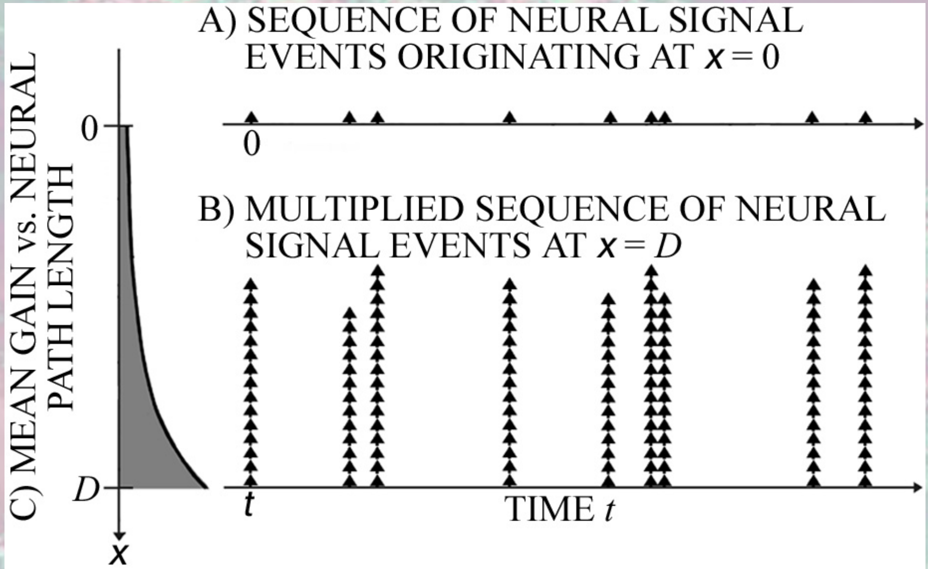
WILLIAM J. MCGILL† AND MALVIN C. TEICH‡

Columbia University

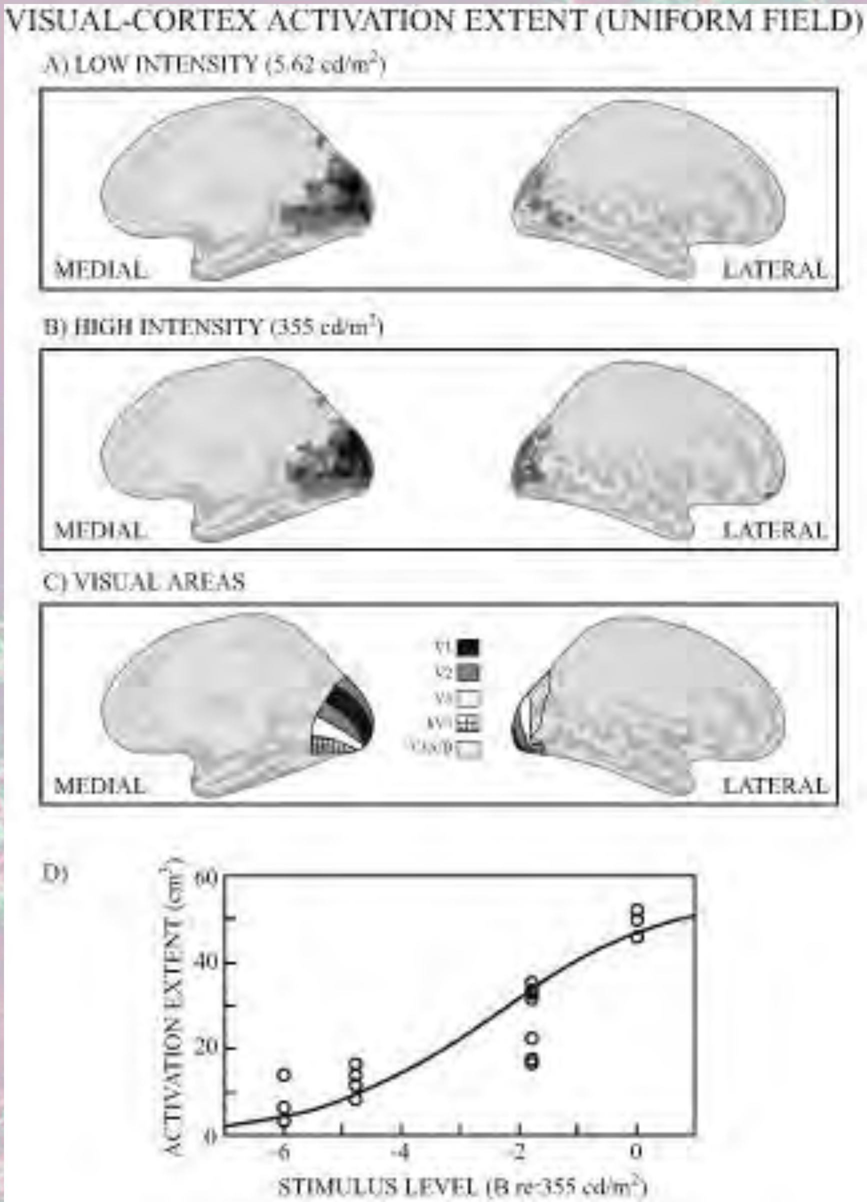


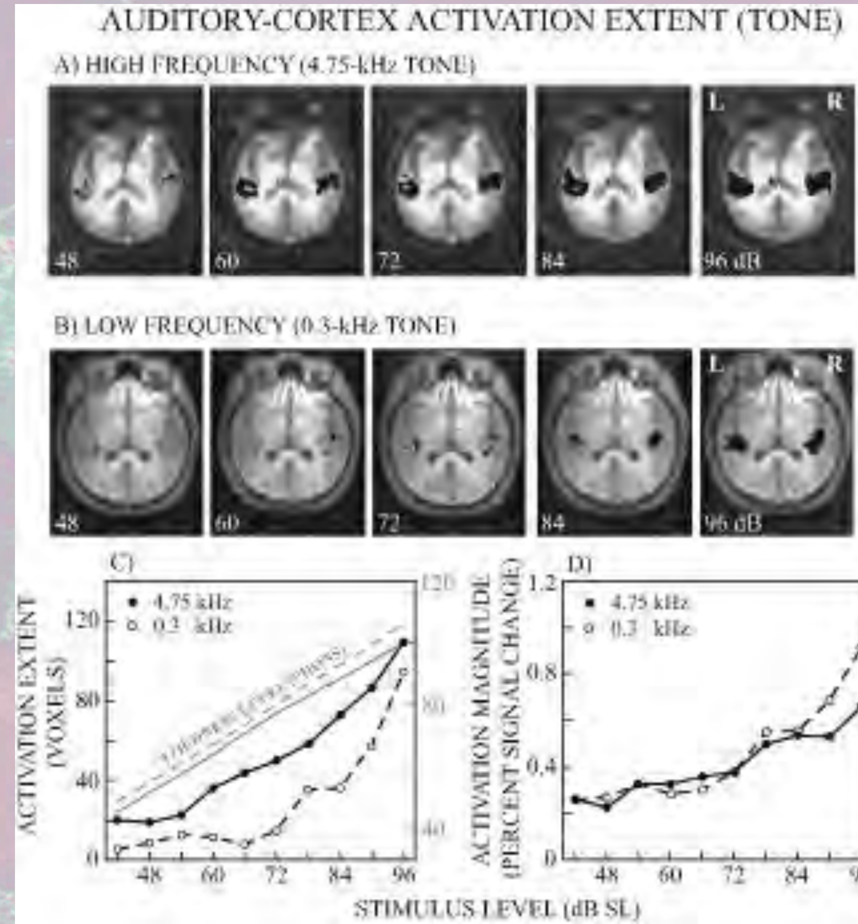
McGill





CORTICAL-ACTIVATION EXTENT VS. STIMULUS INTENSITY





Hart, Hall, and Palmer, “The Sound-Level-Dependent Growth in the Extent of fMRI Activation in Heschl's Gyrus is Different for Low- and High-Frequency Tones,” *Hearing Res*. **179**, 104-112 (2003).

Aguirre, Komáromy, Cideciyan, Brainard, Aleman, Roman, Avants, Gee, Korczykowski, Hauswirth, Acland, Aguirre, and Jacobson, “Canine and Human Visual Cortex Intact and Responsive Despite Early Retinal Blindness from *RPE65* Mutation,” *PLoS Med*. **4**, 1117-1128 (2007).

BOSTON UNIVERSITY

SUPPORT GRATEFULLY ACKNOWLEDGED

BU College of Engineering
Boston University

the photonics center
at boston university

NSF

Army
Research Office

CenSSIS
The Center for Subsurface Sensing and Imaging Systems

The David and Lucile Packard Foundation

ZEISS

DARPA