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Normalizing Transformations for Dead-Time-Modified Poisson Counting Distributions

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Abstract. Approximate normalizing transformations are derived for Poisson counting systems affected by nonparalyzable and paralyzable dead time. In the nonparalyzable case the transformation takes the form of a simple inverse hyperbolic function whereas in the paralyzable case it is an inverse trigonometric function. The results are expected to find use in neural counting, photon counting, and nuclear counting, as well as in queuing theory.

1 Introduction

The dead-time-modified Poisson counting distribution has found application in a number of scientific disciplines including neural counting (Ricciardi and Esposito 1966; Teich et al. 1978; Teich and Diament 1980; Prucnal and Teich 1983; Lachs et al. 1984), photon counting (Bédard 1967; Cantor and Teich 1975; Teich and Cantor 1978; Teich and Vannucci 1978; Vannucci and Teich 1979), and nuclear counting (DeLotto et al. 1964; Müller 1973, 1974; Libert 1976). Many cases have been studied in detail, including nonparalyzable (nonextended) and paralyzable (extended) dead-time counting under blocked, unblocked, and equilibrium conditions of the onset of the counting interval. Müller (1981) has summarized the results of many authors and has compiled a comprehensive and useful bibliography of the effects of dead time on various counting processes. In the biological sciences literature the term refractoriness is often used synonomously with the term dead time.

The object of this paper is to present normalizing transformations for nonparalyzable and paralyzable dead-time-modified Poisson counting distributions. The results are derived in the regime where the number of events recorded during a sampling time is much greater than unity, so that the differences among blocked, unblocked, and equilibrium counters are negligible. The treatment is restricted to homogeneous Poisson counting modified by fixed dead time. In certain circumstances these results will be directly applicable to problems involving stochastic dead time and sick time, i.e., gradual recovery (Cox 1962; Parzen 1962; Teich et al. 1978; Teich and Diament 1980). This will apply when many events are captured in the counting interval. In other circumstances, more specialized results can be derived (Young and Barta 1985).

The normalizing transformation can be a useful tool in dealing with a set of data because it increases the degrees of approximation to which a number of desirable properties hold (Tukey 1957). These include: (1) Rendering nonadditive (signal-dependent) noise additive (signal-independent). This permits traditional signal estimation and detection techniques and measures, based on additive signal-independent Gaussian noise, to be used (Prucnal and Teich 1980; Prucnal and Saleh 1981). (2) Stabilizing the variance (error of variability). This permits conventional measures, such as signal-to-noise ratio and d' to be used (Prucnal and Teich 1980). It thereby allows data to be analyzed more effectively and a detection law to be framed (Prucnal and Teich 1980). (3) Rendering the probability density function of the data more symmetrical and more nearly normal. This provides ease of calculation when the probability density function is of interest. A transformation suitable for improving one degree of approximation often turns out to be suitable for improving either or both of the others (Tukey 1957).

The exact normalizing transformation is known for the lognormal (Pruncal and Teich 1980), chi-squared, and noncentral chi-squared random variables (Saleh 1978). Approximate normalizing transformations have been obtained for various other cases such as the Poisson, binomial, and negative binomial distributions (Mattick et al. 1935; Bartlett 1936; Cochran 1940; Curtiss 1943; Anscombe 1948; Freeman and

Tukey 1950). The square-root transformation is often used for the Poisson (Tukey 1957) and Neyman Type-A (Prucnal and Teich 1982) distributions.

2 Approximate Normalizing Transformation

In cases where the count variance can be expressed as a function of the count mean and some arbitrary set of constants, it is often possible to derive an approximate normalizing transformation (Kendall and Stuart 1966). This is the case for the dead-time-modified Poisson counting distributions.

Consider a random variable x with mean N and variance σ^2 where the variance is expressible as a function of the mean, viz.,

$$\sigma^2 = D^2(N) \tag{1}$$

with $D(\cdot)$ a known function. A point transformation $y = g(x)$ is desired such that the variance is rendered approximately independent of the mean count. The resulting random variable will be nearly normally distributed. Since dead-time-modified Poisson counting distributions are generated from renewal processes, they are asymptotically normal to begin with, but, of course, the variance is a function of the mean (Parzen 1962).

The normalizing transformation can be chosen to be monotonic (Prucnal and Teich 1980) which implies that the inverse transformation exists. The approximate solution (yielding $\sigma^2 \approx 1$) is provided by the expression

$$g(x) \approx k \int \frac{dN}{D(N)}\bigg|_{x=N}, \tag{2}$$

where k is a constant (Kendall and Stuart 1966; Prucnal 1980; Prucnal and Saleh 1981). It is apparent that Eq. (2) leads to the square-root transformation for Poisson and Neyman Type-A counting distributions since then $D(N) = \sigma = N^{1/2}$.

3 Nonparalyzable Dead Time

The asymptotic mean and variance for the nonparalyzable dead-time-modified Poisson counting distribution are given by (Feller 1948; Parzen 1962; Müller 1974)

$$N \approx \langle n \rangle / (1 + \langle n \rangle \tau / T) \tag{3}$$

and

$$\sigma^2 \approx \langle n \rangle / (1 + \langle n \rangle \tau / T)^3, \tag{4}$$

respectively, where $\langle n \rangle$ is the mean initial (unmodified) Poisson count, τ is the dead time, and T is the counting time. Combining Eqs. (3) and (4) to eliminate $\langle n \rangle$ provides

$$\sigma \approx N^{1/2}(1 - N\tau/T), \tag{5}$$

where $N\tau/T < 1$. Inserting Eq. (5) into Eq. (2), with the substitutions $w = N^{1/2}$ and $a = (T/\tau)^{1/2}$, leads to the expression (Dwight 1961, p. 35, Eq. 140.1)

$$g \approx 2ka^2 \int \frac{dw}{a^2 - w^2} = 2ka^2[(1/a)\tanh^{-1}(w/a)]\,. \tag{6}$$

Note that $a^2 > w^2$ since $N\tau/T < 1$. Finally, therefore, the normalizing transformation for nonparalyzable dead-time-modified Poisson counting is given by

$$g(x) \approx 2k(T/\tau)^{1/2} \tanh^{-1}(x\tau/T)^{1/2}\,. \tag{7}$$

The limiting case of vanishing dead time ($N\tau/T \to 0$) is recovered by using the first term of the Taylor series expansion (Dwight 1961, p. 166, Eq. 708)

$$\tanh^{-1} v = v + v^3/3 + v^5/5 + \dots, \tag{8}$$

where $v = (x\tau/T)^{1/2}$. In this special case

$$g(x, \tau = 0) = 2kx^{1/2}, \tag{9}$$

as for the unmodified Poisson distribution.

4 Paralyzable Dead Time

For a paralyzable dead-time-modified Poisson counting distribution, the mean and variance are given by (Feller 1948; Mueller 1954; Parzen 1962; Müller 1974)

$$N_\pi \approx \langle n \rangle \exp(-\langle n \rangle \tau / T), \tag{10}$$

and

$$\sigma_\pi^2 \approx \langle n \rangle \exp(-\langle n \rangle \tau / T) - 2(\tau/T)\langle n \rangle^2 \exp(-2\langle n \rangle \tau / T), \tag{11}$$

respectively, where again $\langle n \rangle$ is the mean initial (unmodified) Poisson count, τ is now the paralyzable dead time, and T is the counting interval. Expressing Eq. (11) in terms of Eq. (10) to eliminate $\langle n \rangle$ provides

$$\sigma_\pi \approx N_\pi^{1/2}(1 - 2N_\pi \tau/T)^{1/2}, \tag{12}$$

where $2N_\pi\tau/T < 1$. Inserting Eq. (12) into Eq. (2), and with the help of the substitutions $u = N_\pi^{1/2}$ and $b = (T/2\tau)^{1/2}$, we obtain (Dwight 1961, p. 67, Eq. 320.01)

$$g_\pi \approx 2kb \int \frac{du}{(b^2 - u^2)^{1/2}} = 2kb \sin^{-1}(u/b)\,. \tag{13}$$

Here $b^2 > u^2$ since $2N_\pi\tau/T < 1$. The normalizing transformation for the paralyzable dead-time-modified Poisson counter therefore turns out to be

$$g_\pi(x) \approx 2k(T/2\tau)^{1/2} \sin^{-1}(2x\tau/T)^{1/2}. \tag{14}$$

The character of Eq. (14) is not unlike that of Eq. (7) for the nonparalyzable counter.

In this case, the limiting result for vanishing dead time ($2N_\pi\tau/T \to 0$) is obtained by substituting in Eq. (14) the first term of the Taylor series expansion (Dwight 1961, p. 118, Eq. 501)

$$\sin^{-1} s = s + s^3/6 + 3s^5/40 + \ldots, \tag{15}$$

where $s = (2x\tau/T)^{1/2}$. This leads to

$$g_\pi(x, \tau = 0) = 2kx^{1/2}, \tag{16}$$

again in accord with the unmodified Poisson distribution and with Eq. (9).

5 Discussion

We have previously shown that the normalizing transformation can be a useful tool in determining the detection (increment-threshold) law for sensory systems (Prucnal and Teich 1980). At low stimulus levels, dead-time-modified Poisson counting leads approximately to the square-root transformation, as illustrated by the limits derived here. As a result, something close to the deVries-Rose increment-threshold law emerges (Prucnal and Teich 1980). At high stimulus levels, the result for the nonparalyzable counter turns out to be Weber's detection law (Teich and Lachs 1979, 1983). The variance-stabilizing transformations presented here, which are appropriate for neural systems obeying dead-time-modified Poisson counting, provide a direct path to the detection law for arbitrary stimulus levels.

In many neural systems, saturation and refractoriness are simultaneously present (Teich et al. 1978; Lachs et al. 1984). In cases such as these, the normalizing transformation can be useful in disentangling the two effects although this can also be accomplished by direct calculation as illustrated by Teich et al. (1978) for data from the cat's retinal ganglion cell.

A great deal of neural data can be understood best on the presumption that the refractoriness is stochastic (Teich et al. 1978; Young and Barta 1985) or relative (Teich and Diament 1980), rather than fixed as assumed here. Normalizing transformations for these situations can also be derived. Indeed Young and Barta (1985) recently obtained the normalizing transformation for a Poisson process modified by exponentially distributed nonparalyzable dead time.

In terms of future generalizations, it might be of interest to examine the normalizing transformation for the general type-p counter, which reduces to nonparalyzable and paralyzable behavior as special cases (Albert and Nelson 1953; Parzen 1962).

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