Demand-Invariant Price Relationships and Market Outcomes in Competitive Private Commons

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We introduce a private commons model that consists of network providers who serve a fixed primary demand and price strategically to improve their revenues from an additional secondary demand. For *general* forms of secondary demand, we establish the existence and uniqueness of two characteristic prices: the break-even price and the market sharing price. We show that the market sharing price is *always* greater than the break-even price, leading to a price interval in which a provider is both profitable and willing to share the demand. Making use of this result, we give insight into the nature of market outcomes.

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1. INTRODUCTION

With the increasing use of wireless devices and a fixed bandwidth of radio spectrum, it is important to increase the utilization of spectrum in order to meet the demand. One of the underlying inefficiencies in the spectrum market is the licensing regulations. Therefore, major efforts are underway to make radio spectrum licensing more flexible, hence allowing license holders (*e.g.*, network providers) to lease spectrum to third parties [Akyildiz et al. 2006; Bykowsky 2003; Chapin and Lehr 2007]. In an effort to promote more efficient usage of the electromagnetic spectrum, the FCC is promoting new paradigms for spectrum sharing.

One such paradigm is the *Private Commons*, which is deemed both "commercially viable and technologically feasible" [Buddhikot 2007]. This paradigm supports spectrum transactions, where ownership of spectrum remains with the license holder providing service to its primary users, but this provider may also provide spectrum access to secondary users for a fee. The Amazon's Kindle model can be viewed as an early realization of this paradigm, in which owners of Kindle e-readers make secondary use of AT&T network to retrieve contents from the cloud. Other precursors include machine to machine (M2M) communication and mobile virtual network operators (MVNOs), such as Republic Wireless, that mainly rely on Wi-Fi and utilize the licensed spectrum of a cellular network as a fallback.

Private commons hold significant potential to increase spectrum utilization. For instance, cellular networks are generally over-provisioned to cope with short-term spikes in their loads. Large-scale measurement studies in the US and in Germany indeed indicate that the majority of base stations in crowded areas, such as city centers, remain underloaded by its contracted users at all times [McHenry and McCloskey 2005; Michalopoulou et al. 2011]. Another measurement based study by Kone et. al. [2012] indicates that conservative policies that minimize interference to primary users (such as one proposed by Jung and Liu [2012]) result in spectrum inefficiencies, where only 20-30% of the available spectrum is extracted for secondary use. Such studies suggest that providing spot-on service to secondary users could increase spectrum utilization levels, thus translating into increased revenues for the provider.

A network providing secondary spectrum access has two major challenges to resolve. The first such challenge is to keep a profitable margin by making the correct strategic pricing decision. The difficulty of this challenge lies with the uncertainty in the demand response to the advertised price, which is

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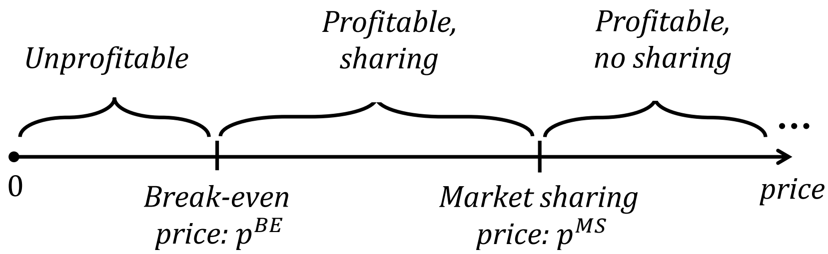


Fig. 1. Illustration of profitability and market sharing price intervals

generally stipulated by a so-called *demand function*. However, the exact form of this function is hard to characterize (since it is derived from specific assumptions on the amount of utility consumers associate with the provisions) and may also be time-varying.

The second challenge is the market competition that a provider faces in spectrum offerings. When several providers offer secondary spectrum provisions, this opens up the possibility of market sharing by advertising the same price. However, it is not clear whether such an action is favorable over trying to capture the entire market by slightly lowering the price in turn. The outcomes of market competition also determine whether these new spectrum sharing technologies will become monopolized or foster a free market competition that benefits the users.

The main goal of our paper is to provide insight, applicable to *general* demand functions, into the market outcomes of a game involving multiple providers offering secondary spectrum access in their private commons. We consider an uncoordinated access setting, where each provider accepts secondary users to enhance its revenue while minimizing the loss due to blocked primary users. Toward this end, we identify two price thresholds playing a critical role for each provider, and establish a fundamental relationship between them. Next, we consider a competitive scenario and seek to answer the question whether a single provider will win the entire secondary market or several providers will choose to share the market. Either way, how do network parameters such as traffic intensities and network capacities affect this market outcome? Are there certain parameters that do not depend on the specific shape of the demand function? Can we identify a unique set of prices that can be used to define a market equilibrium?

The contributions of this paper are as follows: We start with identifying and analyzing two characteristic prices which dictate provider behavior in secondary spectrum markets. First, we prove the existence of a unique *break-even* price p^{BE} that guarantees a positive profit as long as a provider sets its price above it. Next, we derive another unique threshold price, called *market sharing* price p^{MS}, below which a provider finds it desirable to share secondary demand with another provider. We then establish that for *general* demand functions, there always exists a price interval in which a network provider is profitable and is willing to share the market as illustrated in Fig. 1. Even though the expressions for the market sharing and break-even prices are implicit, we prove that the market sharing price is always strictly greater than the break-even price, regardless of the demand function.

Next, we consider a duopoly competition where network providers make pricing decisions to maximize their revenues. We formally establish the best response strategy of each provider and list the possible market outcomes in the form of Nash equilibrium (NE) : i) if the market sharing intervals overlap, then the providers end up sharing the market; ii) if the market sharing intervals do not overlap, then the provider with the lower break-even price captures the entire market, which reflects the result of a price war. The equilibria prices under the first case are possibly much higher than the break-even prices of each provider, while under the second case the equilibrium price is roughly equal to the higher break-even price. We provide an interpretation and possible refinement of our results by taking into consideration *payoff dominant strategies* and demand redistribution according to observed Quality of Service (QoS) metrics. We also discuss the possibility of extending our duopoly results to multiple providers.

The rest of the paper is organized as follows. In Section 2, we survey previous work. In Section 3, we present the network model used to conduct our analysis. Next, in Section 4, we establish the market sharing interval and derive the break-even and the market sharing prices. In Section 5 we establish the best response of a provider based on the break-even and market sharing prices and then list the market outcomes as NE. We provide a brief refinement of our model in Section 6. We conclude the paper in Section 7.

2. RELATED WORK

Competition in a spectrum market takes place at two different levels. The first level of competition is for the spectrum itself, as a resource to lease from a license holder (or the government). This level of competition is analyzed commonly through auction methods. The second level of competition arises after the said leasing of the spectrum, and consists of providers competing to offer services to end-users.

The majority of papers on secondary spectrum markets are concerned with the first level of competition. For instance, in the works by Duan et. al. [2010], Kasbekar and Sarkar [2012], Ren et. al. [2011], Sengupta and Chatterjee [2009], Xing et. al. [2007] and Yang et. al. [2013] game theoretic approaches to spectrum auctioning and leasing are analyzed. Papers by Chun and La [2013], Gao et. al. [2013], Kash et. al. [2013], Sheng and Liu [2013] and Zhu et. al. [2013] are concerned with the effectiveness of the employed sales mechanisms. The set-up in all of these papers is different from ours as they are concerned with a spectrum owner leasing its spectrum to a secondary party based on availability. On the other hand, we focus on the second level of competition, where the leased spectrum is available as a consumption commodity for a specific provision.

The benefits of cooperation or forming coalitions at the auction level are analyzed by Berry et. al. [2013] as a means to increasing coverage and by Xiao et. al. [2012] to better allocate the spectrum. While forming coalitions have benefits, the wireless communications market is highly concentrated[1] [Commission 2013] and this brings antitrust concerns as well. In our model, while we indicate the possibility of market sharing, this behavior is entirely the result of revenue driven market dynamics rather than intentional cooperation.

Several papers analyze the two levels of competition in a combined fashion, where both the auction side and the service side of the competition are considered over different time scales [Ileri et al. 2005; Kim et al. 2011; Maille and Tuffin 2010]. The first focuses on the auctioning side of the competition while the second paper considers different but substitute technologies. The third paper considers a spatial distribution of single channels that turn on and off.

Profitability in secondary spectrum markets is studied in a work by Alanyali et al. [2011], where a pricing policy that guarantees profitability under a monopolistic framework is provided. However, this paper does not mention market competition. Niyato and Hossain [2008] derive market equilibrium pricing by taking into consideration the demand and supply dynamics of spectrum auctions. However, the model uses a very specific secondary demand based on the utility from owning the spectrum and how much it costs to lease the spectrum. On the other hand, our results hold for general demand functions. Drawing conclusions under general demand functions generally requires a more elaborate analysis, as illustrated by several papers [Allon and Gurvich 2010; Andrews et al. 2013; Besbes and Zeevi 2009].

A paper by Fortetsanakis et. al. [2012] considers the second level of competition, where providers offer what the authors call the *Flex Service*. The simulation based results indicate that the welfare of the market increases through the use of a central database which collects information about pricing and quality of service. This work relies on explicit demand and utility functions. Our results hold without making such assumptions.

In our previous work [Kavurmacioglu et al. 2012], we analyze provider competition for secondary demand in private commons under the set-up of a *coordinated access* policy that throttles secondary demand, unlike the uncoordinated access policy studied here. It is shown that the implementation of an optimal coordinated access policy by each provider leads to a price war, irrespective of the demand function. While the same work points out, via an example, the possibility of different market outcomes (i.e., market sharing) under uncoordinated access, no comprehensive analysis is provided. In this paper we provide a broader investigation of the uncoordinated case through a fluid traffic model.

[1]The current wireless market has a Herfindahl-Hirschman Index (HHI) of approximately 2800. Any value of this index greater than 2500 indicates a highly concentrated market (*i.e.,* few firms with large market shares).

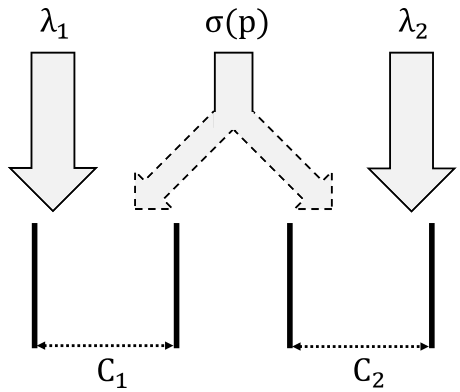


Fig. 2. Market model: Two providers $i = 1, 2$, each with capacity C_i and fixed primary demand λ_i, compete for secondary demand stipulated by a general function of price p, $\sigma(p)$.

3. MODEL

We consider two spectrum providers, where each provider i has a capacity C_i and a primary demand of volume λ_i, which generates a revenue of K_i units per service. These providers compete for a stream of secondary demand, whose volume depends of their pricing of secondary service as illustrated in Figure 2 [2]. We assume a traffic model where if provider i receives a total demand of volume λ_i, then it can accommodate the volume $\min(C_i, \lambda_i)$. The excess demand $\max(\lambda_i - C_i, 0)$ does not generate any revenue for the provider.

The total demand for provider i consists of its primary demand λ_i and, depending on its pricing and the pricing of its competitor, a secondary demand σ_i. We shall assume that the two demand types access the capacity in an uncoordinated fashion, as suggested by documentation on private commons [Buddhikot 2007][3].). In this context, primary users could be viewed as high paying legacy users rather than users with higher priority. Specifically, the two types of demand share capacity *on equal basis*, such that if the demand of provider i is composed of two types with respective volumes λ_i and σ_i, then the overflow volume of *each type* is proportional to the intensity of demand of that type. That is, in view of our previous assumption, a fraction $\min\left(1, \frac{C_i}{\lambda_i + \sigma_i}\right)$ of each type of demand is actually accommodated. The steady-state primary and secondary demands, λ_i and σ_i, and the overflow assumption are consistent with fluid models. Such models have widely been used in the literature to characterize network traffic at the flow level [Kelly and Williams 2004].

We denote the price that provider i charges per unit of serviced secondary demand by p_i. The volume of the secondary demand is assumed to be determined by the minimum price $\min(p_1, p_2)$ stipulated by the two providers. Specifically, the volume of secondary demand is $\sigma(\min(p_1, p_2))$, where $\sigma(\cdot)$ is the *demand function*. We make the mild assumption that this function is differentiable and non-increasing ($\frac{\partial}{\partial p}\sigma(p) \leq 0$). We shall also assume that there exists a positive demand when the service is offered for free ($\sigma(0) > 0$) and the demand eventually becomes zero as the price becomes arbitrarily high ($\lim_{p\to\infty} \sigma(p) = 0$).

It is assumed that the secondary demand is attracted to the provider that charges the lowest price. This behavior can be explained by *price aversion*, a concept employed in marketing management [Tellis and Gaeth 1990]. In the case when both providers charge the same price, the resulting secondary demand splits between the two providers according to an arbitrary but fixed probability vector $\boldsymbol{\alpha} = [\alpha_1, \alpha_2]$ such that $\alpha_1 + \alpha_2 = 1$ and $\alpha_i > 0$, $i = 1, 2$. Namely, in that case, each provider i receives a secondary demand of volume $\alpha_i \sigma(p_i)$. We will relax this assumption in Section 6, where instead of being randomly assigned the secondary demand will be split between the providers according to the accommodation levels.

[2]In order to keep the model generic, we shall not adopt a particular choice of units for capacity and demand at this point. Rather, we provide a discussion of possible choices at the end of this section.
[3]While the model considered in this paper is applicable in Private Commons, it does not necessarily represent the only way to implement it.

If provider i receives a secondary demand of volume $\sigma(p_i)$, its overall revenue is given by:

$$W_i(p_i, \sigma(p_i)) \triangleq \; p_i\sigma(p_i) \min\left(1, \frac{C_i}{\lambda_i + \sigma(p_i)}\right) + K_i\lambda_i \min\left(1, \frac{C_i}{\lambda_i + \sigma(p_i)}\right). \quad (1)$$

In this case, the secondary *profit* (i.e., increment in revenue from secondary access) of the provider is:

$$\Pi_i(p_i, \sigma(p_i)) \triangleq W_i(p_i, \sigma(p_i)) - W_i(0, 0). \quad (2)$$

Since the secondary demand that a provider receives depends on the prices of both providers, so does the profit of the provider. We define the *reward* $R_i(p_i, p_{-i})$ of provider i as its profit when it charges secondary access p_i and its competitor charges p_{-i} units. Namely,

$$R_i(p_i, p_{-i}) \triangleq \begin{cases} \Pi_i(p_i, \sigma(p_i)) & \text{if } p_i < p_{-i} \\ \Pi_i(p_i, \alpha_i\sigma(p_i)) & \text{if } p_i = p_{-i} \\ \Pi_i(p_i, 0) & \text{if } p_i > p_{-i}. \end{cases} \quad (3)$$

In the interest of space, the discussion of this paper is limited to the case when each provider's network is underloaded prior to inclusion of any secondary demand, that is $\lambda_i < C_i$, but can be overloaded for low enough prices, that is $\lambda_i + \sigma(0) > C_i$. Though the omitted cases warrant their own respective analyses, those are arguably less challenging and practical. For instance, if the maximum possible total demand does not exceed the network capacity (i.e., $\lambda_i + \sigma(0) \leq C_i$), then the network can accommodate the entire demand at any price. On the other hand, if the primary demand already exceeds the capacity (i.e., $\lambda_i \geq C_i$), then the revenue per serviced secondary demand would need to match or exceed the revenue per serviced primary demand (i.e., $p_i \geq K_i$).

Discussion. We provide next a possible interpretation of our model. The service capacity C_i can represent the number of sub-carriers in an OFDM modulation scheme used in LTE or the number of radio channels[4] available for assignment for voice or data traffic in common 3G standards [Paul et al. 2011]. The steady primary and secondary demands, λ_i and $\sigma(p)$, and the overflow assumption are consistent with fluid models [Anick et al. 1982]. Such models have widely been used in the literature to characterize network traffic at the flow level [Fred et al. 2001; Kelly and Williams 2004; Hassidim et al. 2013]. This assumption is substantiated by traffic measurements in cellular networks, which show that mean arrival rates do not show significant variations over the course of an hour [Paul et al. 2011; Willkomm et al. 2008]. Obviously, specific values of λ_i and $\sigma(p)$ depend on the hour of the day or day of the week.

4. CHARACTERISTIC PRICES & MARKET SHARING INTERVAL

In this section we present two characteristic prices and demand-invariant price relationships in a secondary spectrum markets. This section focuses on the viewpoint of a single provider. Therefore for simplicity, we omit the use of index i from of our notation throughout this section.

We define the break-even price $p^{BE}(\alpha)$ as the price at which the profit of a provider is zero when it attracts a fraction $0 < \alpha \leq 1$ of the total demand, namely $\Pi(p^{BE}, \alpha\sigma(p^{BE})) = 0$. We start off by providing a formal definition of a break-even price:

Definition 4.1 ***(Break-Even Price)*** *A price $p^{BE}(\alpha) \geq 0$ is called a break-even price if it satisfies the following conditions:*

$$\Pi(p^{BE}, \alpha\sigma(p^{BE})) = 0 \qquad \textit{and} \qquad \alpha\sigma(p^{BE}) > 0.$$

Note that the latter condition in the above definition is to rule out any price that does not generate any secondary demand.

We next define the *market sharing price* $p^{MS}(\alpha)$, that asserts whether a provider finds it desirable to share the secondary demand or not. Specifically, let

$$\Delta W(p) \triangleq W(p, \alpha\sigma(p)) - W(p, \sigma(p)).$$

[4]This radio channel refers to any radio resource allocated to the user such as code, frequency or time slot.

Definition 4.2 ***(Market Sharing Price)*** *A price $p^{MS}(\alpha) \geq 0$ is called a market sharing price if the following is true:*

$$\Delta W(p) = 0 \qquad \textit{for } p = p^{MS}(\alpha).$$

These two prices characterize two important incentives for a network provider. We will show that the break-even price determines provider profitability, where any price set greater is guaranteed to result in a positive profit. We will also establish that, analogous to the relationship between the break-even price and provider profitability, a provider finds it undesirable to share the secondary demand at prices above the market sharing price, whereas the opposite is true for prices below the market sharing price. Having defined the break-even and market sharing prices, we can proceed with stating our main results in the following theorem:

Theorem 4.1 ***(Market Sharing Interval)*** *For any secondary demand function, satisfying the assumptions described in Section 3 and for all values of $\alpha : 0 < \alpha \leq 1$, there exists a price interval*

$$(\mathcal{P}) \equiv (p^{BE}(\alpha), p^{MS}(\alpha)),$$

such that for all $p \in (\mathcal{P})$:

(1) $\Pi(p, \sigma(p)) > 0$,
(2) $\Pi(p, \alpha\sigma(p)) > \Pi(p, \sigma(p))$.

Theorem 4.1 states that no matter the specific shape of a secondary demand function, the existence of the price interval $(\mathcal{P})$ at which a network provider is profitable and finds it preferable to share the secondary demand is guaranteed. In order to prove Theorem 4.1 we will first provide formulations for break-even and market sharing prices in Sections 4.1 and 4.2 respectively. Afterwards, we bring the proof of Theorem 4.1 in Section 4.3.

4.1. Profitability and Break-Even Price

In this section we seek to analyze a provider's profit and the resulting break-even price. Our result applies both to the cases when a network provider serves the entire secondary demand (i.e., $\alpha = 1$) and when it shares the market with another provider (i.e, $\alpha < 1$).

Since a break-even price is a measure of a provider's competitive ability in a price war, characterizing this price is important. The following lemma restricts the price interval on which a break-even price when the provider captures the entire secondary demand (*i.e.,* monopoly) lies:

Lemma 4.1 *For a given α such that $\lambda + \alpha\sigma(0) > C$, there exists a price $\bar{p}^{\alpha}$, which is the minimum price that satisfies $\alpha\sigma(p) = C - \lambda$. Then, any break-even price $p^{BE}(\alpha)$ satisfies the following inequality:*

(1) $p^{BE}(\alpha) \leq \bar{p}^{\alpha}$ *for any demand function* $\sigma(p)$.
(2)

$$\lambda + \alpha\sigma(p^{BE}(\alpha)) \geq C. \tag{4}$$

PROOF. See Appendix A. □

An intuitive explanation to Lemma 4.1 is that for all prices p such that $\lambda + \alpha\sigma(p) < C$, the overflow of either type of demand is zero. Thus, there is no associated penalty with serving additional secondary demand. However, once the excess demand becomes positive, a provider observes a trade-off between the revenue brought in by the secondary demand versus the potential revenue lost from the unserviced primary demand. The break-even price reflects the price at which both sides of this trade-off are equal.

Lemma 4.1 demonstrates that for all such values of α, including the monopolistic case when $\alpha = 1$, we can limit our analysis to those prices that satisfy (4). At these prices the fraction of both types of demand being accommodated is $C/(\lambda + \alpha\sigma(p))$. Then, we can remove the $\min$ operators from Eq. (1) and simplify Eq. (2) for the profit as follows:

$$\Pi(p, \alpha\sigma(p)) = \alpha\sigma(p)p \cdot \frac{C}{\lambda + \alpha\sigma(p)} + \lambda K \left(\frac{C}{\lambda + \alpha\sigma(p)} - 1 \right). \tag{5}$$

The following theorem, leveraging our previous results from Lemma 4.1 and Eq. (5), provides an equation that allows the computation of the break-even price $p^{BE}(\alpha)$ for the aforementioned values of α. The theorem also establishes the uniqueness of this price and the region of profitable prices.

Theorem 4.2 ***(Break-Even Price)***

(1) *For a given* $0 < \alpha \leq 1$*, such that* $\lambda + \alpha\sigma(0) > C$*:*

(a) *A break-even price* $p^{BE}(\alpha)$ *is a solution to the following equation*[5]*:*

$$p = \frac{(\alpha\sigma(p) + \lambda - C)\lambda K}{C\alpha\sigma(p)}. \tag{6}$$

(b) *The break-even price* $p^{BE}(\alpha)$ *is unique.*

(c) *The profit of a provider is such that:*

$$\begin{aligned} \Pi(p, \alpha\sigma(p)) > 0 \qquad & \textit{if } p > p^{BE}(\alpha) \\ \Pi(p, \alpha\sigma(p)) < 0 \qquad & \textit{if } p < p^{BE}(\alpha). \end{aligned}$$

(2) *For a given* $0 < \alpha < 1$*, such that* $\lambda + \alpha\sigma(0) \leq C$*, the break-even price* $p^{BE}(\alpha)$ *is 0.*

PROOF. (1) (a) We know that at a break-even price the profit is given by Eq. (5). In order to ensure $\Pi(p, \alpha\sigma(p)) = 0$, it can be verified through simple algebra that a price p needs to satisfy the following equation:

$$p = \frac{(\alpha\sigma(p) + \lambda - C)\lambda K}{C\alpha\sigma(p)}.$$

Furthermore, we know that at price $p^{BE}(\alpha)$, secondary demand will be positive by combining inequality (49) and the fact that $\lambda < C$:

$$\sigma(p^{BE}(\alpha)) \geq \sigma(\bar{p}) = C - \lambda > 0.$$

(b) We will proceed by demonstrating that the left hand side of Eq. (6) is strictly increasing with respect to p and the right hand side is non-increasing with respect to p, hence meaning that this equality only holds at a single value of p. Since the left hand side of Eq. (6) is p itself, we only need to prove that the right hand side is non-increasing. Under the assumption that $\sigma(p)$ is a differentiable and non-increasing function of p, taking the derivative of the right hand side with respect to p yields:

$$\begin{aligned} \frac{\partial}{\partial p}\left(\frac{(\alpha\sigma(p) + \lambda - C)\lambda K}{C\alpha\sigma(p)}\right) &= \left(\frac{1}{\alpha\sigma(p)} - \frac{\lambda + \alpha\sigma(p) - C}{\alpha^2\sigma^2(p)}\right)\alpha\sigma'(p)\left(\frac{\lambda K}{C}\right) \\ &= \left(\frac{C - \lambda}{\alpha^2\sigma^2(p)}\right)\alpha\sigma'(p)\left(\frac{\lambda K}{C}\right) \leq 0. \end{aligned} \tag{7}$$

Eq. (7) holds because $\lambda < C$ and $\sigma'(p^{BE}(\alpha)) \leq 0$.

We also know that the lhs of Eq. (6) is continuous in p, which follows from the differentiability of the secondary demand $\sigma(p)$. Therefore, there can only be at most one solution for $p^{BE}(\alpha)$ that satisfies Eq. (6).

(c) From Eq. (5), it can verified that in order for $\Pi(p, \alpha\sigma(p)) > 0$ to hold, p needs to satisfy the following inequality:

$$p > \frac{(\alpha\sigma(p) + \lambda - C)\lambda K}{C\alpha\sigma(p)}.$$

In part (b) of our proof, we have demonstrated that the right hand side of Eq. (6) is non-increasing with respect to p. Therefore for $p' > p^{BE}(\alpha)$:

$$\frac{(\alpha\sigma(p^{BE}(\alpha)) + \lambda - C)\lambda K}{C\alpha\sigma(p^{BE}(\alpha))} \geq \frac{(\alpha\sigma(p') + \lambda - C)\lambda K}{C\alpha\sigma(p')}.$$

[5]This implicit equation can be solved with well-established fixed point iterations, such as Newton's Method.

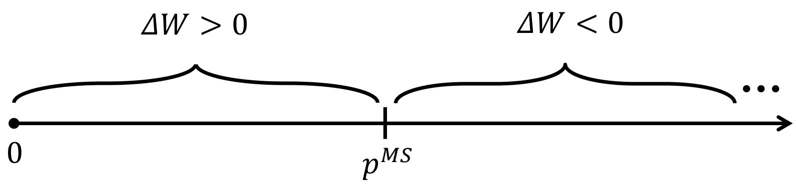


Fig. 3. Market sharing revenue change regions in Theorem 4.3 with respect to the market sharing price $p^{MS}(\alpha)$.

Then, since $p^{BE}(\alpha)$ is the only value that satisfies Eq. (6),

$$p' > p^{BE}(\alpha) = \frac{(\alpha\sigma(p^{BE}(\alpha)) + \lambda - C)\lambda K}{C\alpha\sigma(p^{BE}(\alpha))} \geq \frac{(\alpha\sigma(p') + \lambda - C)\lambda K}{C\alpha\sigma('p)}.$$

To show that $\Pi(p', \alpha\sigma(p')) < 0$ when $p' < p^{BE}(\alpha)$, the same argument follows in the reverse direction.

(2) For a given $0 < \alpha < 1$, such that $\lambda + \alpha\sigma(0) \leq C$, Eq. (2) simplifies to the following:

$$\Pi(p, \alpha\sigma(p)) = p\alpha\sigma(p).$$

Since $\sigma(0) > 0$ by assumption, the only price that satisfies both equations provided Definition 4.1 is $p = 0$. □

In the next lemma, we establish a useful bound on the break-even price $p^{BE}(1)$.

Lemma 4.2 *The break-even price when not sharing the secondary demand (*i.e., $\alpha = 1$*) is strictly smaller than the revenue generated by primary demand:*

$$p^{BE}(1) < K.$$

PROOF. See Appendix A. □

In general, there is no explicit expression for the break-even price for general demand functions. However, it allows us to characterize two distinct price regimes by identifying whether or not a price p generates a profit for the provider for any amount of secondary demand. We next provide an example with a simple demand function, where obtaining an explicit expression is rather straightforward.

***Example* 4.1** We illustrate the relationship between the break-even price when $\alpha = 1$ (i.e., one provider captures the entire secondary demand) and network parameters under a constant elasticity secondary demand function, $\sigma(p) = \frac{\sigma_0}{p}$, where $\sigma_0 > 0$ is a constant.

Under this given demand we can simplify Eq. (6) and obtain the following explicit formula:

$$p^{BE}(1) = \frac{\sigma_0 \lambda K}{C\sigma_0 + \lambda K(C - \lambda)}. \tag{8}$$

We have effectively formulated and characterized the unique break-even price that determines a network provider's profitability. However, profitability alone is not enough to determine a market outcome. As was explained in the network model section, matching prices affects the reward a provider faces in a non-linear fashion. In the next section, we take into account the results of a provider choosing to share the market.

4.2. Market Sharing

We now turn our attention to the effects market sharing has on a provider's revenue. In the next theorem, we present our result on how market sharing affects a provider's profit. The theorem establishes the existence and uniqueness of the market sharing price $p^{MS}(\alpha)$ and provides an implicit equation to compute it. It also states that increased profit is achieved if and only if $p < p^{MS}(\alpha)$.

Theorem 4.3 ***(Market Sharing Price)*** *For any network provider there exists a unique market sharing price* $p^{MS}(\alpha)$*, which satisfies the following:*

(1) If $\lambda + \alpha\sigma(K) \le C$*,* $p^{MS}(\alpha)$ *is the solution to:*

$$p = \frac{(\lambda + \sigma(p) - C)\,\lambda K}{(C - \alpha(\lambda + \sigma(p)))\,\sigma(p)}. \tag{9}$$

(2) If $\lambda + \alpha\sigma(K) > C$*,*

$$p^{MS}(\alpha) = K. \tag{10}$$

and for any given $p^{MS}(\alpha)$ *the following is true:*

$$\Delta W(p) > 0 \qquad \textit{for } p < p^{MS}(\alpha), \tag{11}$$

$$\Delta W(p) < 0 \qquad \textit{for } p > p^{MS}(\alpha). \tag{12}$$

Before we prove Theorem 4.3, we first establish several useful results that will later facilitate our proof. Since the general revenue function of a provider involves $\min$ operators, we need to make use of some auxiliary prices that will simplify the expressions of $W(p, \sigma(p))$ and $W(p, \alpha\sigma(p))$. In Lemma 4.1 we had already defined $\bar{p}^{\alpha}$ to be an auxiliary price that satisfies the equality $\lambda + \alpha\sigma(\bar{p}^{\alpha}) = C$. In this section we provide another such auxiliary price to simplify our analysis. We let $\bar{p}$ denote the price that satisfies the following equation:

$$\lambda + \sigma(\bar{p}) = C. \tag{13}$$

Since we assume that the secondary demand $\sigma(p)$ is non-increasing in p for all $0 < \alpha < 1$ it follows that $\bar{p}^{\alpha} < \bar{p}$, which is illustrated for a generic demand function in Figure 4.

By defining these prices we have effectively divided prices into three separate regions, i.e. $[0, \bar{p}^{\alpha})$, $[\bar{p}^{\alpha}, \bar{p})$, $[\bar{p}, \infty)$, in each of which we have a simplified revenue function. Now, we can start our analysis on how the revenue changes depending on which region a given price value p falls in.

a) We first consider the price region $\{p : p \ge \bar{p}\}$. Note that the price inequality corresponds to when the total demand under price p does not exceed the provider's service capacity. In the following lemma we establish that in this region, it is never optimal for a provider to choose market sharing.

Lemma 4.3 *Assume* $p \ge \bar{p}$*, then*

$$\Delta W(p) < 0. \tag{14}$$

PROOF. Note that our assumption $p \ge \bar{p}$ is equivalent to stating that $\lambda + \sigma(p) < C$. Since $\bar{p} > \bar{p}^{\alpha}$, it is also true that $p > \bar{p}^{\alpha}$. Then, the total arrival under market sharing is also less than provider i capacity (i.e., $\lambda + \alpha\sigma(p) < C$). Simplifying Eq. (1) under these assumptions, we get:

$$\Delta W = W(p, \alpha\sigma(p)) - W(p, \sigma(p)) = \Big(\alpha\sigma(p)p + \lambda K\Big) - \Big(\sigma(p)p + \lambda K\Big) = \alpha\sigma(p)p - \sigma(p)p < 0.$$

Therefore we conclude that Eq. (14) holds. □

b) Next, we cover the price region $\{p : p < \bar{p}^{\alpha}\}$. Since price values need to be non-negative, we do not consider the case $\bar{p}^{\alpha} = 0$. In this price interval, there are two cases two consider. If the value of K happens to be in this region, then the revenue change is positive for price values below K and negative for price values above K. If K does not fall in this price interval, then the revenue change is always positive and thus a provider will always find it desirable to share the market. We formalize these results in the following lemma:

Lemma 4.4 *Assume* $\bar{p}^{\alpha} > 0$ *and* $p < \bar{p}^{\alpha}$*, then*
(a) If $\bar{p}^{\alpha} \ge K$*:*

$$\Delta W(p) > 0 \qquad \textit{if } p < K; \tag{15}$$

$$\Delta W(p) = 0 \qquad \textit{if } p = K; \tag{16}$$

$$\Delta W(p) < 0 \qquad \textit{if } p > K. \tag{17}$$

(b) If $\bar{p}^{\alpha} < K$*:*

$$\Delta W(p) > 0 \quad \forall p < \bar{p}^{\alpha}. \tag{18}$$

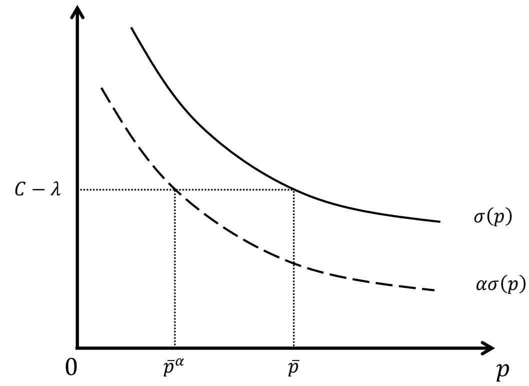


Fig. 4. An illustration of the prices $\bar{p}$ and $\bar{p}^{\alpha}$ under a generic secondary demand.

PROOF. Note that our assumption $p < \bar{p}^{\alpha}$ is equivalent to stating that:

$$\lambda + \alpha\sigma(p) \geq C.$$

Since $\bar{p}^{\alpha} < \bar{p}$, it must also be true that $p < \bar{p}$. Then the combined demand without market sharing is greater than the provider's capacity (i.e., $\lambda+\sigma(p) \geq C$). Simplifying Eq. (1) under these assumptions, we obtain:

$$\Delta W = W(p, \alpha\sigma(p)) - W(p, \sigma(p)) = \frac{\alpha\sigma(p)pC}{\alpha\sigma(p)+\lambda} + \frac{\lambda KC}{\alpha\sigma(p)+\lambda} - \frac{\sigma(p)pC}{\sigma(p)+\lambda} - \frac{\lambda KC}{\sigma(p)+\lambda}.$$

After rearrangement we get:

$$\Delta W = \frac{(1-\alpha)\sigma(p)\lambda C}{(\alpha\sigma(p)+\lambda)(\sigma(p)+\lambda)}(K-p). \tag{19}$$

Eq. (19) only takes on the value zero when $p = K$. Additionally for price values $p < K$, ΔW is positive and for $p > K$, ΔW is negative. □

Remark 4.1 Lemma 4.4 considers prices for which the reduced secondary demand, when combined with the primary demand, exceeds the capacity of a provider. In that case, this provider can increase its revenue at prices up to K if $\bar{p}^{\alpha} \geq K$ or all prices p if $\bar{p}^{\alpha} < K$, by choosing to share the market with another provider. On the other hand, if $\bar{p}^{\alpha} \geq K$, choosing to share the market decreases the revenue at prices greater than K.

c) Finally, we cover the price region between the regions covered in parts a) and b), such that $\{p : \bar{p}^{\alpha} \leq p < \bar{p}\}$. Note that these are price values such that the combined demand of the primary and secondary types exceed the service capacity without market sharing and do not exceed the service capacity with market sharing. Once again, similar to the previous case, the revenue change depends on the relationship between K and how this price interval is defined. If $K \geq \bar{p}^{\alpha}$, then the market sharing price lies on this interval and the revenue change is negative for price values above and positive for price values below. Otherwise, the revenue change is always in the negative direction and market sharing is not desirable. We present the following lemma in this light:

Lemma 4.5 *Assume $\bar{p}^{\alpha} \leq p < \bar{p}$. Then,*
1) If $\bar{p}^{\alpha} \leq K$:

$$\begin{aligned} \Delta W(p) > 0 \qquad & \textit{if } p < p^{MS}(\alpha); \\ \Delta W(p) \leq 0 \qquad & \textit{if } p \geq p^{MS}(\alpha), \end{aligned} \tag{20}$$

where $p^{MS}(\alpha)$ denotes the solution to the following equation:

$$p = \frac{(\lambda + \sigma(p) - C)\,\lambda K}{(C - \alpha(\lambda + \sigma(p)))\,\sigma(p)}. \tag{21}$$

2) If $\bar{p}^{\alpha} > K$:

$$\Delta W(p) < 0. \tag{22}$$

PROOF. Note that $\bar{p}^{\alpha} \leq p < \bar{p}$ is equivalent to stating that $\lambda + \sigma(p) > C$ and $\lambda + \alpha\sigma(p) \leq C$. Under these conditions the revenue change function becomes:

$$\Delta W(p) = \Big(\alpha\sigma(p)p + \lambda K\Big) - \Big(\sigma(p)p\frac{C}{\lambda + \sigma(p)} + \lambda K\frac{C}{\lambda + \sigma(p)}\Big).$$

Regrouping yields:

$$\Delta W(p) = \Big(\frac{\alpha(\lambda + \sigma(p)) - C}{\lambda + \sigma(p)}\sigma(p)p\Big) + \Big(\frac{\lambda + \sigma(p) - C}{\lambda + \sigma(p)}\lambda K\Big).$$

(1) Noting that $\alpha(\lambda + \sigma(p)) < C$, it can be verified that $\Delta W(p) = 0$ is satisfied by the solution of the following implicit equation:

$$p = \frac{(\lambda + \sigma(p) - C)\lambda K}{(C - \alpha(\lambda + \sigma(p)))\sigma(p)}. \tag{23}$$

Furthermore, one can check that Eq. (23) is satisfied by a unique price p. Multiplying both sides of Eq. (23) with the first term in the denominator we obtain:

$$(C - \alpha(\lambda + \sigma(p)))p = \frac{(\lambda + \sigma(p) - C)\lambda K}{\sigma(p)}. \tag{24}$$

Taking the derivative of the left hand side of Eq. (24) we get:

$$\begin{aligned}&\frac{\partial}{\partial p}(C - \alpha(\lambda + \sigma(p)))p = \\ &\qquad (C - \alpha(\lambda + \sigma(p))) - \alpha\sigma'(p)p > 0.\end{aligned}$$

Taking the derivative of the right hand of Eq. (24) side yields:

$$\frac{(\lambda + \sigma(p) - C)\lambda K}{\sigma(p)} = \Big(\frac{1}{\sigma(p)} - \frac{\lambda + \sigma(p) - C}{\sigma^2(p)}\Big)\sigma'(p)\lambda K = \Big(\frac{C - \lambda}{\sigma^2(p)}\Big)\sigma'(p)\lambda K \leq 0.$$

One side of the equation is strictly increasing with p, while the other is non-increasing in p. Since both sides are continuous in p, we conclude that equality (23) holds for a unique value of p.
If $\bar{p}^{\alpha} \leq K$, Lemma 4.4 states that $\Delta W(p) > 0$ for price values $p < \bar{p}^{\alpha}$. By Lemma 4.3 we have $\Delta W(p) < 0$ for $p \geq \bar{p}$. Therefore, it must be that $p^{MS}(\alpha) \in [\bar{p}^{\alpha}, \bar{p})$. Since $\Delta W(p) = 0$ only when $p = p^{MS}(\alpha)$, by continuity of revenue it follows that $\Delta W(p) > 0$ for all $p < p^{MS}(\alpha)$ and $\Delta W(p) < 0$ for all $p > p^{MS}(\alpha)$.

(2) In the previous part of our proof we have demonstrated that on the price interval $[\bar{p}^{\alpha}, \bar{p}]$, the only possible price that sets $\Delta W(p) = 0$ is given by:

$$p = \frac{(\lambda + \sigma(p) - C)\lambda K}{(C - \alpha(\lambda + \sigma(p)))\sigma(p)}. \tag{25}$$

We will show that if $\bar{p}^{\alpha} > K$, the solution to Eq. (25) lies outside the price interval $[\bar{p}^{\alpha}, \bar{p})$. Let p^* denote a particular solution to Eq. (25). Assume $p^* \in [\bar{p}^{\alpha}, \bar{p})$, which means that $p^* > K$. Taking the ratio of $\frac{p^*}{K}$ and substituting the right hand side of Eq. (25) for p^* yields:

$$\frac{(\lambda + \sigma(p^*) - C)\lambda}{(C - \alpha(\lambda + \sigma(p^*)))\sigma(p^*)} > 1. \tag{26}$$

After some rearrangement we get:

$$\begin{aligned}\lambda(\lambda + \sigma(p^*)) &> C(\lambda + \sigma(p^*)) - \alpha\sigma(p^*)(\lambda + \sigma(p^*)),\\ \lambda &> C - \alpha\sigma(p^*).\end{aligned} \tag{27}$$

which is a contradiction to our initial assumption $p^* \in [\bar{p}^{\alpha}, \bar{p}]$. Therefore, no value of p yields $\Delta W(p) = 0$ on the price interval $[\bar{p}^{\alpha}, \bar{p}]$. Additionally, since $\bar{p}^{\alpha} > K$, Lemma 4.4 states that $\Delta W(\bar{p}^{\alpha}) < 0$. Due to the continuity of the revenue $W(p)$ and the fact that there are no zero crossings in this interval, it must also be true that $\Delta W(p) < 0$ for $p \in [\bar{p}^{\alpha}, \bar{p}]$. □

Having analyzed how the revenue changes under market sharing for the three price intervals we have defined, we can finally move on to proving Theorem 4.3:

Proof of Theorem 4.3

(1) *Case 1* - $\lambda + \alpha\sigma(K) \leq C$: By the way we have defined $\bar{p}^\alpha$, this case is equivalent to stating that $\bar{p}^\alpha \leq K$. Then by Lemma 4.3 and Lemma 4.4 we have that:

$$\Delta W(p) < 0 \qquad \text{for } p \geq \bar{p}, \tag{28}$$

$$\Delta W(p) > 0 \qquad \text{for } p < \bar{p}^\alpha. \tag{29}$$

Therefore the market sharing price $p^{MS}(\alpha)$ must lie on the price interval $[\bar{p}^\alpha, \bar{p})$. Lemma 4.5 states that $p^{MS}(\alpha)$ satisfies Eq. (21) such that $\Delta W(p^{MS}(\alpha)) = 0$ and for $p \in [\bar{p}^\alpha, \bar{p})$:

$$\Delta W(p) < 0 \qquad \text{if } p > p^{MS}(\alpha), \tag{30}$$

$$\Delta W(p) > 0 \qquad \text{if } p < p^{MS}(\alpha). \tag{31}$$

Combining Eq.'s (28) to (31) we obtain the results stated in the proposition.

(2) *Case 2-* $\lambda + \alpha\sigma(K) > C$: By the way we have defined $\bar{p}^\alpha$, this case is equivalent to stating that $\bar{p}^\alpha > K$. Then by Lemma 4.3 and Lemma 4.5 we have that:

$$\Delta W(p) < 0 \text{ for } p \geq \bar{p}^\alpha. \tag{32}$$

Therefore the market sharing price $p^{MS}(\alpha)$ must belong to the price interval $[0, \bar{p}^\alpha)$. Lemma 4.4 states that the revenue change is equal to zero when $p = K$, therefore we conclude that the market sharing price $p^{MS}(\alpha) = K$. Additionally, we have that:

$$\Delta W(p) < 0 \qquad \text{if } p > p^{MS}(\alpha), \tag{33}$$

$$\Delta W(p) > 0 \qquad \text{if } p < p^{MS}(\alpha). \tag{34}$$

Combining Eq.'s (32) to (34) we obtain the results stated in the theorem. □

Theorem 4.3 yields a rather non-straightforward result such that for any network provider there exists a unique price which acts as a threshold value: market sharing at all prices greater than this threshold results in a profit decrease, while at prices below this threshold the network provider is guaranteed a profit increase by decreasing its secondary demand. In this way, it serves a similar function to that of the break-even price: It further divides the price ranges into two regimes but this time by identifying when serving the reduced secondary demand generates more profit that serving the full demand.

In the next lemma, we establish an upper bound on the market sharing price, similar to what we did in Lemma 4.2.

Lemma 4.6 *The market sharing price is less than or equal to the revenue generated by primary demand:*

$$p^{MS}(\alpha) \leq K.$$

PROOF. See Appendix A. □

***Example* 4.2** We illustrate the relationship between the market sharing price and network parameters under the same constant elasticity secondary demand function we used before, $\sigma(p) = \sigma_0/p$.

Under this given demand and assuming $\sigma_0 \leq (C - \lambda)K/\alpha$ such that $\lambda + \alpha\sigma(K) \leq C$, we obtain from Eq. (9) the following explicit formula for the market sharing price:

$$p^{MS}(\alpha) = \frac{\lambda K \sigma_0 + \alpha\sigma_0^2}{\sigma_0(C - \alpha\lambda) + \lambda K(C - \lambda)}. \tag{35}$$

If $\sigma_0 > (C - \lambda)K/\alpha$, then $p^{MS}(\alpha) = K$ by Eq. (10).

Theorems 4.2 and 4.3 provide implicit equations for the break-even price $p^{BE}(\alpha)$ and market sharing price $p^{MS}(\alpha)$ that depend on the demand function $\sigma(p)$. Strikingly, one can show through careful analysis that the ratio of $p^{BE}(\alpha)$ to $p^{MS}(\alpha)$ is strictly smaller than 1 for any demand function.

4.3. Proof of Theorem 4.1

(1) If $p^{BE}(\alpha) > 0$:

(a) Assume $p^{MS}(\alpha) = K$. In Lemma 4.2 we have established that $p^{BE}(1) < K$. Let us rearrange Eq. (6) as follows:

$$p = \frac{\lambda K}{C} - \frac{\lambda K(C-\lambda)}{C\alpha\sigma(p)}$$

One can observe that the right hand side of Eq. (6) is increasing with α since $\lambda < C$. Therefore, the solution to the implicit equation that yields the break-even price is increasing with α. Hence, we have

$$p^{BE}(\alpha) < p^{BE}(1) \quad \forall \alpha \in [0,1) \tag{36}$$

Combining Eq. (36) with the results of Lemma 4.2 we conclude that $p^{BE}(\alpha) < K$.

(b) Assume $p^{MS}(\alpha)$ is given by the solution to the implicit equation in Eq. (21). We will prove the inequality by contradiction. Assume:

$$p^{MS}(\alpha) \le p^{BE}(1).$$

Since secondary demand is non-increasing in p it follows that $\sigma(p^{MS}(\alpha)) \ge \sigma(p^{BE}(1))$. Taking the ratio between Eq. (6) and Eq. (21) yields:

$$\frac{p^{BE}(1)}{p^{MS}(\alpha)} = \frac{(\sigma(p^{BE}(1)) + \lambda - C)\sigma(p^{MS}(\alpha))}{(\sigma(p^{MS}(\alpha)) + \lambda - C)\sigma(p^{BE}(1))} \cdot \frac{(C - \alpha(\lambda + \sigma(p^{MS}(\alpha)))}{C}. \tag{37}$$

The first fraction in Eq. (37) is less than or equal to 1 while the second is strictly less than 1. Furthermore, we know that both fractions must be positive since $\sigma(p^{BE}(1)) \ge C - \lambda$ by Lemma 4.1 and $p^{MS}(\alpha) < \bar{p}$. We have:

$$p^{BE}(1) < p^{MS}(\alpha), \tag{38}$$

which contradicts our initial assumption that $p^{MS}(\alpha) \le p^{BE}(1)$. Hence, it must be true that $p^{MS}(\alpha) > p^{BE}(1)$. In Eq. (36) we have established that $p^{BE}(\alpha) < p^{BE}(1)$ for $\alpha < 1$. Hence it is true for all values of $\alpha \in (0,1]$ that $p^{MS}(\alpha) > p^{BE}(\alpha)$.

(2) If $p^{BE}(\alpha) = 0$, we can show that the market sharing price is strictly greater than zero in both cases. $p^{MS}(\alpha) = K$ is self-explanatory and by Eq. (21), we conclude $p^{MS}(\alpha) > 0$ as $\sigma(p^{MS}(\alpha)) + \lambda - C > 0$.

5. DUOPOLY COMPETITION

In the previous sections we have identified a provider's competitive ability in a price war through establishing the break-even price and its incentive to share the market through the market sharing price. However, spectrum markets do not consist of a single provider, but rather several providers competing with each other. Therefore, our previous results, while being important, are not enough to determine the outcome of a secondary spectrum market. In this section, we consider the simplest oligopoly possible, a duopoly where two providers compete to enhance their profits by first capturing and then serving the secondary demand. To identify a market equilibrium, we utilize the concept of *Nash equilibrium* (NE) from game theory. Since NE are classically determined by *best response* functions, we will first seek to establish the best response dynamics of provider i to a fixed competitor price p_{-i}, where the notation $-i$ signifies the competing provider.

Definition 5.1 ***(Best Response)*** *Given two providers, provider i's best response to competitor's pricing decision p_{-i} is the payoff maximizing strategy such that:*

$$p_i^{BR}(p_{-i}) = \arg\max_{p_i} R_i(p_i, p_{-i}). \tag{39}$$

Definition 5.2 ***(Nash Equilibrium)*** *A pricing strategy profile (p_1^*, p_2^*) is a Nash equilibrium (NE) if and only if both prices are a best response to each other such that:*

$$p_1^* = p_1^{BR}(p_2^*) \quad \textit{and} \quad p_2^* = p_2^{BR}(p_1^*). \tag{40}$$

Facing a competitor price p_{-i}, the strategies available to provider i consist of either matching this price and sharing the secondary demand or not matching it and trying to capture all of the secondary

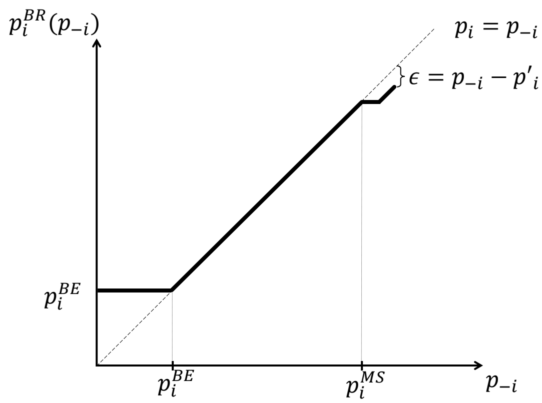


Fig. 5. Best response of network provider i: a) When the competitor price is below p_i^{BE}, the provider sets its price to p_i^{BE}; b) When the competitor price is within the market sharing interval, the provider matches the price; c) When the competitor price is above p_i^{MS}, the provider sets its price slightly lower.

demand. While setting the price below or above the competitor's price follows a rather straightforward approach, the case of matching the competitor's price requires a more detailed analysis due to the discontinuity in the profit function. The next lemma states that if it is possible to increase the profit by capturing all of the secondary demand $\sigma(p_i)$ at a certain price p_i, then it is also desirable to capture the secondary demand at a slightly lower price $p'_i < p_i$. We will then utilize this result in establishing provider i's best response for prices $p_i > p_i^{MS}(\alpha)$.

Lemma 5.1 *For any p_i such that $\Delta W_i(p_i) < 0$ holds, there exists a price p'_i such that $p_i^{MS}(\alpha) < p'_i < p_i$ and*

$$W_i(p'_i, \sigma(p'_i)) > W_i(p_i, \alpha_i \sigma(p_i)). \tag{41}$$

PROOF. See Appenix A. □

The next theorem presents provider i's best response, which we shall utilize later to determine NE.

Theorem 5.1 ***(Best Response)*** *Provider i best response to its competing provider pricing decision p_{-i} is:*

$$p_i^{BR}(p_{-i}) = \begin{cases} p_i^m(p_{-i}) & \text{for } p_{-i} > p_i^{MS}(\alpha) \\ p_{-i} & \text{for } p_i^{BE}(\alpha_i) \leq p_{-i} \leq p_i^{MS}(\alpha) \\ p_i^{BE} & \text{for } p_{-i} < p_i^{BE}(\alpha_i), \end{cases}$$

where $p_i^m(p_{-i}) < p_{-i}$ satisfies Eq. (41) in Lemma 5.1 to the optimality such that

$$W_i(p_i^m, \sigma(p_i^m)) = \max_{p_i \in (p_i^{MS}(\alpha), p_{-i})} W_i(p_i, \sigma(p_i)). \tag{42}$$

Remark 5.1 *The exact value of $p_i^m(p_{-i})$ depends on where the revenue is being maximized over the interval $(p_i^{MS}(\alpha), p_{-i})$. If the revenue is monotonically increasing up until p_{-i}, we can simplify Eq. (42) to the following:*

$$p_i^m(p_{-i}) = p_{-i} - \epsilon,$$

where ϵ is a sufficiently small discretization step, which is used when working with continuous prices. This assumption is a well-known approach used in game theory [Osbourne 2004] because otherwise, a best response does not technically exist. On the other hand, it is possible that provider i's revenue attains a maximum at a lower price point, in which case $p_i^m(p_{-i})$ is as given in Eq. (42) and its exact value depends on the price elasticity of secondary demand.

PROOF. We will consider each price condition described in Theorem 4.3 separately.

(1) In the first price condition, such that $\Delta W_i(p_{-i}) < 0$, provider i can either choose to match, lower or increase its price. Lowering the price such that $p'_i < p_{-i}$ is clearly better than price matching ($p_i = p_{-i}$)

since we have demonstrated in Lemma 5.1 that:

$$W_i(p_i', \sigma(p_i')) > W_i(p_{-i}, \alpha_i \sigma(p_{-i})).$$

Lowering the price to p_i' is also better than increasing the price to $p_i > p_{-i}$ since the following is true:

$$W_i(p_i', \sigma(p_i')) > 0 = W_i(p_i, p_{-i}), \quad \text{for all } p_i > p_{-i}.$$

Hence, lowering the price to p_i' is the best response of provider i.

(2) In the second competitor price condition such that $p_i^{BE}(\alpha_i) \leq p_{-i} \leq p_i^{MS}(\alpha)$, we know that the following holds:

$$\Delta W_i(p_{-i}) = W_i(p_{-i}, \alpha_i \sigma(p_{-i})) - W_i(p_{-i}, \sigma(p_{-i})) \geq 0. \tag{43}$$

Selecting a price above the competitor's price such that $p_i > p_{-i}$ does not attract any secondary demand and therefore yields a profit of zero. Thus matching p_{-i} is better than increasing the price to $p_i > p_{-i}$:

$$\Pi_i(p_i, 0) = 0 \leq \Pi_i(p_{-i}, \alpha_i \sigma(p_{-i})), \quad \forall p_i > p_{-i}.$$

Next, we compare matching the price at p_{-i} to lowering the price to any price $\{p_i : p_i < p_{-i}\}$. We seek to find the price that maximizes the revenue function $W_i(p_i, \sigma(p_i))$ on the interval $[0, p_{-i}]$. We know from Lemmas 4.4 and 4.5 that $p_i^{MS}(\alpha) < \bar{p}$. Hence any price p on the interval $[0, p_{-i}]$ where $p_{-i} < p_i^{MS}(\alpha)$ satisfies $\lambda_i + \sigma(p) > C_i$. Simplifying Eq. (1) and by taking the derivative with respect to p_i we can show that:

$$\begin{aligned}
\frac{\partial}{\partial p_i} W_i(p_i, \sigma(p_i)) =& \frac{\partial}{\partial p_i} \left(\sigma(p_i) p_i \frac{C_i}{\lambda_i + \sigma(p_i)} + \lambda_i K_i \frac{C_i}{\lambda_i + \sigma(p_i)} \right) \\
&= (\sigma(p_i) + \sigma'(p_i) p_i) \frac{C_i}{\lambda_i + \sigma(p_i)} - \sigma'(p_i) \sigma(p_i) p_i \frac{C_i}{(\lambda_i + \sigma(p_i))^2} - \sigma'(p_i) \lambda_i K_i \frac{C_i}{(\lambda_i + \sigma(p_i))^2}.
\end{aligned}$$

Regrouping the terms yields:

$$\frac{\partial}{\partial p_i} W_i(p_i, \sigma(p_i)) = \sigma(p_i) \frac{C_i}{\lambda_i + \sigma(p_i)} + \lambda_i C_i \sigma'(p_i) \frac{p_i - K_i}{(\lambda_i + \sigma(p_i))^2} > 0,$$

for $p_i \leq K_i$ since $\sigma'(p_i) \leq 0$.

We also know from Lemma 4.6 that $p_i^{MS}(\alpha) \leq K_i$. Therefore, the revenue maximizing price (which is also profit maximizing) is given by $p_i = p_{-i}$ such that for all $0 \leq p_i \leq p_{-i}$:

$$W_i(p_{-i}, \sigma(p_{-i})) \geq W_i(p_i, \sigma(p_i)).$$

By Equation (43), it follows that for all $p_i^{BE}(\alpha_i) \leq p_i \leq p_{-i}$, which demonstrates that matching the price at p_{-i} is better than lowering it to any $p_i < p_{-i}$:

$$W_i(p_{-i}, \alpha_i \sigma(p_{-i}) \geq W_i(p_i, \sigma(p_i)).$$

Hence, we conclude that $p_i^{BR}(p_{-i}) = p_i$.

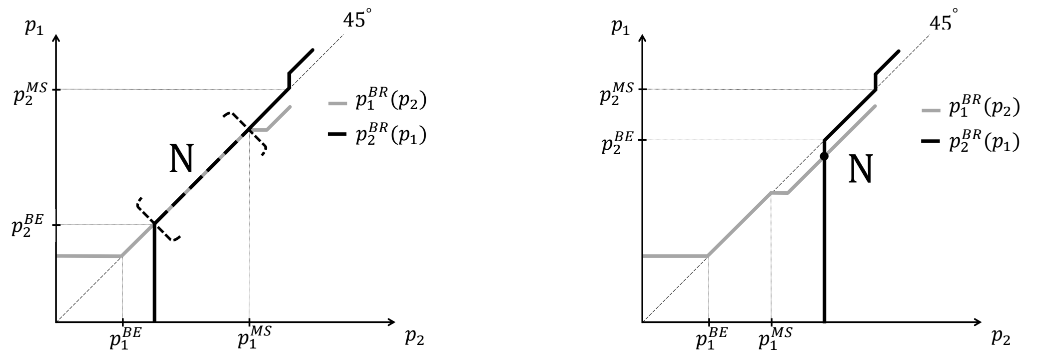
(3) Lastly, we consider the case when $p_{-i} < p_i^{BE}$. Fortunately, this case can be quickly analyzed through the definition of the break-even price. If provider i chooses to match or lower its price by definition of the break-even price we have that:

$$\Pi_i(p_i, p_{-i}) < 0, \quad \text{for all } p_i \leq p_{-i}.$$

At any price $p_i > p_{-i}$ provider i profit will be zero. However setting the price to p_i^{BE} prevents the other provider from increasing its price further, thus we establish it as the best response. □

Theorem 5.1 establishes that for any network provider, a price interval, in which market sharing is the best response, is guaranteed to exist. Above this price interval, a provider will lower its price below the competitor's price, as in a typical price war. Below this price interval, profitability conditions from Section 4.1 are violated. While this interval is guaranteed to exist, whether the market equilibrium is established in this interval warrants further analysis. In the next theorem, we determine the different market outcomes by providing the resulting NE from the best response functions of the two providers.

Theorem 5.2 ***(Nash Equilibrium)*** *In a market with two network providers, a pricing strategy profile* (p_1^*, p_2^*) *is a NE such that:*



(a) The placement of NE on the best response curves when market sharing intervals overlap, corresponding to part 1 of Theorem 5.2, where $p_1^{BE} < p_2^{BE}$ and $p_1^{MS} < p_2^{MS}$.

(b) The placement of NE on the best response curves when market sharing intervals do not overlap, corresponding to part 2 of Theorem 5.2, where $p_1^{MS} < p_2^{BE}$.

Fig. 6. Illustration of the two possible types of market outcomes.

(*1*) *If* $\max(p_1^{BE}(\alpha_1), p_2^{BE}(\alpha_2)) \le \min(p_1^{MS}, p_2^{MS})$*, then* $p_1^* = p_2^*$*, and for* $i = 1, 2$

$$p_i^* \in [\max(p_1^{BE}(\alpha_1), p_2^{BE}(\alpha_2)), \min(p_1^{MS}(\alpha_1), p_2^{MS}(\alpha_2))].$$

(*2*) *If* $\max(p_1^{BE}(\alpha_1), p_2^{BE}(\alpha_2)) > \min(p_1^{MS}(\alpha_1), p_2^{MS}(\alpha_1))$ *and without loss of generality* $p_2^{BE}(\alpha_2) < p_1^{BE}(\alpha_1)$

$$p_1^* = p_1^{BE}(\alpha_1) \text{ and } p_2^* = p_2^m(p_1^{BE}(\alpha_1)).$$

where $p_i^m(p_{-i}) < p_{-i}$ *is defined as in Theorem 5.1.*

PROOF. See Appendix B. □

Next, we discuss the implications of Theorem 5.2 and provide examples that illustrate our results:

Interpretation of the NE. As stated in Theorem 5.2, the exact price profiles that give the NE depend on the relationship between the market sharing intervals of the two providers. If two price intervals overlap, as illustrated in part (a) of Fig. 6, any equal price pair in that interval will give us a NE. As a result, two providers share the market and set their prices at a value above their respective break-even prices but always less than the smaller of the two market sharing prices, a value which is guaranteed to be no greater than K_i, the primary reward collected by provider i.

On the other hand, if the market sharing price intervals of the two providers do not intersect, as illustrated in part (b) of Fig. 6, the market outcome is the same as the result of a price war, where the provider with the lower break-even price captures all of the secondary demand by pricing slightly below its competitor's break-even price. The losing provider cannot match this price without making a negative profit. In this case, even though both providers find it desirable to go into market sharing as the prices approach their break-even prices, the gap between the two market sharing intervals does not allow them to converge to a market sharing point.

Examples. In the following two examples, we seek to illustrate different market outcomes depending on the placement of the market sharing intervals on the price line. In the first example, we will use a constant elasticity demand function as in our previous examples. In the second example, we will use an exponentially decreasing demand to illustrate the fact that our results hold over general demand functions. Both types of demand functions are commonly used in the economics literature [Talluri and Ryzin 2004].

***Example* 5.1** Suppose the secondary demand is given by $\sigma(p) = 20/p$. We consider two network providers whose parameters are:

$$(\lambda_1, C_1, K_1, \alpha_1) = (6, 10, 4, 0.5) \quad \text{and} \quad (\lambda_2, C_2, K_2, \alpha_2) = (5, 10, 4, 0.5).$$

Given these parameters it follows that $\lambda_i + \alpha_i\sigma(0) > C_i$ and $\lambda_i + \alpha_i\sigma(K) < C_i$ for $i = 1, 2$. Under these conditions, by making use of the explicit formulas provided in Eqs. (8) (substituting $\sigma(p)$ with $\alpha\sigma(p)$)

and (35), we obtain the following break-even and market sharing prices of both providers:

$$p_1^{BE}(0.5) = 1.225, \quad p_1^{MS}(0.5) = 2.881,$$
$$p_2^{BE}(0.5) = 1.000, \quad p_2^{MS}(0.5) = 2.400.$$

Clearly $p_1^{BE}(0.5) > p_2^{BE}(0.5)$ and $p_2^{MS}(0.5) < p_1^{MS}(0.5)$. Furthermore, it is also true that $p_1^{BE}(0.5) < p_2^{MS}(0.5)$. Therefore, both providers' market sharing price intervals are overlapping. Then, part one of Theorem 5.2 states that all NE price profiles (p_1^*, p_2^*) have the form: $p_1^* = p_2^* \in$ and lie in the price interval $[1.225, 2.400]$.

***Example* 5.2** In this example we consider an exponentially decreasing secondary demand given by $\sigma(p) = 20e^{-0.2p}$. This time we consider two similarly loaded providers with significantly different primary rewards. We choose the network parameters of these providers as such:

$$(\lambda_1, C_1, K_1, \alpha_1) = (6, 10, 6, 0.5) \quad \text{and} \quad (\lambda_2, C_2, K_2, \alpha_2) = (8, 10, 14, 0.5).$$

Notice that this time provider 2 has a higher primary demand and a higher associated reward collected. Once again, network parameters and the secondary demand satisfy $\lambda_i + \alpha_i\sigma(0) > C_i$ and $\lambda_i + \alpha_i\sigma(K) < C_i$ for $i = 1, 2$. Solving for the Eq. (6) in Theorem 4.2 and Eq. (9) in Theorem 4.3, we obtain the following break-even and market sharing prices of both providers:

$$p_1^{BE}(0.5) = 2.098, \quad p_1^{MS}(0.5) = 4.984,$$
$$p_2^{BE}(0.5) = 5.050, \quad p_2^{MS}(0.5) = 9.241.$$

Clearly $p_1^{BE}(0.5) < p_2^{BE}(0.5)$ and $p_2^{MS}(0.5) > p_1^{MS}(0.5)$. However, this time $p_2^{BE}(0.5) > p_1^{MS}(0.5)$. Therefore, the market sharing price intervals of the two providers do not intersect. As a result, these two providers will go into a price war and provider 1, having the lower break-even price will be the winner. In this light, part 2 of Theorem 5.2 states that the NE is given by $p_1^* = 5.050 - \epsilon$ and $p_2^* = 5.050$.

Best Response Dynamics. While Theorem 5.2 states that the NE exist and gives the pricing profiles of such, depending on the initial conditions one might never reach that equilibrium if best response dynamics change the prices in a different direction. In our case, the convergence to the NE is guaranteed from the way best response dynamics work. In both cases, for any price above the described NE prices, the best response dynamics lowers the price as each provider tries to capture the secondary demand by setting their price lower than the competitor's. For any prices below the NE, since this yields a negative profit for at least one provider, the best response dynamics now work to increase the prices to the break-even price of each provider, which in turn fall in the range of the NE given by Theorem 5.2.

Payoff Dominant Strategy Refinement. In part (1) of Theorem 5.2 we identified a price range in which all possible NE could lie. While all price pairs are viable NE , it is desirable to be able to characterize the market outcome through a single price pair. A possible refinement of the case when facing multiple NE is through the consideration of Payoff Dominant Strategy (PDS) equilibrium:

Definition 5.3 *Let $\mathcal{S}$ denote the set of price pairs $\{(p_1^*, p_2^*) : p_1^* = p_2^*\}$ that give the NE in part (1) of Theorem 5.2. Then, the PDS equilibrium (p_1^D, p_2^D) is a NE with the following refinement condition:*

$$R_i(p_1^D, p_2^D) = \max_{(p_1,p_2)\in\mathcal{S}} R_i(p_1, p_2) \quad \textit{for} \quad i = 1, 2,$$

In other words, when multiple NE are present, a PDS yields the maximum possible payoff for both providers [Straub 1995]. Using this condition we can identify the PDS equilibrium $(p_1^D, p_2^D) \in \mathcal{S}$. Since the prices in $\mathcal{S}$ are equal, we know from Eq. (3) that the payoff is equal to the profit under reduced demand. $(R_i(p_1, p_2) = \Pi_i(p_i, \alpha_i\sigma(p_i)))$. If $\sigma'(p) < 0$, let $\hat{p}$ denote the solution to:

$$p = -\sigma(p)/\sigma'(p). \tag{44}$$

Otherwise we set $\hat{p} = \infty$. Note that Eq. (44) corresponds to the price elasticity of demand. Through careful analysis, we can state the following:

Theorem 5.3 *For relatively inelastic demand such that $\hat{p} > \max(p_1^{MS}(\alpha_1), p_2^{MS}(\alpha_2))$, there exists a unique PDS equilibrium (p_1^D, p_2^D) given by:*

$$p_1^D = p_2^D = \min(p_1^{MS}(\alpha_1), p_2^{MS}(\alpha_2)) \tag{45}$$

PROOF. See Appendix C. □

6. QUALITY OF SERVICE EXTENSION

In our model we have made the assumption that secondary users always choose the lowest price advertised and when the prices are the same arriving secondary traffic randomly choose a provider. While we have argued that *price aversion* might be a possible explanation for choosing the lower price, Quality of Service (QoS) might also have an impact on the customer's decision process. In this subsection we extend our model to take QoS into consideration.

We consider a simple QoS performance metric: the acceptance rate of the incoming traffic. Then we extend our model as follows: When both providers charge the same price and the secondary demand at this price is sufficiently large that the total demand in the market exceeds the total capacity (i.e., $\lambda_1 + \lambda_2 + \sigma(p) > C_1 + C_2$), secondary demand is split between the two providers according to a vector $\boldsymbol{\alpha} = [\alpha_1, \alpha_2]$ such that $\alpha_1 + \alpha_2 = 1$, $\alpha_1, \alpha_2 > 0$ and satisfying the following equality:

$$\frac{C_1}{\lambda_1 + \alpha_1\sigma(p)} = \frac{C_2}{\lambda_2 + \alpha_2\sigma(p)}. \tag{46}$$

Namely, instead of randomly choosing a provider, secondary demand distributes itself in a fashion that the accommodation level it faces is homogeneous across both providers. In the case of two providers, let $\alpha_1 = \alpha$ and $\alpha_2 = 1 - \alpha$. Then we can obtain an explicit expression for α:

$$\alpha = \frac{C_1(\lambda_2 + \sigma(p)) - C_2\lambda_1}{(C_1 + C_2)\sigma(p)}. \tag{47}$$

and substituting Eq. (47) for α in $\alpha\sigma(p)$ we obtain:

$$\alpha\sigma(p) = \beta_1 + \gamma_1\sigma(p), \tag{48}$$

where $\beta_i = \frac{C_i\lambda_{-i} - C_{-i}\lambda_i}{C_i + C_{-i}}$ and $\gamma_i = \frac{C_i}{C_i + C_{-i}}$ for $i = 1, 2$.

Under this new model, the previous results stated in our theorems still hold. Since we consider all values of $\alpha \in (0, 1]$, we can simply replace $\alpha\sigma(p)$ in our equations with Eq. (48). We illustrate this result in the following example and extend our analysis in Appendix D.

***Example* 6.1** Suppose the secondary demand is given by $\sigma(p) = 30e^{-10p}$. We consider two network providers whose parameters are:

$$(\lambda_1, C_1, K_1) = (10, 20, 1) \quad \text{and} \quad (\lambda_2, C_2, K_2) = (8, 10, 1).$$

Suppose that secondary demand is split between the providers in a way that satisfies Eq. (46). Then, using Eq. (48), we have the following reduced demand functions:

$$\alpha_1\sigma(p) = 2\sigma(p)/3 + 2 \quad \text{and} \quad \alpha_2\sigma(p) = \sigma(p)/3 - 2.$$

Observe that $\alpha_1\sigma(p) + \alpha_2\sigma(p) = \sigma(p)$. We can check that $\lambda_i + \alpha_i\sigma(0) > C_i$ and $\lambda_i + \alpha_i\sigma(K) < C_i$ for $i = 1, 2$. Under these conditions, we need to use Eq. (6) for calculating the break-even price and (9) for the market sharing price for both providers. Doing the necessary calculations, one finds:

$$p_1^{BE} = 0.184, \quad p_1^{MS} = 0.529,$$
$$p_2^{BE} = 0.317, \quad p_2^{MS} = 0.417.$$

Clearly $p_1^{BE} < p_2^{BE}$ and $p_1^{MS} > p_2^{MS}$. Thus, providers 2's market sharing interval is a subset of provider 1's. Then, part one of Theorem 5.2 states that all NE price profiles (p_1^*, p_2^*) have the form $p_1^* = p_2^*$ and lie in the price interval $[0.162, 0.235]$.

7. CONCLUSION

In this paper, we investigated critical price values that determine the outcomes in a secondary spectrum market where multiple providers compete to attract secondary demand. We focused on an uncoordinated access setting for secondary spectrum under private commons, and carried out our analysis based on a fluid model.

Since market outcomes are determined by break-even and market sharing prices, we carefully defined and analyzed these two characteristic prices. We demonstrated existence and uniqueness of these prices for each provider, under general demand functions. We further provided implicit formulas to compute both of these prices as a function of the system parameters. The results of the paper show that below the break-even price, no secondary user will be admitted. Similarly, in between the break-even and market-sharing prices, it is possible that only a fraction of the total secondary demand will be admitted by each provider. Thus, a provider treats secondary users equally to primary users, only when they pay a high enough price (i.e., above the market sharing price).

A significant result is the existence of a market sharing interval under general forms of demand functions, implying that the incentive to share a spectrum market always precedes the incentive to exit the market due to negative profits. Using the notions of best response and Nash equilibrium, we then built on our results and showed the emergence of two markedly different possible market outcomes, depending on the secondary demand function $\sigma(p)$ and the network parameters of each provider (i.e, the service capacity C, primary demand λ, and primary reward K).

If the market sharing price intervals of the two providers intersect, as described in part one of Theorem 5.2, then the providers converge to a price profile where they will share the market. All prices falling between the maximum break-even price and minimum market sharing price among the two providers are possible NE. On the other hand, if the market sharing price intervals do not intersect, as described in part two of Theorem 5.2, then the NE reflects a price war wherein the winning provider sets its price slightly below the break-even price of its competitor and gets all the profit.

We have presented several refinements on our results, particularly through the consideration of the Payoff Dominant Strategy (PDS) equilibrium and the Quality of Service (QoS) extensions. Through PDS we were able to characterize the market outcome through a unique price pair for both cases in Theorem 5.2. We have also introduced a QoS extension, in which the secondary demand's decision were affected by the accommodation levels. We observed that our previous results held under this QoS model, suggesting their applicability in real world markets.

While it is possible to extend our analysis of the duopoly to any number of providers, there are several challenges to overcome. Since we considered a monopoly, network providers would either share the market at a certain fraction or not. When more providers enter the market, there rises a need to consider many different sharing scenarios. Furthermore, as the market share of a provider grows or shrinks due to other providers entering or exiting the market, this changes the break-even and market sharing prices. Since the calculation of these prices depend on the secondary demand function, it is challenging to predict whether these changes would alter the market dynamics. Specifically, one can envision a case where as a result of a provider exiting the market the characteristic prices of the remaining providers increase to a point that the exiting provider would re-enter the market, only to once exit again. Such cyclic behavior in the market could pose serious barriers to the analysis. We plan to consider the case of multiple providers in the future.

In summary, this paper is aimed to shed light into demand-invariant governing price relationships and the resulting outcomes of a spectrum market in a private commons competition under uncoordinated access. Future work could focus on the regulatory implications of the results (e.g., whether market sharing at a high price, even though inadvertent, may raise antitrust concerns amongst policy makers), and the extension of the results to oligopolies.

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Appendix A - Lemmas

PROOF OF LEMMA 4.1. The existence of $\bar{p}^{\alpha}$ follows from the assumption $\lambda+\alpha\sigma(0) > C$ and that the demand is non-increasing with the limit $\lim_{p\to\infty}\sigma(p) = 0$.

(1) Let p' be such a price that $\alpha\sigma(p')+\lambda \le C$. Since we know that secondary demand is non-increasing in p it also follows that p' must satisfy the following inequality: $p' \ge \bar{p}^{\alpha}$. We know that setting price equal to p' results in a non-negative profit since by Eq.'s (1) and (2) we have that:

$$\Pi(p', \alpha\sigma(p')) = \alpha\sigma(p')p' \ge 0.$$

Given that any price greater than or equal $\bar{p}^{\alpha}$ yields a non-negative profit for a provider, we can conclude that $\bar{p}^{\alpha}$ is an upper bound on the break-even price $p^{BE}(\alpha)$ (i.e., $p^{BE}(\alpha) \le \bar{p}^{\alpha}$).

(2) From part 1 of our proof we know that:

$$p^{BE}(\alpha) \le \bar{p}^{\alpha}.$$

Then, through our assumption that the secondary demand is non-increasing in p, the following is also true:

$$\alpha\sigma(p^{BE}(\alpha)) \ge \alpha\sigma(\bar{p}^{\alpha}). \tag{49}$$

Thus, from Eq. (49) and the definition of $\bar{p}^{\alpha}$ we obtain: $\lambda + \alpha\sigma(p^{BE}(\alpha)) \ge C$. □

PROOF OF LEMMA 4.2. We can check this claim by taking a look at the right hand side of Eq. (6):

$$\frac{(\sigma(p) + \lambda - C)\,\lambda K}{C\sigma(p)}.$$

In order for the claim to hold, we need $(\sigma(p) + \lambda - C)\,\lambda < C\sigma(p)$, which can be rewritten as: $\lambda(\lambda+\sigma(p)) < C(\lambda + \sigma(p))$. This is true under our initial assumption $\lambda < C$. □

PROOF OF LEMMA 4.6. The inequality holds when $p_i^{MS}(\alpha) = K_i$. When $p_i^{MS}(\alpha)$ is given by the solution to the implicit equation in Eq. (21), we prove it by showing the following:

$$p_i^{MS} = \frac{(\lambda_i + \sigma(p_i^{MS}(\alpha)) - C_i)\lambda_i K_i}{(C_i - \alpha_i(\lambda_i + \sigma(p_i^{MS}(\alpha))))\sigma(p_i^{MS}(\alpha))} \le K_i.$$

After some simple algebra and regrouping of terms we get:

$$\lambda_i \le C_i - \alpha_i\sigma(p_i^{MS}(\alpha)).$$

Note that if $p_i^{MS}(\alpha)$ is given by the solution to the implicit equation in Eq. (21), by Lemma 4.5 it also follows that $p_i^{MS}(\alpha) \in [\bar{p}_i^{\alpha}, \bar{p}_i)$. From the way we have defined $\bar{p}_i^{\alpha}$ in Lemma 4.1 we conclude that $\lambda + \alpha\sigma(p_i^{MS}(\alpha)) \le C$. □

PROOF OF LEMMA 5.1. Since we know that $W_i(x, \sigma(x))$ is differentiable in x, we can always a pick a price $q_i < p_i$ such that on the interval $[q_i, p_i)$, the function $W_i(x, \sigma(x))$ is either monotonically increasing, constant or monotonically decreasing with respect to x[6].

We break our proof into two cases:

(1) Assume that for a given q_i such that $q_i < p_i$, the following is true for any $\hat{p}_i \in [q_i, p_i)$:

$$W_i(\hat{p}_i, \sigma(\hat{p}_i)) \ge W_i(p_i, \sigma(p_i)).$$

Then it follows by our assumption $\Delta W_i(p_i) < 0$ that $W_i(\hat{p}_i, \sigma(\hat{p}_i)) > W_i(p_i, \alpha_i\sigma(p_i))$, and $p_i' = \hat{p}_i$.

(2) Assume for a given q_i such that $q_i < p_i$, the following is true for all $\hat{p}_i \in [q_i, p_i)$:

$$W_i(\hat{p}_i, \sigma(\hat{p}_i)) < W_i(p_i, \sigma(p_i)). \tag{50}$$

Then by the definition of continuity, the following can be stated for $W_i(p_i, \sigma(p_i))$: $\forall\epsilon > 0,\ \exists\delta(\epsilon, p_i) > 0$ s.t. if $|p_i - \hat{p}_i| < \delta$ then

$$|W_i(p_i, \sigma(p_i)) - W_i(\hat{p}_i, \sigma(\hat{p}_i))| < \epsilon.$$

[6]It should be noted that differentiability is not a necessary condition for this statement; local monotonocity of $W_i(x, \sigma(x))$ would suffice. However, as we need differentiability elsewhere in the paper, we simply use it here as well.

Making use of Eq. (50) and our assumption that $\hat{p}_i \in [q_i, p_i)$, we can remove the absolute value from the previous equation and simplify it to:

$$W_i(p_i, \sigma(p_i)) - W_i(\hat{p}_i, \sigma(\hat{p}_i)) < \epsilon. \tag{51}$$

Taking $\epsilon = W_i(p_i, \sigma(p_i)) - W_i(p_i, \alpha_i \sigma(p_i))$ and cancelling the terms $W_i(p_i, \sigma(p_i))$ on both sides of the inequality (51) we obtain:

$$-W_i(\hat{p}_i, \sigma(\hat{p}_i)) < -W_i(p_i, \alpha_i \sigma(p_i)),$$
$$W_i(\hat{p}_i, \sigma(\hat{p}_i)) > W_i(p_i, \alpha_i \sigma(p_i)),$$

and $p_i' = \hat{p}$. $\square$

Appendix B - Nash Equilibrium

PROOF OF THEOREM 5.2.

(1) Without loss of generality, assume that $p_1^{BE}(\alpha_1) < p_2^{BE}(\alpha_2)$. Now suppose $p_1^{MS}(\alpha_1) > p_2^{MS}(\alpha_2)$, such that we have the following relationship between the break-even and market sharing prices:

$$p_1^{BE}(\alpha_1) < p_2^{BE}(\alpha_2) < p_2^{MS}(\alpha_2) < p_1^{MS}(\alpha_1).$$

We will establish NE by determining when $p_1^* = p_1^{BR}(p_2^{BR}(p_1^*))$. In order to do so we first give provider 2's best response:

$$p_2^{BR}(p_1^*) = \begin{cases} p_2^{BE}(\alpha_2) & \text{for } p_1^* < p_2^{BE}(\alpha_2) \\ p_1^* & \text{for } p_2^{BE}(\alpha_2) \leq p_1^* \leq p_2^{MS}(\alpha_2) \\ p_2^m(p_1^*) & \text{for } p_1^* > p_2^{MS}(\alpha_2), \end{cases} \tag{52}$$

where $p_2^m(p_1^*)$ satisies Eq. (42) in Theorem 5.1.

We can now formulate provider 1's best response to provider 2's best response:

$$p_1^{BR}(p_2^{BR}(p_1^*)) = \begin{cases} p_1^{BE}(\alpha_1) & \text{for } p_1^* < p_1^{BE}(\alpha_1) \\ p_2^{BE}(\alpha_2) & \text{for } p_1^{BE}(\alpha_1) \leq p_1^* < p_2^{BE}(\alpha_2) \\ p_1^* & \text{for } p_2^{BE}(\alpha_2) \leq p_1^* \leq p_2^{MS}(\alpha_2) \\ p_2^m(p_1^*) & \text{for } p_2^{MS}(\alpha_2) < p_1^* \leq p_1^{MS}(\alpha_1) \\ p_1^m(p_2^*) & \text{for } p_1^* > p_1^{MS}(\alpha_1). \end{cases} \tag{53}$$

Therefore the only price interval where

$$p_1^* = p_1^{BR}(p_2^{BR}(p_1^*))$$

can be satisfied is $[p_2^{BE}(\alpha_2), p_2^{MS}(\alpha_2)]$ and from Eq. (52) in this interval we have that $p_2^* = p_1^*$, hence giving us the NE. The other case where $p_1^{MS}(\alpha_1) \leq p_2^{MS}(\alpha_2)$ can be proven following the same argument.

(2) Suppose that $p_1^{BE}(\alpha_1) > p_2^{MS}(\alpha_2)$. Then, by Theorem 4.1, we also know the following relationship between the break-even and market sharing prices of both providers:

$$p_2^{BE}(\alpha_2) < p_2^{MS}(\alpha_2) < p_1^{BE}(\alpha_1) < p_1^{MS}(\alpha_1).$$

This time we will establish NE by determining when $p_2^* = p_2^{BR}(p_1^{BR}(p_2^*))$. In order to do so we first give provider 1's best response:

$$p_1^{BR}(p_2^*) = \begin{cases} p_1^{BE}(\alpha_1) & \text{for } p_1^* < p_1^{BE}(\alpha_1) \\ p_2^* & \text{for } p_1^{BE}(\alpha_1) \leq p_2^* \leq p_1^{MS}(\alpha_1) \\ p_1^m(p_2^*) & \text{for } p_2^* > p_1^{MS}(\alpha_1). \end{cases} \tag{54}$$

We can now formulate provider 2's best response to provider 1's best response:

$$p_2^{BR}(p_1^{BR}(p_2^*)) = \begin{cases} p_2^{BE}(\alpha_2) & \text{for } p_2^* < p_2^{BE}(\alpha_2) \\ p_1^{BE}(\alpha_1) & \text{for } p_2^{BE}(\alpha_2) \leq p_2^* \leq p_2^{MS}(\alpha_2) \\ p_2^m(p_1^{BE}(\alpha_1)) & \text{for } p_2^{MS}(\alpha_2) < p_2^* < p_1^{BE}(\alpha_1) \\ p_2^m(p_2^*) & \text{for } p_1^{BE}(\alpha_1) \leq p_2^* \leq p_1^{MS}(\alpha_1) \\ p_2^m(p_1^m(p_2^*)) & \text{for } p_2^* > p_1^{MS}(\alpha_1). \end{cases} \tag{55}$$

A careful look yields the result that the only time $p_2^* = p_2^{BR}(p_1^{BR}(p_2^*))$ is possible when $p_2^* = p_2^m(p_1^{BE}(\alpha_1))$, given in the third pricing interval in Eq. (55). From Eq. (54) we have that

$$p_1^* = p_1^{BR}(p_2^m(p_1^{BE}(\alpha_1))) = p_1^{BE}(\alpha_1), \tag{56}$$

thus completing the pricing strategy profile of the only NE possible in this case. Note that since $p_1^{BE}(\alpha_1) > p_2^{MS}(\alpha_2)$, it follows from Theorem 4.1 that $p_1^{BE}(\alpha_1) > p_2^{BE}(1)$. Since ϵ can be chosen arbitrarily small, we can extend this result to $p_1^{BE}(\alpha_1) - \epsilon > p_2^{BE}(1)$, therefore provider 2 is profitable at this NE as a monopoly. We can use the same argument to prove the case when $p_1^{BE}(\alpha_1) \leq p_2^{MS}(\alpha_2)$. □

Appendix C - Payoff Dominant Strategy (PDS) Equilibrium

PROOF OF THEOREM 5.3.

Combining Lemmas 4.4, 4.5 and Theorem 4.3 we know the following:

(i) If p_i^{MS} is given by Eq. (9), then $\bar{p}_i^{\alpha_i} \leq p_i^{MS}(\alpha_i) < \bar{p}_i$.
(ii) If $p_i^{MS}(\alpha_i) = K_i$, then $p_i^{MS} < \bar{p}_i^{\alpha_i}$.

Therefore, we need to consider two different formulations of the profit $\Pi_i(p_i, \alpha_i\sigma(p_i))$. One can observe from Eq. (2) that:

$$\frac{\partial}{\partial p_i} W_i(p_i, \alpha_i\sigma(p_i)) = \frac{\partial}{\partial p_i} \Pi_i(p_i, \alpha_i\sigma(p_i)).$$

Therefore, we will use the derivative of revenue with respect to price in our calculations instead of profit.

Case 1 - $p_i < \bar{p}_i^{\alpha_i}$

The price condition is equivalent to stating that $\lambda_i + \alpha_i\sigma(p) > C_i$. Simplifying Eq. (1) and by taking the derivative with respect to p_i we can show that:

$$\begin{aligned}\frac{\partial}{\partial p_i} W_i(p_i, \alpha_i\sigma(p_i)) =& \frac{\partial}{\partial p_i}\left(\alpha_i\sigma(p_i)p_i\frac{C_i}{\lambda_i + \alpha_i\sigma(p_i)} + \lambda_i K_i \frac{C_i}{\lambda_i + \alpha_i\sigma(p_i)}\right)\\ =& \alpha_i(\sigma(p_i) + \sigma'(p_i)p_i)\frac{C_i}{\lambda_i + \alpha_i\sigma(p_i)} - \alpha_i\sigma(p_i)p_i\frac{C_i\alpha_i\sigma'(p_i)}{(\lambda_i + \alpha_i\sigma(p_i))^2} - \lambda_i K_i\frac{C_i\alpha_i\sigma'(p_i)}{(\lambda_i + \alpha_i\sigma(p_i))^2}.\end{aligned}$$

Regrouping the terms yields:

$$\frac{\partial}{\partial p_i} W_i(p_i, \alpha_i\sigma(p_i)) = \alpha_i\sigma(p_i)\frac{C_i}{\lambda_i + \alpha_i\sigma(p_i)} + \lambda_i C_i\alpha_i\sigma'(p_i)\frac{p_i - K_i}{(\lambda_i + \alpha_i\sigma(p_i))^2} > 0, \tag{57}$$

for $p_i \leq K_i$ since $\sigma'(p_i) \leq 0$.

Case 2 - $p_i \geq \bar{p}_i^{\alpha_i}$

Simplifying Eq. (1) and by taking the derivative with respect to p_i we can show that:

$$\begin{aligned}\frac{\partial}{\partial p_i} W_i(p_i, \alpha_i\sigma(p_i)) =& \frac{\partial}{\partial p_i}\left(\alpha_i\sigma(p_i)p_i + \lambda_i K_i\right)\\ =& \alpha_i\left(\sigma(p_i) + \sigma'(p_i)p_i\right).\end{aligned}$$

If $\sigma'(p_i) = 0$, then $\frac{\partial}{\partial p_i} W_i(p_i, \alpha_i\sigma(p_i)) > 0$ for all $p_i \geq \bar{p}_i^{\alpha_i}$. On the other hand, if $\sigma'(p_i) < 0$ we have the following:

$$\frac{\partial}{\partial p_i} W_i(p_i, \alpha_i\sigma(p_i)) \begin{cases} > 0 & \text{if } p_i < \hat{p} \\ = 0 & \text{if } p_i = \hat{p} \\ < 0 & \text{if } p_i > \hat{p}, \end{cases} \tag{58}$$

where $\hat{p}$ denotes the solution to:

$$p = -\sigma(p)/\sigma'(p).$$

Note that $\hat{p}$ is the same for both providers. Now we consider the cases $(p_1^{MS}(\alpha_1) = K_1, p_2^{MS}(\alpha_2) = K_2)$, $(p_i^{MS}(\alpha_i) = K_i, p_{-i}^{MS} = g_{-i}(p_{-i}))$ and $(p_1^{MS} = g_1(p_1), p_2^{MS} = g_2(p_2))$ separately, where $g_i(p_i)$ represents the right hand side of Eq. (9).

(1) Assume $p_1^{MS}(\alpha_1) = K_1, p_2^{MS}(\alpha_2) = K_2$. Recalling condition (ii) in the beginning of our proof, we have:

$$p_i^{MS}(\alpha_i) < \bar{p}_i^{\alpha_i}, \quad \text{for} \quad i = 1, 2. \tag{59}$$

From Eq. (57) we know that for $p < \bar{p}_i^{\alpha_i}$ the profit is increasing on the interval $[0, K_i]$. Therefore, both providers obtain their maximum revenue rates at their respective market sharing prices. Then, the PDS equilibrium is:

$$p_1^D = p_2^D = \min(K_1, K_2) = \min(p_1^{MS}(\alpha_1), p_2^{MS}(\alpha_2)). \tag{60}$$

(2) Assume $p_1^{MS}(\alpha_1) = K_1, p_2^{MS}(\alpha_2) = g_2(p_2)$. From Eq. (57) we know that provider 1's payoff is maximized at K_1. Recalling condition (i), we have $p_2^{MS}(\alpha_2) > \bar{p}_2^{\alpha_2}$. Then from Eq. (58) we know that provider 2's payoff is increasing until $\hat{p}$. Since we assume that $\hat{p} > p_2^{MS}(\alpha_2)$, and we consider price strategy profiles that are upper bounded by $\min(p_1^{MS}(\alpha_1), p_2^{MS}(\alpha_2))$, the PDS equilibrium is given by:

$$p_1^D = p_2^D = \min(p_1^{MS}(\alpha_1), p_2^{MS}(\alpha_2)). \tag{61}$$

(3) $p_1^{MS}(\alpha_1) = g_1(p_1), p_2^{MS}(\alpha_2) = g_2(p_2)$. From Eq. (58) we conclude that both providers' profits are increasing until $\hat{p}$. Once again recalling our assumption that $\hat{p} > \max(p_1^{MS}(\alpha_1), p_2^{MS}(\alpha_2))$ and the upper bound $\min(p_1^{MS}(\alpha_1), p_2^{MS}(\alpha_2))$ on $\mathcal{S}$, we conclude that the PDS equilibrium is given by:

$$p_1^D = p_2^D = \min(p_1^{MS}(\alpha_1), p_2^{MS}(\alpha_2)). \tag{62}$$

□

Appendix D - Quality of Service (QoS) Extension

Before we begin our proof of Theorem 4.2 we need to revisit the two prices we have created before: $\bar{p}$ and $\bar{p}^{\alpha}$. By definition $\bar{p}$ is the same, while substituting Eq. (47) for α we get the new following relationship:

$$\lambda_1 + \alpha_1 \sigma(\bar{p}^{\alpha}) = C_1 \iff \lambda_1 + \lambda_2 + \sigma(\bar{p}^{\alpha}) = C_1 + C_2. \tag{63}$$

Therefore, our previous result $p^{BE}(\alpha) \leq \bar{p}^{\alpha}$ from Lemma 4.1 is equivalent to the following:

$$\lambda_1 + \lambda_2 + \sigma(p^{BE}(\alpha)) \geq C_1 + C_2 \tag{64}$$

PROOF OF THEOREM 4.2 REVISITED.

Parts 1 & 3 of the proof remain unchanged. The following is a revision of part 2 in our proof:

(2) Let us rearrange the terms in Eq. (6) and reintroduce index i to get the following:

$$\frac{C_i}{\lambda_i K} p - 1 = \frac{\lambda_i - C_i}{\alpha_i \sigma(p)}. \tag{65}$$

We will proceed by demonstrating that the left hand side of Eq. (65) is strictly increasing with respect to p and the right hand side is non-increasing with respect to p, hence meaning that this equality only holds at a single value of p. Since the left hand side of Eq. (6) linearly increasing in p, we only need to prove that the right hand side is non-increasing. Under the assumption that $\sigma(p)$ is a differentiable and non-increasing function of p, substituting

$$\alpha_i \sigma(p) = \beta_i + \gamma_i \sigma(p), \tag{66}$$

and taking the derivative of the right hand side with respect to p yields:

$$\frac{\partial}{\partial p}\left(\frac{\lambda_i - C_i}{\beta_i + \gamma_i \sigma(p)}\right) = \gamma_i \frac{C_i - \lambda_i}{(\beta_i + \gamma_i \sigma(p))^2} \sigma'(p) \leq 0. \tag{67}$$

Eq. (67) holds because $\lambda_i < C_i$ and $\sigma'(p) \leq 0$. Therefore, there can only be at most one solution for $p^{BE}(\alpha)$ that satisfies Eq. (6). □

PROOF OF THEOREM 4.3 REVISITED. From the way we have defined the new distribution vector $\boldsymbol{\alpha}$ in Eq. (47), our model does not extend to prices are greater than $\bar{p}^{\alpha}$ as the total demand does not exceed the total market capacity, and can be fully accommodated. Therefore, we keep our assumption of the random splitting of the secondary demand for these price values. Lemmas 4.3 and 4.5 remain unchanged. We need to revisit the price values where $p : p < \bar{p}^{\alpha}$, and update the corresponding Lemma 4.4.

Lemma 7.1 *Assume $\bar{p}^\alpha > 0$ and $p < \bar{p}^\alpha$, then*

(1) *If $\bar{p}^\alpha \geq K$:*

$$\Delta W_i(p) > 0 \qquad \text{if } p < K; \tag{68}$$

$$\Delta W_i(p) = 0 \qquad \text{if } p = K; \tag{69}$$

$$\Delta W_i(p) < 0 \qquad \text{if } p > K. \tag{70}$$

(2) *If $\bar{p}^\alpha < K$:*

$$\Delta W(p) > 0 \quad \forall p < \bar{p}^\alpha. \tag{71}$$

PROOF. Note that our assumption $p < \bar{p}^\alpha$ is equivalent to stating that:

$$\lambda_1 + \lambda_2 + \sigma(p) > C_1 + C_2$$

Since $\bar{p}^\alpha < \bar{p}$, it must also be true that $p < \bar{p}$. Then the combined demand without market sharing is greater than the provider's capacity (i.e., $\lambda_i + \sigma(p) \geq C_i$). Simplifying Eq. (1) under these assumptions, we obtain:

$$\Delta W_i = W_i(p, \alpha_i\sigma(p)) - W_i(p, \sigma(p)) = \frac{\alpha_i\sigma(p)pC_i}{\alpha_i\sigma(p) + \lambda_i} + \frac{\lambda_i K C_i}{\alpha_i\sigma(p) + \lambda_i} - \frac{\sigma(p)pC_i}{\sigma(p) + \lambda_i} - \frac{\lambda_i K C_i}{\sigma(p) + \lambda_i}.$$

After rearrangement and substituting $\alpha_i\sigma(p)$ with $\beta_i + \gamma_i\sigma(p)$ we get:

$$\Delta W_i = \frac{(\beta_{-i} + \gamma_{-i})\sigma(p))\sigma(p)\lambda_i C_i}{((\beta_i + \gamma_i)\sigma(p) + \lambda)(\sigma(p) + \lambda)}(K - p). \tag{72}$$

Eq. (72) only takes on the value zero when $p = K$. Additionally for price values $p < K$, ΔW_i is positive and for $p > K$, ΔW_i is negative. $\square$

Since the results of Lemma 7.1 and Lemma 4.4 are the same, the revised proof of Theorem 4.2 remains the same as before.

Having demonstrated that the main results stated in Theorems 4.2 and 4.3 hold under the extended model, we revisit Theorem 4.1.

PROOF OF THEOREM 4.1 REVISITED. The only part of the proof we need to revisit is for prices $p < \bar{p}^\alpha$. If the market sharing and the break-even prices are in the price range $[0, \bar{p}^\alpha)$, then from Lemma 4.4 we conclude that $p^{MS}(\alpha) = K$. Further, by substituting the right hand side of Eq. (47) for α in Eq. (6), we can demonstrate that $p^{BE}(\alpha)$ is given by the solution to the following equation:

$$p = \frac{(\lambda_1 + \lambda_2 + \sigma(p) - C_1 - C_2)\,\lambda_1 K}{C_1(\lambda_2 + \sigma(p)) - C_2\lambda_1}.$$

Then we need to demonstrate that:

$$\frac{(\lambda_1 + \lambda_2 + \sigma(p) - C_1 - C_2)\,\lambda_1 K}{C_1(\lambda_2 + \sigma(p)) - C_2\lambda_1} < K.$$

After rearranging and collecting the terms we obtain the following:

$$\lambda_1(\lambda_1 + \lambda_2 + \sigma(p)) - C_2\lambda_1 < C_1(\lambda_1 + \lambda_2 + \sigma(p)) - C_2\lambda_1,$$

which is true for since $\lambda_1 < C_1$ in our initial assumptions. $\square$

The results stated in Section 5, once the break-even and market sharing prices are determined, do not depend on the specific value α_i takes. The results stated Lemma 5.1 depends of continuity of the price p_i and the sign of the revenue change $\Delta W_i(p_i)$. The proof Theorem 5.1 builds on Lemma 5.1 and utilities the revenue rate without sharing ($W_i(p_i, \sigma(p_i))$). The proof of Theorem 5.2 is based on the game theoretic interpretation of the results stated in Theorem 5.1. All of these results hold as long as α_i takes on a value in the interval $[0, 1)$, which our extended model does not violate.