# STATE-VECTOR FORMALISM FOR INTRAPERSONAL, INTERPERSONAL, AND GROUP INTERACTIONS[1]

*by Malvin Carl Teich*

*Columbia University*

**This article deals with information processing subsystems of the human organism and group. While a number of specific (but generally unrelated) mathematical models exist for various processes in the realm of social psychology and psychiatry, the predictive power of these models usually covers only a small range of phenomena. The overall objective of this work is to provide a vehicle through which a number of existing models in mathematical psychology, and various as yet unmathematized processes in social psychology and psychiatry, can be brought together constructively and on a uniform mathematical basis. Particular attention is given to impression formation, choice, and the psychotherapeutic interaction. A probabilistic and relativistic Dirac-type state-vector formalism provides the overall framework. The ultimate goal of constructing such a system is to increase our knowledge of the dynamic laws governing human behavior. It is clear from our study that there is an essential similarity among the psychoanalytic, gestalt, and behavioral psychotherapies.**

Key Words: human systems, organism, group, information processing, interpersonal interaction, impression formation, psychotherapy, human group interactions, choice.

## INTRODUCTION

The concept of "psychic energy," first introduced by Freud, was inspired by the successes of Newtonian physics and thermodynamics, which were lively fields of study in the nineteenth century. These pre-quantum physical models for nature could perhaps be best described as "deterministic," i.e., given a completely specified system at one instant of time $t_0$, a measurement may be made at a later time, $t_1$, with no uncertainty. While it appears that the concept of psychic energy has not yet proved useful in any practical way, it provided perhaps the first attempt to quantitatively characterize the mental state by a parameter which was presumed to be somehow measurable.

The advent of twentieth century physics has shed new light on the workings of nature and has provided us with a picture of the world which may be described as "probabilistic" rather than deterministic. This way of viewing nature provides a more general basis for describing human behavior as well, since the probabilistic model is more flexible and provides many more options or degrees of freedom than does the deterministic model. Furthermore, as will be seen subsequently, the latter may be obtained as a special case of the former so that no loss of generality is encountered by assuming a probabilistic model.

In this section, we briefly describe what we consider to be crucial scientific realizations since the early 1900s, which are sufficiently general to be of importance in our study. The probabilistic concept in wave mechanics and the effect of the observer on a measurement performed on a system were first set forth by Max Born and by Werner Heisenberg, respectively, in 1926 and 1927. This followed the pioneering work of Niels Bohr and Erwin Schrödinger in quantum physics (see Jammer, 1974). The importance of the probabilistic concept has, more

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recently, come to be accepted as an integral part of *classical* science as well, although it plays a less fundamental role there. The mathematician Norbert Wiener (1964) generalized the concept of deterministic classical dynamics to arrive at a far-reaching result which permits uncertainties in an initial classical measurement, yet reduces to the deterministic result in the absence of such uncertainty. The use of a probabilistic formalism in the social sciences is supported by psychological observations that individuals are not perfectly consistent in their preferences, even under constant or identical conditions (Thurstone, 1927; Luce, 1959; Tversky, 1969; Tversky, 1972a).

While Schrödinger's equation, Heisenberg's uncertainty principle, and Bohr's complementarity interpretation substantially altered the course of modern philosophy, literal connections to the human realm have been weak at best. Taking an existing equation of physics directly over to the realm of human behavior simply has not provided useful results. Indeed, there is no particular reason to believe that the role of probability in psychology is of the same fundamental character as it is in quantum physics; it is far more likely that probability enters psychology in much the same way that it enters classical physics, as a result of our inability to completely characterize a complex system. Yet direct analogies, particularly with the uncertainty principle, the definition of the information bit, and entropy, have been made by many. Donald Griesinger (1974), for example, has recently constructed such a model using the Schrödinger equation. Since it is even more specific than Freud's use of the conservation of energy, it is subject to exacting tests which have, unfortunately, not been applied. A great deal of the work in the literature which examines the interrelation between natural science and human behavior, it appears, suffers from the effects of just such a direct transposition. Other examples include the work of Rothstein (1965), and Houghton and his co-workers (Houghton, 1968; Carroll & Houghton, 1970).

One article of particular interest is the review paper by Ayalah Meir (1969), who discerns several different approaches to the search for a general theory of behavior. Dr. Meir presents several very good suggestions in general form for the application of systems theory to psychiatry, and we have tried to understand our work within the context of his overall broad outline. More recently, Goldman (1976) wrote a marvelous monograph dealing with relationships between physics, biology, psychology, and sociology. The reader is particularly directed to chapter 13 of Goldman's work, which deals with what the author calls duology (the combined fields of psychology and sociology). We find Goldman's approach fascinating, and consider the work presented here to be an operational complement to Goldman's development. The work of Kurt Lewin (1935, 1936) and his colleagues and students (Deutsch, 1968) represented an early attempt to obtain an overall field-theoretical model for behavior. Lewin's research concentrated on generating a spatial or topological model for the "life space." While Lewin's work provided some broad insights into psychological behavior, it failed to be truly useful from a mathematical point of view since it provided few operational definitions.

The physicist Dirac (1930), using the state-vector formalism, provided an important advance in describing the evolution of a probabilistic physical system in general terms, without reference to any specific coordinate system. This is important because it allows a system to be generally described yet experimentally observed in any one of a large number of "representations," corresponding to different observational conditions. An appropriate analog is the generality of a vector relationship such as $\mathbf{F} = d\mathbf{p}/dt$ (force is the time-derivative of momentum) as opposed to a manifestation of this general result in a specific representation such as Cartesian or polar coordinates. Dirac's system is described by what is called the "state vector," and it offers both the general rule and the specific description.

Another of Dirac's important contributions to quantum mechanics was to include the invariance requirements of Einstein's special theory of relativity for inertial frames of reference. In psychology, similarly, the relative nature of the person-person interaction takes on importance. Lewin (1936), for example, referred to quasi-physical, quasi–social, and quasi–conceptual facts within his life space, portraying his belief that the individual must be studied in interrelation with the group to which he belongs. Many of the psychoanalytic theories such as Harry Stack Sullivan's (1953), and most existential theories, define interactions relatively. A more or less nonrelative interaction, such as is required by ideal classical psychoanalysis, is easily obtained as a special or limiting case of the more general formulation which takes relative interaction into account. Gestalt psychology has emphasized a study of the whole, in analogy with "collective phenomena" which occur in physical systems. Indeed, there is general agreement among psychologists (as among physicists) that the observer, in the process of observing an event, affects its course (Bachrach, 1962, p. 33; Weick, 1968). Gergen and his co-workers (1965, 1972) have emphasized that the healthy individual wears many masks of identity depending upon the social situation in which he finds himself.

It should be noted that Robert Leighton published an article some time ago (Leighton, 1971) about the difficulty of reaching conclusions in panel discussions of scientists. This paper essentially defined what we refer to as a "theoretical concept space," ascribed certain rules to the various interactions of state vectors (representing theoretical ideas and experimental facts in physics) in this space, and developed a "calculus" for their behavior. While this paper was intended, in part, as a not altogether serious statement of the author's exasperation at participating in panel discussions, it also provided an example of dynamic interaction in a space with humanistic coordinates.

An interesting exposition of multidimensional subjective spaces for psychological scaling (Coombs, 1950) has been presented by Micko and Fischer (1970). The axes of the affine space defined by these authors represent subjective attributes, and are taken to be orthogonal for subjectively independent attributes. Positively or negatively correlated attributes are represented by axes at angles $<\pi/2$ or $>\pi/2$ with each other. These authors derive a number of metrics from given rules of combination and discuss these in terms of shifts of attention. Their model is basically deterministic, however, inasmuch as the magnitude of the projection along a given axis represents the strength of that attribute. They acknowledge that a probabilistic formulation is likely more realistic and propose that the magnitude of this projection may instead represent the relative frequency of an attribute in a random sample of such a space. Nevertheless, the foregoing treatment is directed to multidimensional scaling and as such does not contain or presuppose any dynamic law or time development which is the requisite for a predictive theory. Other authors have also used restricted classes of subjective spaces (Houghton, 1968), but in general an adequate framework for dealing with the quantities defined on these spaces is not provided.

Based on the foregoing, we propose a probabilistic, relativistic state–vector formalism for representing interpersonal interactions. It appears that this framework has the requisite generality for supportively coordinating a number of existing specific theories in psychology within a single mathematical framework. From a systems science point of view, we can say that the primary emphasis of this work is on information processing subsystems of the human organism and group, although it may be possible to extend it to the levels of organization and society.

## FORMALISM

In this section we present the elements of the state–vector formalism introduced above. Following Leighton (who likened every theoretical idea and every experimental fact in an area of physics to a vector in a multidimensional, inhomogeneous space), and Micko and Fischer, we allow various physiological and psychological characteristics, concepts, and behaviors (thoughts, memories, attitudes, perceptions, verbal responses, decisions, actions, etc.) to be represented, each by a subspace of axes, in a Dirac–type affine multidimensional space. As a trivial example, a body temperature of 37.0°C. is represented by a given axis, whereas a body temperature of 37.1°C. is represented along another axis, orthogonal to the first but within the same subspace.

More complex characteristics may be represented by linguistic variables (Zadeh, 1975), e.g., aggression, in which case the axes within the subspace would comprise the term set of the linguistic values (aggressive, not aggressive, reasonably aggressive, very aggressive, etc.). We then allow an individual to be represented as a single state vector in this space, the magnitude of the projection of this state vector along any axis representing the square root of the probability (or level of "participation," or "possibility") that the variable takes on the value represented by that axis. Thus we accept the general probabilistic approach, used by Born, Heisenberg, Bohr, and Dirac for quantum systems, by Wiener for classical systems, and by Thurstone, Luce, and others for psychological systems. Furthermore, for the most part we work in a generalized format in which we are not required to specify the precise nature of the coordinate system unless we wish to make observations in a given representation. We follow Micko and Fischer insofar as we may allow nonorthogonal axes to represent subjectively correlated attributes, but in contrast to their system, we provide a subspace of psychological space with a multiplicity of axes for each attribute. Each axis in this subspace represents the outcome of a possible experiment or measurement, performed by a given experimenter. The magnitude of the projection along the given axis is a measure of the likelihood of obtaining that particular result. This is the approach used for quantum systems which are probabilistic by nature (Dirac, 1930; Prugovečki, 1971). It allows for the consideration of more general ensemble averages rather than time averages implied in the work of Micko and Fischer.

More specifically, we represent the subject $\phi$ as a single state vector $|\phi\rangle$, called phi "ket," and defined on a ket psychological space as described above. There is also defined another state vector $\langle\phi|$, called phi "bra," representing the entity observed by the subject. Note that the complete bra-ket (bracket) $\langle\phi|\phi\rangle$ represents a scalar quantity (corresponding to how well the bra and ket match up with each other on the average) since the inner product defined by the complete bracket is a generalization of the ordinary dot product between two vectors. An incomplete bracket represents a state vector. The magnitude of the overall ket or bra usually carries no meaning for a space of fixed dimensionality, only the *direction*, and hence the projection along any one of the axes, carries meaning. It is usual to normalize the kets and bras such that their length is unity. The formalism easily reduces to the deterministic state vector when the projection of $|\phi\rangle$ is unity along one and only one of the axes making up the subspace of a particular characteristic. In that case, the characteristic is not probabilistic since it yields a given response with probability unity.

In order to determine whether the state-vector formalism is indeed useful, and to provide a basis for defining reasonable quantities, it is necessary to examine a number of existing mathematical models for social behavior. We must insure that the formalism is sufficiently general to permit the representation of processes already studied, and thereby to allow existing results to contribute to the study of the dynamic law and the time evolution of the social system. (Most mathematical models of behavior which have been used in social psychology are interpretations of an existing verbal theory. Rosenberg (1968) discusses the advantages and disadvantages of using mathematical models in a social psychological context, and presents an overall view of existing work in the areas of impression formation, attitude change, interaction processes, and conformity. In line with the objectives of this work, and following Rosenberg, we concentrate on prescriptive models. Normative and purely explicational models are not considered.)

## Impression formation

In developing the state-vector formalism, we may make use of existing psychomathematical models for impression formation, the process by which an individual transforms a multiplicity of observations and hearsay about another person into a set of interpersonal attitudes and perceptions. The work of Anderson (1962, 1964, 1965a) and Levy and Richter (1963) seems to support the proposition that a subject's overall impression is reasonably well described by an average combination of the affective values of the individual stimulus items. While other authors' results appear to fit models such as the summation model (Fishbein & Hunter, 1964) or an intermediate logarithmic model (Manis, Gleason, & Dawes, 1966), it appears that in first approximation, the outcomes of cognitive interactions can be predicted from a weighted averaging of interacting elements. Use of this weighted-average model can be described in terms of the state-vector formalism as follows.

Let us consider an arbitrary representation, consisting of normalized basis vectors labeled by $|i\rangle$, which represent individual stimulus items. Since the identity operator $I$ may be expressed in terms of the outer products of a complete set of basis vectors,

$$I = \textstyle\sum_i |i\rangle\langle i|, \tag{1}$$

a state vector may be represented in terms of its projections on the various fundamental basis vectors or axes $|i\rangle$ comprising the space. Thus,

$$|\phi\rangle = \textstyle\sum_i |i\rangle\langle i|\phi\rangle = (\langle 1|\phi\rangle)|1\rangle + (\langle 2|\phi\rangle)|2\rangle + (\langle 3|\phi\rangle)|3\rangle + \cdots. \tag{2}$$

We define the impression imparted by a person, $\langle\psi|$, to the subject $|\phi\rangle$, (the image formed by $|\phi\rangle$ of $\langle\psi|$) at any instant of time as the collection of scalar elements of the form

$$\epsilon_i = \langle\psi|i\rangle\langle i|\phi^{\text{ideal}}\rangle. \tag{3}$$

Here $|\phi^{\text{ideal}}\rangle$ represents the fantasy ideal which the subject sets up for comparison, and the $i^{\text{th}}$ element in (3) refers to attribute $i$. The ket $|\phi^{\text{ideal}}\rangle$ is equivalent to an appropriately weighted version of the subject's own ket $|\phi\rangle$:

$$|i\rangle\langle i|\phi^{\text{ideal}}\rangle = a_i|i\rangle\langle i|\phi\rangle, \tag{4}$$

where the $a_i$ are approximate weight factors (Anderson, 1964). Thus, the scalar elements in (3) may be rewritten as

$$\epsilon_i = a_i\langle\psi|i\rangle\langle i|\phi\rangle. \tag{5}$$

According to the rules for the weighted average model in impression formation, we need simply to add these various individual contributions to arrive at a scalar quantity for the overall impression, $Im$:

$$Im = \textstyle\sum_i \langle\psi|i\rangle\langle i|\phi^{\text{ideal}}\rangle = \textstyle\sum_i a_i\langle\psi|i\rangle\langle i|\phi\rangle. \tag{6}$$

Since all vectors are assumed to be normalized, this summation automatically represents an average. In the case where all $a_i = 1$, by using the identity represented by (1) we obtain the simple result

$$Im = \langle\psi|\phi\rangle. \tag{7}$$

In this case, Eq. (4) tells us that $|\phi^{\text{ideal}}\rangle = |\phi\rangle$ so that the subject's impression may be simply described as the inner or dot product of the subject's ket with the bra observed. The overall impression then simply represents how well the observed person matches up with the subject. Impression formation, as described above, involves an interpersonal interaction, and is intrinsically more complex than an intrapersonal process such as choice (which will be considered subsequently). It is important to note that we do not imply a one-to-one correspondence between bras and kets in this case, as is usual in quantum mechanics. Clearly $\langle\psi|\phi\rangle$ is distinct from $\langle\phi|\psi\rangle$. For intrapersonal interactions, on the other hand, this distinction is not important and we assume that $|\phi\rangle$ and $\langle\phi|$ are conjugate vectors related on a one-to-one basis.

## Measurement and observation

Inasmuch as the above results theoretically account for the detection of all possible attributes within the subject's cognitive structure at a particular moment in time (rather than just a limited set as presented by an experimenter), they are pertinent to the domain of person perception (Tagiuri, 1968). A measurement of the elements in Eqs. (3) and (5), or the scalar quantity in Eq. (6) at any instant of time will generally require a transformation or mapping to a semantic, physiological, or behavioral space (see, for example, Scott, 1968). In this sense, the state-vector formalism provides a mathematical structure for the state of the subject in a particular interaction. This may be combined constructively with a related system for measurement, such as the Osgood semantic differential technique (Osgood, Suci, & Tannenbaum, 1957). It is the direct representation of the subject's state, however, which provides a convenient framework for examining the processes involved in cognitive dynamics (time development).

It is axiomatic in this formulation that the vector $\langle\psi|$ can never be measured in isolation. That is, in order to measure a state vector we must include an observer (which may be the subject himself), and it is entirely possible (and indeed likely) that two observers $|\phi\rangle$ and $|\phi'\rangle$ will see different versions of $\langle\psi|$ in the subset $S$. Thus in general

$$\langle\psi|\phi\rangle \neq \langle\psi\phi'\rangle, \tag{8}$$

representing the concept of relativity so important in interpersonal (Sullivan, 1953) and existential psychologies (Jaspers, 1963; Frankl, 1957). The required pairing of state vectors described above provides the rationale for the magnitude of the projection onto a basis vector representing the *square root* of the probability.

The results considered to this point are applicable at a given instant of time; yet, we must account for the change that an individual undergoes either by himself through his own physical and thought processes $|\phi(t)\rangle$, or under an external influence. To account for the change with time, we introduce an operator $\hat{T}_\psi$, which causes the state vector $|\phi(t_1)\rangle$ to rotate in its space, thereby changing its projection along various axes. This indicates that the operator $\hat{T}_\psi$ has in some way changed the response that will be evoked from $|\phi\rangle$. This operator carries the subscript $\psi$ to indicate that it is $\langle\psi|$ which is operating or effecting a change in $|\phi\rangle$. For example, a psychotherapist $\langle\psi|$ may cause a change in a patient $|\phi\rangle$ which may be expressed as

$$\hat{T}_\psi|\phi\rangle \rightarrow |\phi'\rangle. \tag{9}$$

Or, the patient may produce a change in himself: $\hat{T}_\phi|\phi\rangle \rightarrow |\phi'\rangle$. Changes may also be induced by rotating and changing the size of the vector space itself, or by the process of a subject being observed. The importance of the method of observation has been emphasized by Weick. When it is the object to find those methods which elicit a response from the subject without subtantially altering his state of mind and body, we may express this operation within the state–vector formalism as

$$\hat{T}_\psi|\phi\rangle \simeq \tau|\phi\rangle. \tag{10}$$

Here $\tau$ represents a possible response to the measurement operation.

We expect that a precise equality in Eq. (10) will rarely hold; rather, we can hope for minimum disturbance and approximate equality. Operations that behave pretty much according to the minimum disturbance rule include simple everyday interactions in the course of one's work or home life, or measurements made without the subject's knowledge. Such interactions or measurements are not very likely to alter the subject in any substantial way. Operations such as those eliciting the adaptive behavior described by Mead (1934) and Gergen (1965, 1972) are also possible, and are at the opposite end of the spectrum. In general, one cannot make an observation on a subject in a definite state without altering that state for the purposes of the measurement (in which case the observer is promoted to participator). In its turn, of course, the set of subject responses $\tau$ operates back on the observer, $\langle\psi|\hat{\tau}$, which accounts for the introduction of observer bias.

The behavior in an arbitrary representation, and the interrelationship between various representations can also be considered from a more abstract and formal point of view. We may define a calculus that obeys certain axioms and conditions of reasonableness, and inquire of its rules and properties. For example, we may ask whether it is commutative, which would be represented as $\hat{T}_\psi\hat{T}_\phi|\alpha\rangle = \hat{T}_\phi\hat{T}_\psi|\alpha\rangle$, or we may ask if it is associative. The commutivity property relates to the primacy–recency problem and the importance of sequence in impression formation (Asch, 1946; Asch, 1952, pp. 214–217; Anderson, 1965b) and learning (Luce, 1965). We may also wish to explore the insights obtained by considering certain formal operations involving identities. For example, using the previously expressed expansion $|\phi\rangle = \sum_i \langle i|\phi\rangle|i\rangle$, along with $\langle\psi| = \sum_j \langle\psi|j\rangle\langle j|$, we obtain

$$\begin{aligned}\langle\psi|\phi\rangle &= \textstyle\sum_i\sum_j \langle i|\phi\rangle\langle\psi|j\rangle\langle j|i\rangle \\ &= \textstyle\sum_i \langle i|\phi\rangle\langle\psi|i\rangle \\ &= \textstyle\sum_i \langle\psi|i\rangle\langle i|\phi\rangle \\ &= \langle\psi|\phi\rangle.\end{aligned} \tag{11}$$

To obtain this expression, we require orthogonal basis vectors such that $\langle j \mid i \rangle = \delta_{ij}$, and $\sum_i \mid i \rangle \langle i \mid = I$, where $\delta_{ij}$ is the Kroneker delta and $I$ again represents the identity operator. The above example indicates that an intermediate result for the expression $\langle \psi \mid \phi \rangle$, obtained along the way, has a different form and may therefore provide another perspective on the nature of the quantity $\langle \psi \mid \phi \rangle$.

## DISCUSSION

The use of a mathematical formulation for examining dynamic interactions has the virtue that it forces a clear, unambiguous statement of the relationships required by the givens. It specifies precisely what is intended and exposes that which is contradictory, implied, not clear, or assumed in verbal statements.

A useful descriptive mathematical theory can generally be dissected into the following main constituents (Prugovečki, 1971): (1) The formalism: This consists of a set of symbols and rules of deduction which allow statements or propositions to be made. A formalism may or may not have axioms associated with it. (2) The dynamic law: This expresses the time evolution or behavior of the system and is the key component of the theory in that it provides it with its predictive power. (3) The correspondence rules: These assign empirical meaning to symbols appearing in the formalism. Jammer (1974) draws attention also to the presence of primitive (undefined) notions such as system, observable, and state, and he distinguishes carefully between the formalism and the interpretation of the formalism (the latter representing the model or the physical picture, which is more general than just the dynamic law).

Consideration of the interpretation of the formalism leads us quickly to the realm of metaphysics and to contemplation of the nature of reality and consciousness. We do not deal with these questions in any depth here, but simply point out that our approach is more closely related to that of Bohr than to that of Einstein (see Jammer, 1974, pp. 197, 201). Thus we consider the individual, together with the observer, as forming a single system not susceptible to separation into distinct parts. We assume that the question, "What is the nature of a given person?" presupposes reference to a particular observer to be meaningful (the observer may be the person himself). As Kurt Hübner put it in 1971: "For Einstein, relations are defined by substances; for Bohr, substances are defined by relations" (see Jammer, 1974, p. 157). It goes almost without saying that we appeal also to what Schrödinger (1958) called the Principle of Objectivation—the necessary but artificial exclusion of consciousness from our model. Related to this principle is the theory of measurement (see Jammer, 1974, pp. 470–521); we touched upon this only briefly (and truly inadequately) in the previous section.

Nevertheless, the state-vector formalism appears useful for representing a variety of probabilistic interactions useful in psychiatry and social psychology. It has the virtue that it permits the intuitive generation of correspondence rules without great difficulty. Clearly, it is not the only formalism which may be chosen, but it appears to be a useful choice. The key element still lacking in our treatment is a dynamic law (equation of motion) to describe the time development of the ket. Complete knowledge of a psychological system implies obtaining a (probabilistic) solution for $\mid \phi(t) \rangle$ for all $t > 0$, in the presence of an arbitrary stimulus or measurement.

Such an ambitious task clearly cannot be accomplished. Much as we avoided dealing with the nature of consciousness, so too do we avoid representing the detailed development of the ket over any substantial portion of an individual's life (see Becker, 1973, for a remarkable psychophilosophical synthesis dealing with human development). (Perhaps it is not inappropriate at this point to draw the reader's attention to the fascinating similarity between Bohr's 1934 interpretation of complementarity: "We must, in general, be prepared to accept the fact that a complete elucidation of one and the same object may require diverse points of view which defy a unique description" (see Jammer, 1974, pp. 97–98), and Becker's 1973 interpretation of the existential paradox: "There is no way to overcome the *real* dilemma of existence, the one of the mortal animal who at the same time is conscious of his mortality.")

A modest but perhaps achievable aim of

this work is to specify some characteristics of the dynamic law for the ket during the (relatively brief) time interval $[t, t']$. We do this by using various current theories and interactions postulated by psychologists, psychotherapists, and psychoanalysts. In the next section we describe the time evolution of the ket under some very special conditions.

## TIME DEVELOPMENT

At the outset it is useful to tabulate available information for the dynamic behavior of the ket in a common language for a number of specific interactions. To serve as examples, we schematically examine a psychological model for choice and three idealized representations for the psychotherapeutic interaction (which we consider because of its structured and, therefore, simplified nature). A good deal of information is available from the vast literature on case histories and treatment methods (Freedman & Kaplan, 1967; Loew, Grayson, & Loew, 1975).

### Choice

First we consider choice. Within the range of possible responses to a given situation involving choice (or preference), the probability distribution representing a response may be obtained with the use of a particular mathematical model, e.g., the constant utility model (Luce, 1959), the random utility model (Block & Marschak, 1960), or, as proposed by Restle (1961) and Tversky (1972a, 1972b), the elimination model. In Tversky's elimination by aspects (EBA) model, for example, aspects are interpreted as desirable features. The selection of any particular aspect eliminates all alternatives not containing the selected aspect. The EBA model is represented in the state-vector formalism by denoting various aspects by $\langle k|$, $\langle l|$, $\langle m|$, $\cdots$, and by denoting the subject, together with the range of his possible fantasy choices (responses) designated $A$, $B$, $C$, $\cdots$, by the kets $|\phi_A\rangle$, $|\phi_B\rangle$, $|\phi_C\rangle$, $\cdots$. Choice proceeds by selecting an aspect $\langle k|$, and then by testing the brackets $\langle k|\phi_A\rangle^2$, $\langle k|\phi_B\rangle^2$, $\langle k|\phi_C\rangle^2$, $\cdots$. All $|\phi\rangle$ for which $\langle k|\phi\rangle^2 = 0$ are eliminated, leaving those $|\phi\rangle$ for which $\langle k|\phi\rangle^2 > 0$. The remaining $|\phi\rangle$ are then tested against the next aspect $\langle l|$.

Let us assume, as an artificial but instructive example, that the choice concerns the type of new automobile to be purchased. In a simplified model of the process, we might assume that for a particular subject, the aspect $\langle k|$ might represent high gasoline mileage whereas the aspect $\langle l|$ might represent sufficiently large size. Let $|\phi_A\rangle$ represent the subject with the fantasy choice of a large automobile with low gasoline mileage, $|\phi_B\rangle$ a small sports car with high gasoline mileage, and $|\phi_C\rangle$ a medium-size car with high gasoline mileage. Selecting the first aspect $\langle k|$, high gasoline mileage, the subject discovers that $\langle k|\phi_A\rangle^2 = 0$, thus eliminating $|\phi_A\rangle$. Since $\langle k|\phi_B\rangle^2 > 0$ and $\langle k|\phi_C\rangle^2 > 0$, he proceeds to aspect $\langle l|$. Then discovering that $\langle l|\phi_B\rangle^2 = 0$, he settles on choice $C$ which, in our oversimplified example, is a medium-size car with high gasoline mileage. The probabilistic nature of the process arises from the particular aspects tested and from the allowed fantasy choices, both of which will vary with time. It should be noted that choice models other than EBA, some of which are based on the criterion of general scalability (Tversky, 1972a, 1972b), may be of interest. One such example cited by Tversky (1972b) is the additive random aspect (ARA), also a random utility model. In the ARA model, the aspects are represented by random variables and choice is described as a comparison of sums of random variables, while in the EBA model aspects are constants and choice is by sequential elimination.

### Psychotherapeutic interactions

In characterizing several kinds of psychotherapeutic interactions, we first consider the ideal classical psychoanalytic theory (Becker, 1973; Freedman & Kaplan, 1967; Loew, Grayson, & Loew, 1975). The goal of psychoanalysis is to undo repression and to make the unconscious conscious (through free association and transference). Let the quantity $|\phi(t)\rangle$ represent the patient's state vector which contains both conscious and unconscious parts, and let $|\phi_c(t)\rangle$ represent the conscious part of the patient's state vector. Clearly, the size (or dimensionality) $D$ of the vector space $V$ required to represent $|\phi(t)\rangle$ is larger than that required to represent $|\phi_c(t)\rangle$. We assume that both state vectors are represented on the larger space. The steps involved in progress, in the analytic sense of the concept, then involve the schematic sequence illustrated in Table 1. In this example, the state vector of the therapist does not appear directly; in a nonidealized psychoanalytic situation it would clearly play some role.

Next, we consider a more interventional type of therapy (in which case the therapist takes on a more active role). The gestalt therapist, for example, directs and guides the patient (client) by establishing a connection with him and by encouraging him to role play and to experiment. From a theoretical point of view, the emphasis is on actuality, awareness and acceptance, and wholeness (Loew, Grayson, & Loew, 1975). The patient is encouraged to understand that he effects all choices. Progress comes about by integrating divided parts of the self and by increasing connections with the therapist and others. The quantity $|\phi(t)\rangle$ again represents the patient, and here $\langle\psi|$ represents the therapist. We designate by $|\phi_s(t)\rangle$ the ket for any divided (split) part of the self, or for any object or person that the patient might role play. Thus $s$ includes the affective and physical split parts of the patient himself, pertinent objects, the therapist, and pertinent observers such as family members, co-workers, and friends. The schematic sequence representing the therapeutic interaction is presented in Table 2. We assume that all state vectors are represented on the space of largest dimensionality ($D_3$). Clearly, the fantasy and role playing of divided parts of the self, various objects, and various persons imply unconscious determinates, so that $D_3 > D_2$. In Table 2, we see the presence of the external operator $\hat{T}_\psi$, representing the active role of the therapist; this is in contradistinction to the ideal psychoanalytic process which is noninterventional (see Table 1).

The final specific example that we illustrate here is a behavioral therapy such as Wolpe's (1958) counter conditioning. From a theoretical point of view, behavioral therapies are deterministic and mechanistic in their philosophy. The goals of behavioral therapy are to rid the patient (client) of pathological behaviors (which are assumed to be learned and involuntarily acquired undesirable habits of responding to environmental stimuli) and to replace them by more effective behaviors (Loew, Grayson, & Loew, 1975). Behavioral therapists treat symptoms and modify behaviors, taking full responsibility for the outcome. The therapist is therefore active, either directly promoting change in the patient or educating him in methods of control. Nevertheless,

TABLE 1
SCHEMATIC SEQUENCE REPRESENTING PATIENT PROGRESS IN TERMS OF CLASSICAL PSYCHOANALYTIC THEORY.

| | | |
|---|---|---|
| (a) | Patient has certain conscious view of himself at $t = t_1$. | $\epsilon_i = \langle i \mid \phi_c(t_1)\rangle^2$ for all $i$<br>$D(t_1) \equiv D_1$ |
| (b) | Under gentle persuasion of therapist, patient free associates and forms a transference toward the analyst. | $\hat{T}_\phi \mid \phi_c(t_1)\rangle \rightarrow \mid \phi_c(t_2)\rangle$ |
| (c) | With help of interpretation from analyst, patient develops enlarged conscious view of himself at $t = t_2$. | $\langle \phi_c(t_2) \mid \phi(t_2)\rangle > \langle \phi_c(t_1) \mid \phi(t_1)\rangle$<br>$D_2 > D_1$ |
| (d) | As a result, patient alters behavior in a positive way. | $\mid \phi(t_2)\rangle \neq \mid \phi(t_1)\rangle$ |

TABLE 2
SCHEMATIC SEQUENCE REPRESENTING THE THERAPEUTIC INTERACTION FOR A MORE INTERVENTIONAL FORM OF THERAPY, IN THIS CASE GESTALT THERAPY.

| | | |
|---|---|---|
| (a) | Patient behaves in a certain manner at $t = t_1$. | $\mid \phi(t_1)\rangle$<br>$D(t_1) \equiv D_1$ |
| (b) | Therapist directs and guides patient, encouraging him to fantasize and role play split parts of himself and various objects and persons. | $\hat{T}_\psi \mid \phi(t_1)\rangle \rightarrow \mid \phi(t_2)\rangle$<br>$D_2 \simeq D_1$ |
| (c) | Patient role-plays affective and physical split parts, as well as various objects and persons such as therapist and friends. | $\epsilon_i = \langle i \mid \phi_s(t_2)\rangle^2$ for all $i$, $s$ |
| (d) | Patient enlarges his view by examining and integrating split parts; he increases his connection with therapist and others. | $\hat{T}_\phi \mid \phi(t_2)\rangle \rightarrow \mid \phi(t_3)\rangle$<br>$\langle \phi_s(t_2) \mid \phi(t_3)\rangle$<br>$> \langle \phi_s(t_2) \mid \phi(t_2)\rangle$<br>for all $s$<br>$D_3 > D_2$ |
| (e) | As a result, patient alters behavior in a positive way. | $\mid \phi(t_3)\rangle \neq \mid \phi(t_1)\rangle$ |

appeal to the unconscious is implicit in testing, and in such techniques as behavior rehearsal, role playing, scene reinactment, and imagery. Indeed, these techniques are closely related to gestalt therapy. In Table 3, the patient and therapist are represented by $|\phi\rangle$ and $\langle\psi|$, respectively. The external operators $\hat{T}_\psi'$ and $\hat{T}_\psi''$, representing the active behavioral therapist, are observed to act on the vector space (environment) rather than on the ket (patient), which remains stationary.

Such an alteration of $V$ with time has an overall effect similar to a change of $|\phi\rangle$ with time, since $|\phi\rangle$ is defined relative to $V$, but the two processes are distinct. The relationship is similar to the Schrödinger and Heisenberg pictures of the evolution of a quantum system; von Neumann showed that both pictures produce valid and equivalent results in Hilbert space though we understand them differently (see Jammer, 1974). Thus, while there is a great distinction on philosophical grounds between behavioral therapy and psychoanalysis, we can understand how both may work. As a clinical example, both analysis and behavioral therapy can relieve depression and anxiety. It must be kept in mind, however, that while interesting as examples, the models discussed above are idealizations; most practical psychotherapies involve the time evolution of both $|\phi\rangle$ and $V$. The analogy in the quantum mechanics is the "interaction representation," intermediate between the Schrödinger and Heisenberg pictures.

TABLE 3
SCHEMATIC SEQUENCE REPRESENTING THE THERAPEUTIC PROCESS FOR AN IDEALIZED BEHAVIOR THERAPY, PATTERNED ON WOLPE'S COUNTER CONDITIONING.

| | | |
|---|---|---|
| (a) | Patient behaves in a certain manner at $t = t_1$. | $\|\phi(t_1)\rangle$<br>$D(t_1) \equiv D_1$ |
| (b) | Therapist questions, tests, and educates patient; shows sympathy toward and establishes goals with patient; formulates treatment plan for behaviors to be modified. | $\hat{T}_\psi \|\phi(t_1)\rangle \to \|\phi(t_2)\rangle$<br>$D_2 \simeq D_1$ |
| (c) | Therapist teaches patient self-observation and self-control techniques (e.g., thought control, relaxation training) later to be practiced at home. Family and group counseling. | $\hat{T}_\psi'\{V(t_2)\} \to \{V(t_3)\}$<br>$D_3 > D_2$ |
| (d) | Therapist calls upon reciprocal inhibition techniques (e.g., systematic desensitization, assertion training) to link pairs of characteristics. | $\hat{T}_\psi''\{V(t_3)\} \to \{V(t_4)\}$<br>$D_4 > D_3$<br>Orthogonality eliminated between appropriate pairs and rotation of space imposed. |
| (e) | Patient responds by demonstrating changed behavior within altered space. | $\langle i(t_4) \| \phi\rangle^2 \neq \langle i(t_1) \| \phi\rangle^2$<br>$\{V(t_4)\} \neq \{V(t_1)\}$ |

Considering these three psychotherapies in terms of a common mathematical formalism, it is clear that many psychologies may effectively be more alike than they appear. Some may be virtually identical, differing only by the particular subset of characteristics on which they concentrate. In such cases of similar structure we might expect, because of correlation of attributes, that a particular ket rotation relative to a specific subset of characteristics may result in a simultaneous rotation in other subsets as well. A better understanding of such coupling between subsets may serve to provide a more concrete explanation of why different therapies may work with a given patient. More importantly, it could possibly lead to an optimization of the form of therapy for a particular patient. Furthermore, it is evident from Tables 1, 2, and 3 that the psychoanalytic, gestalt, and behavioral therapeutic processes have a very important common denominator: a vector space whose dimensionality increases as the process proceeds. Stated in different words, there is a central thread running through all three approaches, and it is the increase in the patient's repertoire of effective interactions. In psychoanalysis, the increased dimensionality is drawn from the unconscious; in gestalt therapy, from fantasy (which implies the unconscious); and in behavioral therapy, from the therapist and from the altered environment.

## APPLICATIONS

Examples of other applications of the state-vector formalism that might be expected to produce results of interest are the following:

(1) Several sociopsychological processes that depend on time, including attitude change, preference, structural balance, cognitive dissonance, conformity, and persuasion, can be analyzed dynamically.

(2) Since finite and continuous response learning models can be considered with little mathematical difficulty because of their considerably simplified response class (Rosenberg, 1968), a relatively explicit equation of motion can be obtained for a particular (and usually small) subset of the ket $|\phi_m\rangle$. The relationship of $|\phi_m(t)\rangle$ and $|\phi(t)\rangle$ may be investigated by examining correlated attributes. Formats for commutative and noncommutative operators applied to learning (Luce, 1965) are particularly easy to represent in the state–vector formalism.

(3) Several representative patient/therapist interactions, including classical Freudian psychoanalysis, the interpersonal and cultural psychoanalytic theories of Adler, Sullivan, and Horney, the existential psychology of Jaspers (1963), and the behavioral therapies of Wolpe (1958) and Skinner (1953) may be chosen for analysis. Using the details of specific case histories, we can define appropriate operators and vector spaces, obtain correspondence rules, and examine the characteristic time development for the state vector $|\phi\rangle$ and for the vector space $V$, presenting the information in tabular form. Each relationship in the table can be associated with a verbal statement describing it. Realistic representations can be found by referring to specific case histories as indicated above. (Freud, for example, was interventional to a degree as illustrated by his Wolf-man case.) We can then attempt to tie together the various information gathered for the interaction process by searching for mathematical similarities among the various representations. For example, the formats obtained for Freud and Adler can be examined in light of trait correlation (nonorthogonal axes) in an affine vector space which may be used to represent these psychologies. This would clarify the role that different types of therapy may play with a given patient. As another example, we might also look for formal similarities in analytical and existential therapy techniques in light of the relativity of the rotation of the ket with respect to the vector space.

(4) The state–vector formalism can also be applied to the analysis of group therapeutic interactions (Kadis, Krasner, Winick, & Foulkes, 1963). Use can be made of existing mathematical models for small group processes (Coleman, 1960; Horvath, 1965). An application of this kind might be particularly interesting since many simultaneous and interrelated interactions make verbal understanding especially difficult. A particular experiment that it may be useful to conceptualize within this framework is the selection of a new member for an already formed therapy group. By studying the dynamics of the group over a period of time, the therapist could form some idea of the range of subsets $G$ over which certain pairs of subgroups of patients seemed to interact fruitfully. Then, depending on the theory of the group and individual knowledge of the new patient $|\phi\rangle$, the therapist could better decide whether the patient would benefit from and/or contribute to that particular group, at least in terms of its stated goals. Given certain specified criteria for good performance, therefore, one could appropriately constitute a group by choosing patients according to a prescription such as that described above. One obvious difficulty is the determination of the magnitudes of various brackets over a given subset. Nevertheless, a seasoned group therapist has a reasonable knowledge of the interactions among his group members, in which case he should be able to estimate these quantities, at least roughly. The assumption of an ideal therapist is implicit in our remarks; in actuality, he must form part of the interaction, and should therefore properly be included in it. The group situation is further complicated by the effects on a given patient $|\alpha\rangle$ by an interaction between patients $\langle\beta|$ and $\langle\gamma|$, or between the therapist $\langle\psi|$ and a patient $|\delta\rangle$. These effects can phenomenologically be accounted for by terms of the form $\hat{T}_{\beta\gamma}|\alpha\rangle$, where $\hat{T}_{\beta\gamma}$ represents an effective operation performed on $|\alpha\rangle$ by the interaction of $\langle\beta|$ and $\langle\gamma|$. Again, the particular representation to be used in examining such a problem depends on the nature of the group; an analytically oriented group operates differently from a teaching group. And indeed, a great deal of observational information would be necessary in order to carry out such an analysis. Nevertheless, with the

appropriate assumptions and a choice of representation, a computer simulation of a particular group interaction would likely provide some otherwise unobtainable insights.

We should point out that for both group and individual interaction, it may not always be a simple matter to make the appropriate identification between a given psychology and the state-vector formalism, and it is anticipated that reference to case histories will be of great value in an operational understanding of a particular mode of therapy. Our work is preliminary and as such it is possible that, after a good deal of study, certain psychologies may not fit within the confines of the formalism presented here. In that case, the formalism would have to be either generalized, altered, or possibly abandoned for that particular therapy.

Finally, we note that from a general systems theory point of view, the usefulness of the state-vector formalism could possibly be extended from the levels of the human organism and group, as presented here, to the levels of organization, society, and supranational systems.

## CONCLUSION

This work specifies a useful framework which admits the representation of a number of existing, specific theories in mathematical psychology. It is expected to allow for the codification and clarification of the interrelationships among these theories and represents an attempt to bring existing knowledge in mathematical psychology and psychiatry closer together and into a mutally supportive role. It could provide a research tool in psychotherapy leading to new insights, and possibly to understanding patterns in interpersonal interaction that might otherwise be difficult to discern. The concept may eventually be used to specify an optimal (or suboptimal) choice of modality for a particular patient in individual therapy, or possibly to match a patient with a particular school of group therapy. We emphasize that the formalism presented here is primitive by any standard. It is by no means unique, nor it is likely to be an optimal choice. Rather it provides a point of departure in choosing among a wide variety of existing mathematical constructs. A restriction of the class of possible choices can only be achieved by continually folding in new experimental results and insights, in the process narrowing the choice (perhaps through the elimination by aspects model!).

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