SZG Macro 2010: Lecture 2

Introduction to dynamic macroeconomics: Analysis of Saving and Investment

Outline of lecture

A. Investment and capital accumulation

with given output and factor prices: partial equilibrium

B. The Solow model: exogenous saving

C. The Ramsey-Cass-Koopmans model

in discrete time: endogenous saving and investment in general equilibrium

A. Two visions of the demand for capital

- Both visions will assume that there is
  - a constant returns to scale (CRTS) production function at the firm and aggregate level.
  - market in which capital rents at rate "q".
- The production function will be
  - $y=F(k,n)$ in general
  - $y=ak^{1-\alpha}n^{\alpha}$ (Cobb-Douglas form) for examples

A1. Demand for capital with labor given

- A firm maximizes profit, $F(k,n)-qk$, taking as given the quantity of labor (n).
- With labor fixed, there is a diminishing marginal product of capital
- Profit maximization requires $F_k(k,n)-q=0$
- The demand for capital is $k^d=\kappa(q,n)$, depending negatively on the rental price
- Increases in the quantity of labor raise the demand for capital.

Vision 1 in a diagram

A2. Demand for capital with output given

- A firm can also buy labor at wage rate **w**
- A firm minimizes cost **wn+qk** taking factor prices and an output level **y** as given.
- Changes in w and q induce factor substitution (along an isoquant)
- With CRTS, the factor demands are
  - $k^d=\kappa(q/w)y$
  - $n^d=\eta(q/w)y$
- Increases in output (scale) and the wage rate (substitution) raise the demand for capital, increases in the rental rate lower it (substitution).

Vision 2 in a diagram

Investment and the demand for capital

- Suppose that a period is the length of time that it takes for investment (i) to become productive as capital (k).
- The rental market can reallocate the existing capital stock, but cannot change the predetermined quantity.
- Production: $y_t = F(k_t, n_t)$ with k predetermined
- Capital accumulation: $k_{t+1} - k_t = i_t - \delta k_t$, where δ is the depreciation rate.

Investment and the demand for capital

(note that the κ functions are different across visions)

Vision 1:

$$k_{t+1} = \kappa(q_{t+1}, n_{t+1})$$

Vision 2:

$$k_{t+1} = \kappa(q_{t+1} / w_{t+1}) y_{t+1}$$

Either: $$i_t = (k_{t+1} - k_t) + \delta k_t$$

Total = Net investment + replacement investment

Interest and rental rates

- An individual who invests in capital at t earns a return $q_{t+1}-\delta$. (To maintain his capital, he must deduct depreciation).
- The cost of borrowing to invest in capital is r_t (sometimes this is dated as r_{t+1}).
- Absence of profits implies $r_t = q_{t+1}-\delta$

2. Solow’s dynamic model

- Stress on production function of neoclassical form – smooth substitution between factor inputs.
- Cambridge controversy: when is an aggregate production function useful?
- Short-cut of fixed saving rate “s” brought a great deal of tractability.

Four key equations

- Production function $$y_t = F(k_t, n_t)$$

- Marginal products $$w_t = F_n(k, n)$$ $$q_t = F_k(k, n)$$

- Capital accumulation

$$k_{t+1} - k_t = sf(k_t) - \delta k_t$$

$f(k) = F(k,1)$ when assume 1 unit of labor

Stationary point

- Value of capital such that if $k_t=\underline{k}$ then $k_{t+1}=\underline{k}$, which is the condition that $sf(k)=\delta k$
- With Cobb-Douglas

$$0=k_{t+1}-k_t=s[ak_t^{1-\alpha}n^{\alpha}]-\delta k_t$$

$$\Rightarrow \frac{k}{n}=\left(\frac{sa}{\delta}\right)^{(1/\alpha)}$$

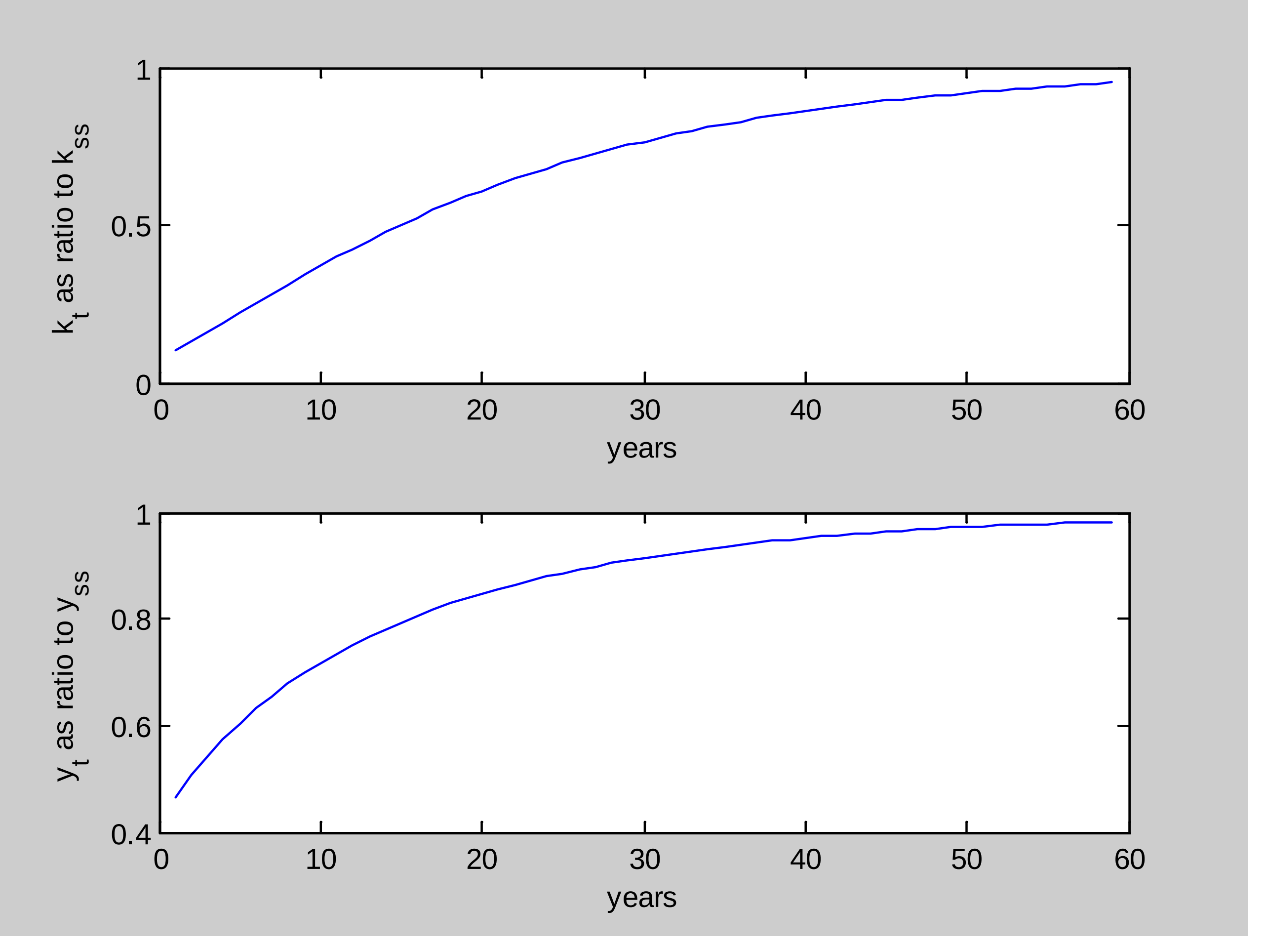
Stationary point: $sf(k)=\delta k$

Golden rule

- Max $c=f(k)-\delta k$ with respect to k.
- Optimal saving rate?

Transitional Dynamics I: A Function

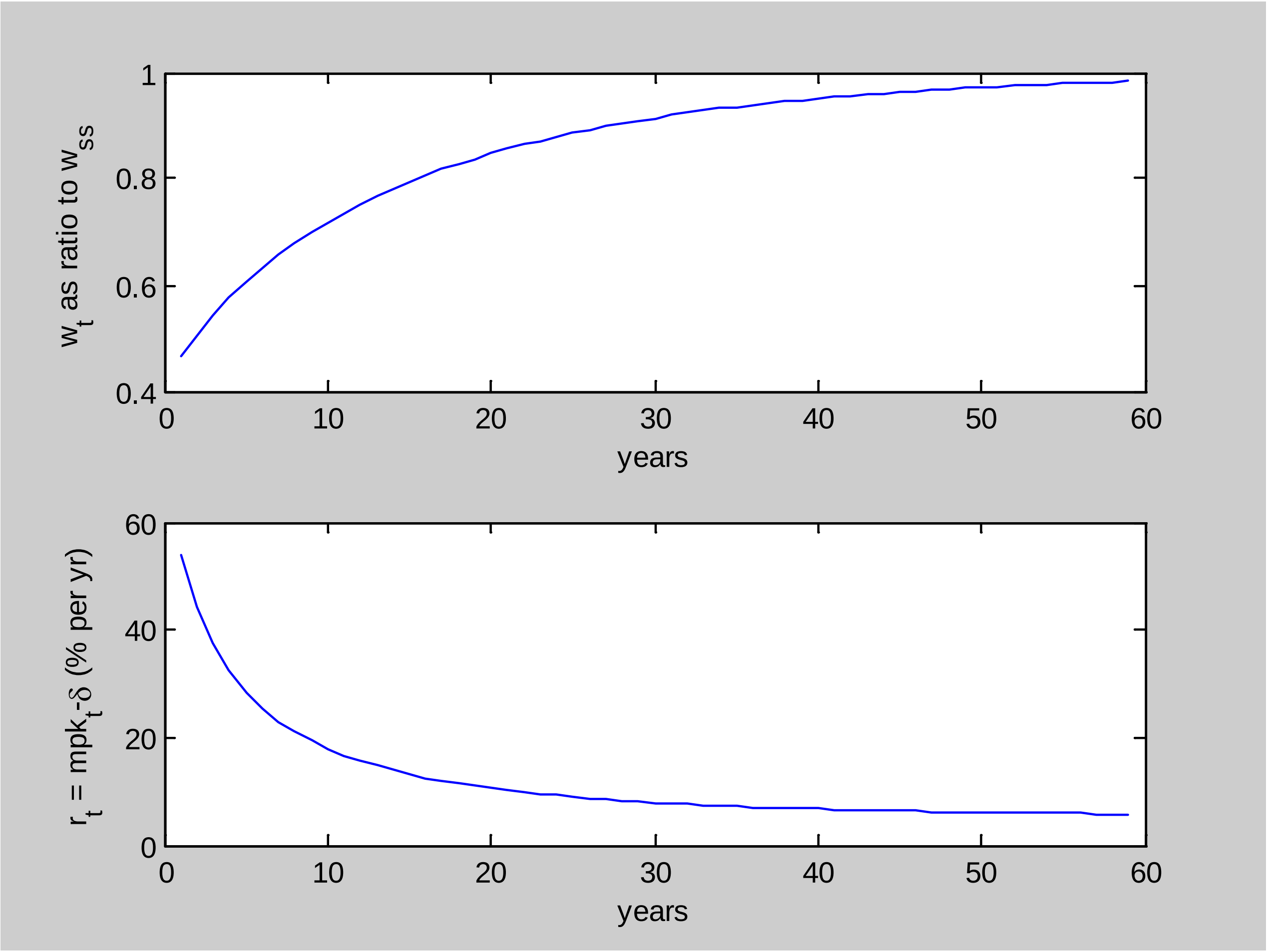
Transitional dynamics II: A Path



Implications for factor prices

- Under CRTS, marginal products depend only on ratio (k/n). Hence
  - Since k grows along transition path and since marginal product of k falls, q (and r) must fall
  - Since k grows along transition path and since marginal product of labor increases in k, w must rise

Implications for factor prices



C. Outline of RCK model discussion

1. The Setup
2. Stationary Points
3. Dynamic paths from low initial k
4. Local dynamics
5. Properties of local, global paths
6. Market interpretations

C1. The Setup

- Social planner maximizes

$$U = \lim_{T \to \infty} U_T = \lim_{T \to \infty} \sum_{t=0}^{T} \beta^t u(c_t)$$

Constraints

- Initial capital given k_0
- Resource constraint each period $\Phi(k_t) - k_{t+1} - c_t \geq 0$
- Terminal capital positive $k_{T+1} \geq 0$

$\Phi(k_t)$ is short-hand for $\mathrm{f}(\mathrm{k}_\mathrm{t}) + (1-\delta)k_t$

Finite horizon Lagrangian

- Multipliers on each constraint

$$L = \sum_{t=0}^{T} \beta^t u(c_t) + \sum_{t=0}^{T} \beta^t \lambda_t [\Phi(k_t) - k_{t+1} - c_t] + \theta [k_{T+1}]$$

- Efficiency condition are same as for infinite horizon problem below, except for last period

Terminal capital issues

- Think about from Kuhn-Tucker perspective: don't leave valuable stuff

$$\frac{\partial L}{\partial k_{T+1}} = -\beta^T \lambda_T + \theta = 0$$

$$\frac{\partial L}{\partial \theta} \geq 0 \qquad k_{T+1} \geq 0 \qquad \frac{\partial L}{\partial \theta} k_{T+1} = 0$$

Transversality condtion (TC): $\beta^T \lambda_T k_{T+1} = 0$

Infinite horizon Lagrangian

$$L = \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$+ \sum_{t=0}^{\infty} \beta^t \lambda_t [\Phi(k_t) - k_{t+1} - c_t]$$

Necessary conditions for optimum

$$c_t : \beta^t [u_c(c_t) - \lambda_t] = 0$$

$$k_{t+1} : \beta^t [-\lambda_t + \beta \Phi_k(k_{t+1}) \lambda_{t+1}] = 0$$

$$\beta^t \lambda_t : \Phi(k_t) - k_{t+1} - c_t = 0$$

$$TC : \lim_{t\to\infty} \beta^t \lambda_t k_{t+1} = 0$$

C2. Stationary points

- One feasible stationary point is to just stay with the consumption level that is "sustainable" with initial capital

$$c_t = f(k_0) - \delta k_0 = \Phi(k_0) - k_0$$

$$U = \frac{1}{1-\beta} u(c)$$

Optimal stationary point

- If the initial capital stock is such that the last condition holds with equality, then there is no contradiction. The stationary point is optimal. Thus occurs when

$$0 = [-1 + \beta\Phi_k(k^*)] \Leftrightarrow f_k(k^*) - \delta - \upsilon = 0$$

- It is optimal to keep consumption and capital constant when the "net return" to capital is equal to the rate of time preference: these levels are called the "modified golden rule" levels of c and k. They are less than the golden rule levels.

Key economic lesson

- When current consumption must be sacrificed to form capital that yields future consumption, it is not optimal to maximize stationary utility u(c).
- Generally, the solution to dynamic optimum problems does *not* involve maximizing the momentary objective u. The exception occurs if does not face real intertemporal trade-offs

Reconsidering the k_0 stationary point

- Is this optimal? It *is* feasible in the sense that it satisfies the resource constraint. And the TC is satisfied too. In terms of the other two conditions, we'd have

$$c_t \; : [u_c(c) - \lambda] = 0$$

$$k_{t+1} : \lambda[-1 + \beta\Phi_k(k_0)] > 0$$

- Constant consumption is not optimal because return to capital formation exceeds time preference. It is desirable to give up current consumption in exchange for future consumption

C. Dynamic paths from $k_0<k^*$

- Assertion: there is only one path from each initial k that (a) satisfies the FOCs and (b) has a limiting value of k^*.

- Easier to understand this feature in continuous time, as in week 2 discussion of this model, because we can draw “phase plane”.

Generating a path

- Start at any shadow price, initial capital
- This implies consumption (from FOC: c)
- Consumption and capital imply next period's capital (resource constraint)
- Current shadow price and next period's capital imply next period's shadow price.
- Continue…

Suboptimality of alternative paths

- Assume that utility and production are both strictly concave

$$u(c) < u(c^*) + u_c(c^*)[c - c^*]$$

$$f(k) < f(k^*) + f_k(k^*)[k - k^*]$$

- This implies that

$$U(\{c_t^*\}_{t=0}^{\infty}) - U(\{c_t\}_{t=0}^{\infty}) > 0$$

Why? Diagram

D. Local dynamics

- Taylor series approximation of FOCs

$$u_c(c_t) = \lambda_t \Rightarrow (u_{cc}(c_t - c^*) = (\lambda_t - \lambda^*)$$

$$k_{t+1} = \Phi(k_t) - c_t \Rightarrow (k_{t+1} - k^*) = \Phi_k(k_t - k^*) - (c_t - c^*)$$

$$\beta\lambda_{t+1}\Phi_k(k_{t+1}) = \lambda_t \Rightarrow$$

$$(\lambda_{t+1} - \lambda^*) + \beta\Phi_{kk}\lambda^*(k_{t+1} - k^*) = (\lambda_t - \lambda^*)$$

Substitute out for "c"

$$(c_t - c^*) = \frac{1}{u_{cc}}(\lambda_t - \lambda^*)$$

$$(\lambda_{t+1} - \lambda^*) + \beta\Phi_{kk}\lambda^*(k_{t+1} - k^*) = (\lambda_t - \lambda^*)$$

$$(k_{t+1} - k^*) = \Phi_k(k_t - k^*) - \frac{1}{u_{cc}}(\lambda_t - \lambda^*)$$

Local dynamics in matrix form (of capital and shadow price, after substituting out for consumption)

$$\begin{bmatrix} 1 & \beta\Phi_{kk}\lambda^* \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (\lambda_{t+1} - \lambda^*) \\ (k_{t+1} - k^*) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -\frac{1}{u_{cc}} & \Phi_k \end{bmatrix} \begin{bmatrix} (\lambda_t - \lambda^*) \\ (k_t - k^*) \end{bmatrix}$$

$$AY_{t+1} = BY_t \Rightarrow Y_{t+1} = MY_t$$

Eigenvalues

$$0 = \left\| \begin{bmatrix} 1 & \beta\Phi_{kk}\lambda^* \\ 0 & 1 \end{bmatrix} z - \begin{bmatrix} 1 & 0 \\ -\frac{1}{u_{cc}} & \Phi_k \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} (z-1) & \beta\Phi_{kk} z \\ \frac{1}{u_{cc}} & z - \Phi_k \end{bmatrix} \right\|$$

$$= (z-1)(z-\Phi_k) - \frac{u_c}{u_{cc}} \beta\Phi_{kk} z$$

Eigenvalues (cont'd)

- The eigenvalues can be "located" in terms of size and the influence of various factors explored, as follows.
- First, note that $1=\beta\Phi_k$ or that $\Phi_k=1+\nu>1$
- Second, note that the equation above can be written as the intersection of a quadratic and a line with positive slope

$$(z-1)(z-(1+\upsilon))=\frac{u_c}{u_{cc}}\beta\Phi_{kk}z$$

Eigenvalues cont'd (graph)

one root is $0<\mu_s<1$ and one root is $\mu_u>1+\nu$

General solutions for k and λ

$$k_t - k^* = q_{ku}\mu_u^t + q_{ks}\mu_s^t$$

$$\lambda_t - \lambda^* = q_{\lambda u}\mu_u^t + q_{\lambda s}\mu_s^t$$

$$\lim_{t\to\infty}[\beta^t \lambda_t k_{t+1}] = 0$$

$$\Rightarrow \lim_{t\to\infty}[\beta^t (\lambda_t - \lambda^*)(k_{t+1} - k^*)] = 0$$

$$\Rightarrow q_{\lambda u} = 0; \quad q_{ku} = 0$$

Stable dynamics

$$k_t - k^* = q_{ks}\mu_s^t = (k_0 - k^*)\mu_s^t = \mu_s(k_{t-1} - k^*)$$

$$\lambda_t - \lambda^* = q_{\lambda s}\mu_s^t = (\lambda_0 - \lambda^*)\mu_s^t$$

$$(k_{t+1} - k^*) = (1+\nu)(k_t - k^*) - \frac{1}{u_{cc}}(\lambda_t - \lambda^*)$$

$$\mu_s(k_0 - k^*) = (1+\nu)(k_0 - k^*) - \frac{1}{u_{cc}}(\lambda_0 - \lambda^*)$$

$$(\lambda_0 - \lambda^*) = u_{cc}[(1-\mu_s) + \nu](k_0 - k^*)$$

E. Properties of global, local paths

- Capital rises from low initial level to optimal stationary level, as in Solow
- Marginal product of capital falls through time (with capital stock), as in Solow. So too does the market rental and interest rate, $r_t=q_{t+1}-\delta$
- Marginal product of labor rises through time (with capital stock), as in Solow,
- Shadow price λ falls from high initial level to optimal stationary level
- Consumption rises from low initial level to optimal stationary level (inversely related to shadow price)
- With power utility, consumption growth rate declines over time (as a result of Fisher's rule)
- Saving rate (i/y) may either rise or fall (or remain constant if there is log utility, Cobb-Douglas production, and δ =1).

Using Fisher's rule

$$c_t - c^* = [(1-\mu_s) + \upsilon](k_t - k^*)$$

$$\frac{c_t - c^*}{c^*} = [(1-\mu_s) + \upsilon][\frac{k^*}{c^*}]\frac{(k_t - k^*)}{k^*}$$

$$\log(c_t / c^*) = [(1-\mu_s) + \upsilon][\frac{k^*}{c^*}]\log(k_t / k^*) = \theta \log(k_t / k^*)$$

$$r_t - \upsilon = \sigma[\log(c_{t+1} / c^*) - \log(c_t / c^*)] = \sigma(\mu_s - 1)\theta \log(k_t / k^*)$$

F. Market interpretations

- Shadow prices from planner's problem can be used to decentralize this "optimal allocation" as a competitive equilibrium for an infinitely lived representative individual
  - Of a Hicksian form with initial date markets

    Intertemporal prices are $P_t=\beta^t p_t=\beta^t\lambda_t$
  - Of a Fisherian form with sequential markets

    Interest rates are $(1+r_t)=\lambda_t/\beta\lambda_{t+1}$

Why are alternative paths suboptimal? Algebra

$$0=\sum_{t=0}^{\infty}\beta^t\lambda_t^*\{[\Phi(k_t^*)-c_t^*-k_{t+1}^*]-[\Phi(k_t)-c_t-k_{t+1}]\}$$

$$=\sum_{t=0}^{\infty}\beta^t\lambda_t^*[c_t-c_t^*]$$

$$+\sum_{t=0}^{\infty}\beta^t\lambda_t^*[\Phi(k_t^*)-\Phi(k_t)+\Phi_k(k_t^*)(k_t-k_t^*)]$$

$$+\sum_{t=0}^{\infty}\beta^t\lambda_t^*[(k_{t+1}-k_{t+1}^*)-\Phi_k(k_t^*)(k_t-k_t^*)]$$

last line is zero due to efficiency and initial feasibility

once terms are "collected" in $(k_t-k_t^*)$

Why? (algebra cont'd)

- Last line is zero because feasible paths must start from the same point k0 and optimal paths have a specific growth in shadow prices (red components equal zero)

$$\sum_{t=0}^{\infty}\beta^t \lambda_t^* [(k_{t+1} - k_{t+1}^*) - \Phi_k(k_t^*)(k_t - k_t^*)]$$

$$= \Phi_k(k_t^*)(k_0 - k_0^*)$$

$$+ \sum_{t=0}^{\infty}\beta^t [\lambda_t^* - \beta\lambda_{t+1}^* \Phi_k(k_{t+1}^*)](k_{t+1} - k_{t+1}^*) = 0$$

Why? The result

- Concavity in u and f deliver implication

$$\text{using } \lambda_t^* = u_c(c_t^*)$$

$$U(\{c_t^*\}_{t=0}^{\infty}) - U(\{c_t\}_{t=0}^{\infty})$$

$$= \{\sum_{t=0}^{\infty} \beta^t [u(c_t^*) - u(c_t) + u_c(c_t^*)(c_t - c_t^*)\}$$

$$+\{\sum_{t=0}^{\infty} \beta^t u_c(c_t^*)[\Phi(k_t^*) - \Phi(k_t) + \Phi_k(k_t^*)(k_t - k_t^*)]\} > 0$$

Additional graph

- Welfare loss in production, consumption