Equilibrium with complete markets

Reading: SL Chapter 8,

"Equilibrium with complete markets"

Key topics

1. Events and histories
2. Preferences and endowments
3. Alternative trading arrangements
   - Initial date markets (Hicks-Arrow-Debreu)
   - Sequential markets (Fisher-Arrow-Lucas-Prescott)
4. Recursive competitive equilibrium
   - Markovian uncertainty
   - Initial date and sequential markets
5. Pricing simple and complex assets
   - Using the “stochastic factor” or “pricing kernel”

1. Events and histories

- Each period ($t \geq 0$), there is a stochastic event s_t, realized from a set of potential events S.
- The history of the economy up to and including date t is defined as $s^t = [s_0, s_1, \ldots, s_t]$.
- Potential confusion over words
  - Bellman: state variable
  - Arrow & Debreu: state of nature
- We'll sharpen distinction below and see its importance.
- LS notation for probabilities

$$\text{unconditional probability: } \pi_t(s^t)$$

$$\text{conditional probability: } \pi_t(s^t \mid s^\tau)$$

2. Preferences and endowments

- Agents i=1,2,…I
- Common utility function u(c) but could depend on name (i) for many results
- Could depart from additive separability for some results
- Initial event assumed known

$$EU^i \mid s_0 = \sum_{t=0}^{\infty} \sum_{s^t} \pi_t(s^t)\beta^t u(c_t^i(s^t))$$

Endowments

- Dynamic endowment economy, as in Fisher and Hicks. But jazzed up to include uncertainty as in Arrow and Debreu.
- Each agent gets an endowment at date t that may depend on the entire history
- Notation for endowments

$$y_t^i(s^t)$$

Feasible allocations

- Total consumption in the population must not exceed total output in any particular history at a particular date.

$$\sum_i c_t^i(s^t) \leq \sum_i y_t^i(s^t)$$

Pareto optimal allocations

- Lagrangian
- Objective + constraints * multipliers

$$W = \sum_i \lambda_i EU^i \mid s_0 = \sum_i \lambda_i \{\sum_{t=0}^{\infty} \sum_{s^t} \pi_t(s^t)\beta^t u(c_t^i(s^t))\}$$

$$L = W + \sum_{t=0}^{\infty} \sum_{s_t} \theta_t(s^t)[\sum_i y_t^i(s^t) - \sum_i c_t^i(s^t)]$$

FOC: $\lambda_i \pi_t(s^t)\beta^t u'(c_t^i(s^t)) = \theta_t(s^t)$

Efficient risk sharing

- Borch-Arrow condition: state of nature by state of nature, the ratio of marginal utilities should be equated across agents

$$1 = \frac{\theta_t(s^t)}{\theta_t(s^t)} = \frac{\lambda_i \pi_t(s^t)\beta^t u'(c_t^i(s^t))}{\lambda_j \pi_t(s^t)\beta^t u'(c_t^j(s^t))}$$

$$= \frac{u'(c_t^i(s^t))}{u'(c_t^j(s^t))}$$

Efficiency in exchange economy

- SL Proposition 1
- Efficient allocations depend
  - only on aggregate not individual endowments
  - only on event $y(s^t)$ not on history
- Why?
  - Separation across time on production side (endowments)
  - Separation across time on utility side (additive separability)

How derive?

- Alternative approach to SL, which is sometimes convenient,

$$c_t^i(s^t) = d\left(\frac{\theta_t(s^t)}{\lambda_i \pi_t(s^t)\beta^t}\right); \quad \text{u'(d(x))=x}$$

$$\theta_t(s^t) : \sum_i y_t^i(s^t) = \sum_i c_t(s^t) = \sum_i d\left(\frac{\theta_t(s^t)}{\lambda_i \pi_t(s^t)\beta^t}\right)$$

$$\pi_t(s^t)\beta^t \overline{\theta}_t(s^t) : y(s^t) = \sum_i d\left(\frac{\overline{\theta}_t(s^t)}{\lambda_i}\right)$$

3. Alternative trading arrangements

- Initial date market trading (date 0): Arrow-Debreu securities. One security for each possible future history

- One period market trading at each date t: Arrow securities. One security for each possible future history next period, conditional on current history. That is, one security for each event s_{t+1}.

Initial date markets BC

- Takes "contingent present discounted value form"

$$\sum_{t=0}^{\infty} \sum_{s_t} q_t^0(s^t)[y_t^i(s^t) - c_t^i(s^t)] \geq 0$$

- Leads to natural FOC (with μ_i the multiplier on individual I's constraint)

$$\pi_t(s^t)\beta^t u'(c_t^i(s^t)) = \mu_i q_t^0(s^t)$$

Competitive allocations

- Any CE is Pareto optimal
- Any Pareto optimum can be supported as a CE with suitable (initial date, stochastic) transfers
- Share PE implications:
  - "risk pooling"/"aggregate endowment"
  - Insensitivity to history

Sequential markets

- Debt limits
- Flow budget constraint

$$c_t^i(s^t) + \sum_{s_{t+1}} a_{t+1}^i(s_{t+1}, s^t) Q_t\,(s_{t+1} \mid s^t) = y_t^i(s^t) + a_t^i(s^t)$$

- Iterative construction of 2-step-ahead price

$$Note: s^{t+1} = (s^t, s_{t+1}) \text{ and } s^{t+2} = (s^t, s_{t+1}, s_{t+2})$$

$$Q_t\,(s_{t+1} \mid s^t) * Q_{t+1}(s_{t+2} \mid s^{t+1} = (s^t, s_{t+1}))$$

Equivalence

- Initial date markets provide a single budget constraint for individuals, with prices attached to every history
- With non-binding natural debt limit, sequential markets do the same (agents need rational expectations about future security prices, as part of the hypothesis).
- Quantities are same when BC is same
- Links between price measures can be established (from IM to SM and from SM to IM)

4. Recursive equilibrium

- Markovian probability restriction

$$s^t = [s^\tau, s_{\tau+1}, \ldots s_t]$$

$$\pi(s^t \mid s^\tau) = \pi(s_t \mid s_{t-1})\pi(s_{t-1} \mid s_{t-2})\ldots\pi(s_{\tau+1} \mid s_\tau)$$

- Endowment restriction

$$y_t^i(s^t) = y^i(s_t)$$

Implications

- Quantities

$$c_t^i(s^t) = c^i(s_t)$$

- Prices (sequential markets)

$$Q_t\,(s_{t+1} \mid s^t) = Q(s_{t+1} \mid s_t)$$

Recursive formulation

- Bellman equation

$$v^i(a,s) = \max_{c,a(s')}\{u(c) + \beta \sum_{s'} \pi(s'|s) v^i(a(s'),s')\}$$

$$c + \sum_{s'} Q(s'|s) a(s') \leq y(s') + a$$

$$c > 0, \qquad -a(s') \leq A^i(s') \ \text{ for all s'}$$

$$\Rightarrow \text{c=h}^i(a,s) \quad \text{and} \quad \text{a(s')=g}^i(a,s,s')$$

- Discuss “state” evolution and state contingent securities

Multiperiod pricing

- Can determine a price relevant for state j period ahead, via a natural recursion

$$Q_j(s_{t+j} \mid s_t) = \sum_{s_{t+1}} Q_1(s_{t+1} \mid s_t) Q_{j-11}(s_{t+j} \mid s_{t+1})$$

- Note role about beliefs about future prices

- Explicit multiperiod simple securities (those contingent on a single state) have to satisfy zero arbitrage profit links

Pricing complex claims

- Many securities have payouts in more than one state: it is conventional to call these complex claims
- If there were simple securities, then these complex securities must be priced from their component parts
- Recursive forward solution

$$p(s_t) = \sum_{s_{t+1}} Q_1(s_{t+1} \mid s_t)[p(s_{t+1}) + d(s_{t+1})]$$

$$= \sum_{j=0}^{\infty} \sum_{s_{t+j}} Q_j(s_{t+j} \mid s_t) d(s_{t+j})$$