Solving Linear Rational Expectations Models

Form of System

- We'll start with

$$A E_t Y_{t+1} = B Y_t$$

- We'll end with

$$Y_{t+1} = G Y_t + H \varepsilon_{t+1}$$

Necessary condition for solvability

- There must be a "z" (scalar number) such that |Az-B| is not zero.
- Weaker than |A| not zero (required for inverse); can have |A|=0 or |B|=0 or both.
- If there is such a z, then one can construct full rank matrices for transforming system
  - T transforms equations
  - V transforms variables.

Transformed System

- General form

$Y^* = VY$

$$A^* E_t Y^*_{t+1} = B^* Y^*_t$$

$$with\ A^* = TAV^{-1}; B^* = TBV^{-1}$$

Form of Transformed System

- Key matrices are block diagonal
- Jordan matrices with stable and unstable eigenvalues as in Blanchard-Kahn (1980)
- "N" is nilpotent (zeros on diagonal and below; ones and zeros above diagonal).

$$A^* = \begin{bmatrix} N & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \text{ and } B^* = \begin{bmatrix} I & 0 & 0 \\ 0 & J_u & 0 \\ 0 & 0 & J_s \end{bmatrix}$$

i
u
s

An aside

- Solutions to |Az-B|=0 are called "generalized eigenvalues of A,B"
- Since roots of the polynomial are not affected by multiplication by arbitrary nonsingular matrices, these are the same as "generalized eigenvalues of A*,B*" i.e., the roots of |A*z-B*|=0.
- With a little work, you can see that there are only as many roots as $n(\mu_u)+n(\mu_s)$, since there are zeros on the diagonal of N. Try the case at right for intuition

$$A^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu_u & 0 \\ 0 & 0 & \mu_s \end{bmatrix}$$

Implications

- Transformed variables which evolve according to separated equation systems

$$Y_t^* = \begin{bmatrix} i_t \\ u_t \\ s_t \end{bmatrix}$$

$N E_t i_{t+1} = i_t$ ⟵ N, I

$E_t u_{t+1} = J_u u_t$ ⟵ I, J_u

$E_t s_{t+1} = J_s s_t$ ⟵ I, S

Three types of “canonical variables”

- Unstable canonical variables

$$E_t u_{t+1} = J_u u_t \Rightarrow u_t = 0$$

so as to avoid explosive behavior

- Stable canonical variables

$$E_t s_{t+1} = J_s s_t$$

- Infinite eigenvalue canonical variables

$$N E_t \dot{i}_{t+1} = \dot{i}_t \Rightarrow \dot{i}_t = 0$$

- Just like unstable

$$0 \, \dot{i}_{t+1} = \bar{i}_t \qquad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} i_{1,t+1} \\ i_{2,t+1} \end{bmatrix}$$

Solving for the variables we really care about

- Partition Y into those variables that are predetermined or have exogenous forecast error (K) and every thing else (Λ)

$$Y_t = \begin{bmatrix} \Lambda_t \\ K_t \end{bmatrix}$$

- Group i and u variables into U

$$U_t = \begin{bmatrix} i_t \\ u_t \end{bmatrix}$$

Partition the variable transformation matrix V and its inverse R

$$\begin{bmatrix} U \\ s \end{bmatrix} = \begin{bmatrix} V_{U\Lambda} & V_{UK} \\ V_{s\Lambda} & V_{sK} \end{bmatrix} \begin{bmatrix} \Lambda \\ K \end{bmatrix}$$

$$\begin{bmatrix} \Lambda \\ K \end{bmatrix} = \begin{bmatrix} R_{\Lambda U} & R_{\Lambda s} \\ R_{KU} & R_{Ks} \end{bmatrix} \begin{bmatrix} U \\ s \end{bmatrix}$$

$$v_t = V_{v\Lambda}\Lambda_t + V_{vK}K_t$$

- Solve for nonpredetermined or exogenous variables given solutions for U = 0.

$$\Lambda_t = -V_{U\Lambda}^{-1}V_{UK}K_t$$

- Need square and nonsingular matrix

Condition #1: counting rule

- Need same number of elements of Λ as unstable and infinite eigenvalues number (number of elements of U).
- But this implies: number of stable eigenvalues = (number of elements of S) number of variables with predetermined or exogenous forecast error (number of elements of K).
- This condition is due to Blanchard and Kahn (1980) in terms of predetermined variables and was generalized to variables that have exogenous forecast errors by Sims and Klein

Condition #2: rank condition

- But it is not enough for the matrix $V_{U\Lambda}$ to be square: it must be nonsingular to be inverted.
- This condition was present in Blanchard-Kahn (1980) and was emphasized in the generalization of King and Watson (1998).
- A parallel condition is in Sims, Klein and the result is reported in Dynare.

Solving for other variables

- Use the reverse transform, the solution for the stable variables, and the solution for the U variables (unstable and infinite cvs)

$$
\begin{aligned}
E_t K_{t+1} &= R_{KU} E_t U_{t+1} + R_{Ks} E_t s_{t+1} \\
&= 0 + R_{Ks} J_s s_t \\
&= R_{Ks} [J_s (V_{s\Lambda} \Lambda_t + V_{sK} K_t)] \\
&= R_{Ks} J_s \left(V_{s\Lambda} V_{U\Lambda}^{-1} V_{UK} + V_{sK} \right) K_t
\end{aligned}
$$

Now we know

$$
\begin{aligned}
\Lambda_{t+1} &= V_{U\Lambda}^{-1} V_{UK} \; K_{t+1} \\
E_t K_{t+1} &= R_{Ks} J_s \left(V_{s\Lambda} V_{U\Lambda}^{-1} V_{UK} + V_{sK} \right) K_t \\
&= G_K Y_t \\
K_{t+1} &= E_t K_{t+1} + H_K \varepsilon_{t+1} \\
&= G_K Y_t + H_K \varepsilon_{t+1} \\
\Lambda_{t+1} &= V_{U\Lambda}^{-1} V_{UK} \; E_t K_{t+1} + V_{U\Lambda}^{-1} V_{UK} H_K \varepsilon_{t+1} \\
&= G_\Lambda Y_t + H_\varepsilon \varepsilon_{t+1}
\end{aligned}
$$

Done!

- System is now in the form reported in main page
- However, this is a theoretical characterization, not a numerical recipe
- Numerical recipes employ the “Generalized Schur decomposition” which makes matrices like N and J upper triangular. This is a numerically more stable approach, but all the logical components are the same.