3.2 EQUALIZING ENERGIES

For the two 10-particle two-state systems, take the total energy shared to be

\[ U = U_A + U_B = 4 \]

The multiplicity for the system \( W(U) \) of given energy \( U \) will be

\[ W(U=4; U_A) = W_A(U_A) W_B(4-U_A) \]

which for the binomial statistics can be written

\[ W(U=4; U_A) = \frac{10!}{U_A!(10-U_A)!} \cdot \frac{10!}{(4-U_A)!(10-4+U_A)!} \]

Evaluating for values of \( U_A \) you find

| \( U_A \) | 0 1 2 3 4 |
|---|---|---|---|---|
| \( W \) | 210 1200 2025 1200 210 |

The highest multiplicity is found when the energy is equally divided between the two two-level systems.

3.3 ENERGY CONVERSION

\[ (100 \text{ kcal})(4.18 \text{ J/cal})\left(\frac{1}{10 \text{ min}}\right)\left(\frac{1 \text{ min}}{60 \text{ s}}\right) = 697.5 \text{ J/s} \]

100,000 cal

\[ \frac{1}{\text{watts}} \]

3.4 KINETIC ENERGY OF A CAR

\[ \text{kinetic energy} = \frac{1}{2} m v^2 = \frac{1}{2} (1700 \text{ kg}) \left[ \left(\frac{100,000 \text{ m}}{\text{hr}}\right)\left(\frac{\text{hr}}{3600 \text{s}}\right) \right] \]

\[ = 655.8 \text{ kJ} \]
3.5 ROOT-MEAN-SQUARE (RMS) VELOCITY OF A GAS

For O₂ gas at T=300K where

\[ \frac{1}{2} k_B T = \frac{1}{2} m \langle v_x^2 \rangle \]

and

\[ M = \frac{32 \text{ g/mol}}{\frac{1}{6.02 \times 10^{23} \text{ mol}^{-1}}} \times \frac{1 \text{ kg}}{1000 \text{ g}} = 5.316 \times 10^{-24} \text{ kg/mole} \]

so that

\[ \langle v_x^2 \rangle = \frac{k_B T}{m} = \frac{(1.38 \times 10^{-23} \text{ J/K}) \text{ (300 K)}}{5.316 \times 10^{-24} \text{ kg/mole}} = 77.9 \times 10^3 \frac{\text{ m}^2}{\text{s}^2} \]

or

\[ \langle v_x^2 \rangle^{\frac{1}{2}} = 279 \frac{\text{ m}}{\text{s}} \]

3.6 EARTH'S ENERGY BALANCE

(a) The energy flux absorbed by each m² is 168 + 324 = 500 W/m² so that the energy flux absorbed by the backyard is

\[ (500 \frac{\text{ W}}{\text{ m}^2}) (1000 \text{ m}^2) \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) = 88.3 \times 10^3 \text{ W} \]

\[ \frac{1}{2} \text{ of day} \quad \text{overcast} \]

(b) A solar cell can convert per m² about 16.167 W/1000 m² = 16.7 W/m². For 2 W one would need \[ \frac{2 W}{16.7 \text{ W/m}^2} = 0.12 \text{ m}^2 \text{ solar cell} \].

(c) In the absence of greenhouse gases, the Earth would lose

\[ 60\% = \frac{324 \text{ W/m}^2}{(324 + 168) \text{ W/m}^2} \text{ back radiation} \]

which would cause a substantial cooling of the Earth (apparently the Earth would be 255 K!).
(d) The energy in a gallon of gasoline can be computed assuming gas is 100% octane \((C_{8}H_{18})\), with density \(1 g/cm^3\), each C-C bond contributing 60 kcal/mol, and 50% efficiency in burning. We have

\[
C_{8}H_{18} = \frac{114 g}{mol}
\]

\[
1\text{ gallon} \left(\frac{3.785 L}{1\text{ gallon}}\right) \left(\frac{1 kg}{1 L}\right) = 3.785\text{ kg octane}
\]

\[
1 g/cm^3 \left(\frac{1 kg}{1000 g}\right) \left(\frac{1000 mL}{1 L}\right) \left(\frac{1 cm^3}{1 mL}\right)
\]

so that

\[
(3.785 \text{ kg octane}) \left(\frac{1\text{ mol}}{114 g}\right) \left(\frac{7\text{ mol C-C bond}}{1\text{ mol}}\right) \left(\frac{251\text{ kJ}}{mol\text{ C-C bond}}\right) = 5.8 \times 10^7 J
\]

And at 50% efficiency \((0.50)(5.8 \times 10^7 J) = 2.9 \times 10^7 J\) that is also

\(2.9 \times 10^7\) Ws or \(8100\) Wh.

3.7 TAKING THE EARTH’S TEMPERATURE

The Stefan-Boltzmann law states that the electromagnetic power, \(P\), is related to the temperature of the object as

\[
P = (5.67 \times 10^{-8} \frac{W}{m^2 K}) T^4
\]

at steady state.

(a) The incoming solar power hitting the Earth is \(P = 342 \frac{W}{m^2}\), so the predicted \(T\) of the Earth is

\[
\left(\frac{342 W}{m^2}\right) \left(\frac{5.67 \times 10^{-8} W}{m^2 K}\right)^{-1} = T = 255 K = -18^\circ C = 2^\circ F
\]

(b) The Earth’s average \(T\) is \(288 K\) so that

\[
P = (5.67 \times 10^{-8} \frac{W}{m^2 K}) (288 K)^4 = 390 \frac{W}{m^2}\]
3.8 Why do elephants live longer than mice?

We assume that the rate of energy expenditure per unit mass of biological tissue is

\[
\frac{\text{basal metabolic rate}}{\text{unit mass}} = \frac{\text{BMR}}{M}
\]

and is proportional to energy flow into the tissue. We further assume that the blood flow to the tissue is proportional to the energy flow and also is correlated with the heart rate of the organism (HR). This implies

\[
\text{HR} \propto \frac{\text{BMR}}{M}
\]

From Figure 3.5 there is a strong argument that

\[
\text{BMR} \propto M^{3/4}
\]

for organisms ranging from single cells, mitochondria to mammals, elephants. As such

\[
\text{HR} \propto \frac{\text{BMR}}{M} \propto \frac{M^{3/4}}{M} \propto M^{-1/4}
\]

indicating HR decreases as the size of the organism increases. If we accept that mammalian species have

\[
1.5 \times 10^7 \text{ heart beats} = \text{HB}
\]

we expect the lifespan of an organism to vary with mass as

\[
L \propto \text{HB} \propto M^{1/4}
\]

A scaling law suggesting elephants live longer than mice.
4.2 DIFFERENTIATING MULTIVARIATE FUNCTIONS

(a) \( f(x,y) = 3x^2 + y^6 \)
\[
\left( \frac{\partial f}{\partial x} \right)_y = 6x \quad \left( \frac{\partial f}{\partial y} \right)_x = 6y^5
\]

(b) \( f(x,y) = x^{10}y^{\frac{1}{2}} \)
\[
\left( \frac{\partial f}{\partial x} \right)_y = 10x^9y^{\frac{1}{2}} \quad \left( \frac{\partial f}{\partial y} \right)_x = \frac{1}{2}x^{10}y^{-\frac{1}{2}}
\]

(c) \( f(x,y) = x + y^2 + 3 \)
\[
\left( \frac{\partial f}{\partial x} \right)_y = 1 \quad \left( \frac{\partial f}{\partial y} \right)_x = 2y
\]

(d) \( f(x,y) = 5x \)
\[
\left( \frac{\partial f}{\partial x} \right)_y = 5 \quad \left( \frac{\partial f}{\partial y} \right)_x = 0
\]

4.4 MAXIMIZING A MULTIVARIATE FUNCTION WITHOUT CONSTRAINTS

For the function
\[
f(x,y,z) = d - (x-a)^2 - (y-b)^2 - (z-c)^2
\]
the maximum is found when
\[
\left( \frac{\partial f}{\partial x} \right)_{y,z} = \left( \frac{\partial f}{\partial y} \right)_{x,z} = \left( \frac{\partial f}{\partial z} \right)_{x,y} = 0
\]
and
\[
\left( \frac{\partial f}{\partial x} \right)_{y,z} = -2(x-a) = 0 \quad x^* = a \quad \left( \frac{\partial f}{\partial y} \right)_{x,z} = -2(y-b) = 0 \quad y^* = b
\]
\[
\left( \frac{\partial f}{\partial z} \right)_{x,y} = -2(z-c) = 0 \quad z^* = c
\]
and the extremum is \((x^*, y^*, z^*) = (a, b, c)\) where \(f(a, b, c) = d\).
We have two independent variables \( x \) and \( y \) with average values \( \bar{x} \) and \( \bar{y} \). The function \( f(x,y) \) is a dependent variable (on \( x \) and \( y \)) with average \( \bar{f} \). We call the deviations from the averages

\[
e_x = x - \bar{x} \quad \varepsilon_y = y - \bar{y} \quad \varepsilon_f = f - \bar{f}
\]

(a) To express the error in \( f \), \( \varepsilon_f \), as a function of \( \varepsilon_x \) and \( \varepsilon_y \) we start from a Taylor series

\[
f(x) = \bar{f} + \left( \frac{df}{dx} \right)_{\bar{x}} (x - \bar{x}) + \left( \frac{df}{dy} \right)_{\bar{y}} (y - \bar{y}) + \text{higher order terms}
\]

so that

\[
\varepsilon_f = f - \bar{f} = \left( \frac{df}{dx} \right)_{\bar{x}} \varepsilon_x + \left( \frac{df}{dy} \right)_{\bar{y}} \varepsilon_y
\]

If the errors \( \varepsilon_x \) and \( \varepsilon_y \) are small, this will do. If they are larger, we need to include higher order terms.

(b) To find the mean-square error in \( f \), \( \langle \varepsilon_f^2 \rangle \), in terms of \( \langle \varepsilon_x^2 \rangle \) and \( \langle \varepsilon_y^2 \rangle \) we can use

\[
\langle \varepsilon_f^2 \rangle = \left[ \left( \frac{df}{dx} \right)_{\bar{x}} \varepsilon_x + \left( \frac{df}{dy} \right)_{\bar{y}} \varepsilon_y \right]^2
\]

\[
= \left[ \left( \frac{df}{dx} \right)_{\bar{x}} \right]^2 \varepsilon_x^2 + \left[ \left( \frac{df}{dy} \right)_{\bar{y}} \right]^2 \varepsilon_y^2 + \left( \frac{df}{dx} \right)_{\bar{x}} \left( \frac{df}{dy} \right)_{\bar{y}} \varepsilon_x \varepsilon_y
\]

and as we average we first note that as the variables \( x \) and \( y \) are independent, it must be that the errors are uncorrelated so...
that
\[ \langle e_x e_y \rangle = 0 \]

errors in \( x \) and \( y \) uncorrelated

and finally our result
\[ \langle e_f^2 \rangle = \left[ \left( \frac{\partial f}{\partial x} \right) \right] \langle e_x^2 \rangle + \left[ \left( \frac{\partial f}{\partial y} \right) \right] \langle e_y^2 \rangle \]

(c) for the ideal gas law
\[ P = \frac{RT}{V} \]

we can take \( f = P \), \( x = T \) and \( y = V \). If there are 10% random errors in \( T \) and \( V \), the errors in \( P \) are expected to be
\[ \langle e_p^2 \rangle = \left[ \left( \frac{\partial P}{\partial T} \right) \right] \langle e_T^2 \rangle + \left[ \left( \frac{\partial P}{\partial V} \right) \right] \langle e_V^2 \rangle \]

\[ \left( \frac{\partial P}{\partial T} \right)_V = \frac{R}{V} \quad \frac{\delta T}{T} = 0.1 \]
\[ \left( \frac{\partial P}{\partial V} \right)_T = -\frac{RT}{V^2} \quad \frac{\delta V}{V} = 0.1 \]

leading to
\[ \langle e_p^2 \rangle = \left( \frac{R}{V} \right)^2 T^2 (0.1)^2 + \left( -\frac{RT}{V^2} \right)^2 V^2 (0.1)^2 \]

\[ = 2 \left( \frac{RT}{V} \right)^2 (0.01) = 0.02 P^2 \]

(d) For \( n \) measurements, each with uncorrelated error, we expect
\[ e_f^2 = n \left( \frac{\partial f}{\partial x} \right)^2 e_x^2 \]

so that \( e_f \propto \sqrt{n} e_x \).
4.9 Small Differences in Large Numbers Can Lead to Non-Sense

Building on our result for 4.8, suppose we have a function

\[ f(x, y) = x - y \]

with \( x = 20 \pm 2 \) and \( y = 19 \pm 2 \). We know that

\[
\langle \varepsilon_f^2 \rangle = \left[ \left( \frac{df}{dx} \right)_y \right]^2 \langle \varepsilon_x^2 \rangle + \left[ \left( \frac{df}{dy} \right)_x \right]^2 \langle \varepsilon_y^2 \rangle
\]

where

\[
\left( \frac{df}{dx} \right)_y = 1 \quad \left( \frac{df}{dy} \right)_x = -1
\]

so that

\[
\langle \varepsilon_f^2 \rangle = \langle \varepsilon_x^2 \rangle + \langle \varepsilon_y^2 \rangle = 4 + 4 = 8
\]

and it follows

\[ \varepsilon_f = \langle \varepsilon_f^2 \rangle^{1/2} = \sqrt{8} = 2.83 \]

so that if we think of

\[ f(x, y) = x - y = (20 \pm 2) - (19 \pm 2) = 1 \pm 2.83 \]

and the error is substantially larger than the difference itself.