Linear Dynamic Models and Forecasting

Reference article:
"Interactions between the multiplier analysis and the principle of acceleration"

Outline

1. The state space system as an approach to working with systems of difference equations
2. Analysis of the system, focusing on the Samuelson model

1. Placing the Samuelson model in state space form

- Consumption function

$$c_t = \alpha y_{t-1}$$

- Investment function

$$i_t = \beta(c_t - c_{t-1})$$

- Implication

$$y_t = \alpha(1+\beta) y_{t-1} - \alpha\beta y_{t-2} + g_t$$

An approach to analyzing systems of difference equations

- We are going to use a standard framework for studying dynamic systems. It is comprised of two groups of equations
- Equations describing how "observables" relate to "states". Each of these takes the form of a linear relationship, so that we have a matrix system of the form (if we abstract from constant and trend terms)

$$Y_{jt} = \pi_{j1} S_{1t} + \pi_{j2} S_{2t} + ... + \pi_{jn} S_{nt}$$

$$Y_t = \begin{bmatrix} Y_{1t} \\ Y_{2t} \\ \dots \\ Y_{mt} \end{bmatrix} = \begin{bmatrix} \pi_{11} & \pi_{12} & & \pi_{1n} \\ \pi_{21} & \pi_{22} & & \pi_{2n} \\ & & & \\ \pi_{m1} & \pi_{m1} & & \pi_{mn} \end{bmatrix} \begin{bmatrix} S_{1t} \\ S_{2t} \\ \dots \\ S_{nt} \end{bmatrix}$$

And the states evolve as

$$S_{jt} = M_{j1} S_{1,t-1} + \dots M_{jm} S_{m,t-1} + G_{j1} e_{1t} + \dots G_{jn} e_{nt}$$

$$S_t = \begin{bmatrix} S_{1t} \\ S_{2t} \\ \dots \\ S_{mt} \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} & & M_{1m} \\ M_{21} & M_{22} & & M_{2m} \\ & & & \\ M_{m1} & M_{m1} & & M_{mm} \end{bmatrix} \begin{bmatrix} S_{1,t-1} \\ S_{2,t-1} \\ \dots \\ S_{m,t-1} \end{bmatrix}$$

$$+ Ge_t \ [more\ on\ this\ \ part\ \ later]$$

General framework

- Can be written in simple manner

$$Y_t = \Pi S_t \qquad S_t = M S_{t-1} + G e_t$$

- Is easy to study and understand
- Can be used for many of different problems
- Highlights important considerations

Solving this system recursively

- Recursive substitution from initial S, given path of e's

$$S_1 = M S_0 + G e_1$$
$$S_2 = M S_1 + G e_2 = M[M S_0 + G e_1] + G e_2$$
$$= M^2 S_0 + M G e_1 + G e_2$$

continuing

$$S_t = M^t S_0 + M^{t-1} Ge_1$$
$$+ M^{t-2} Ge_2 + ...$$
$$+ MGe_{t-1} + Ge_t$$

$$[check : S_t = M^2 S_0 + MGe_1 + Ge_2]$$

So, where we are at t depends on

- The initial state S_0
- The series of e's that take place along the path between 1 and t
- How important these e's are for S (G)
- Various powers of M

- [very similar to first order difference equation, except in vectors…]

Samuelson’s model (without g): state evolution

$$y_t = \alpha(1+\beta) y_{t-1} - \alpha\beta y_{t-2}$$

$$S_t = \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha(1+\beta) & -\alpha\beta \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \end{bmatrix} = MS_{t-1}$$

Samuelson’s model (without g): other variables

$$Y_t = \begin{bmatrix} y_t \\ c_t \\ i_t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \alpha \\ 1 & -\alpha \end{bmatrix} \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix} = \Pi S_t$$

$$i_t = y_t - c_t = y_t - \alpha y_{t-1}$$

Samuelson's model (with g)

Assume $\mathrm{g}_{\mathrm{t}} = g_{t-1} + e_t$

Implies $S_t = \begin{bmatrix} y_t \\ y_{t-1} \\ g_t \end{bmatrix}$

Samuelson's model (with g): states

$$S_t = \begin{bmatrix} y_t \\ y_{t-1} \\ g_t \end{bmatrix}$$

$$= \begin{bmatrix} \alpha(1+\beta) & -\alpha\beta & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \\ g_{t-1} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e_t$$

$$= MS_{t-1} + Ge_t$$

Samuelson's model (with g): other variables

$$Y_t = \begin{bmatrix} g_t \\ y_t \\ c_t \\ i_t \end{bmatrix} \qquad S_t = \begin{bmatrix} y_t \\ y_{t-1} \\ g_t \end{bmatrix}$$

$$\begin{bmatrix} g_t \\ y_t \\ c_t \\ i_t \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 1 & -\alpha & -1 \end{bmatrix} \begin{bmatrix} y_t \\ y_{t-1} \\ g_t \end{bmatrix}$$

2. Analyzing the system

- Qualitative dynamics of system (as in Samuelson's regions) are governed by M.
- Characteristic roots of M are the solutions to the polynomial |Iz-M|=0, where I is an identity matrix with same dimension as M and |X| means "determinant of X".

Samuelson's model (without g)

$$S_t = \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha(1+\beta) & -\alpha\beta \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \end{bmatrix} = MS_{t-1}$$

Finding roots in Samuelson's model

- Calculating |Iz-M|, we find the same result as in Samuelson's footnote #1: roots are solution to quadratic equation

$$0 = \left| \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} z - \begin{bmatrix} (1+\alpha)\beta & -\alpha\beta \\ 1 & 0 \end{bmatrix} \right|$$

$$= \left| \begin{bmatrix} z-(1+\alpha)\beta & \alpha\beta \\ -1 & z \end{bmatrix} \right|$$

$$= z^2 - \alpha(1+\beta)z + \alpha\beta$$

This is a quadratic equation, so

- There are two roots
- There can either be two real roots or two complex roots
- The roots can be of variety of “sizes” in terms of absolute value:
  - Less than one (stable);
  - Equal to one (unit, borderline);
  - Greater than one (unstable)
- Higher order polynomials also have roots with last two properties (real or complex, small or large)

Complex roots

- Will not be a special focus of our discussion, although they were a key part of Samuelson’s analysis (one definition of “business cycles”)
- Lead to oscillatory nature of dynamic responses (as in Samuelson’s table 1)
- Can either be stable (damped) or unstable (explosive)

3. Stochastic Systems such as vector autoregressions

- Suppose that we treat the e_t in the state system above as a series of unpredictable random variables, so that our expectation (forecast) of e_{t+j} given information at t is zero. We write this as $E_t e_{t+j} = 0$
- Then, it is easy to produce a forecast (compute a conditional expectation)

And this forecast takes a simple form

- In words, the forecast depends on where we start (initial S) and the dynamic forces captured by M.

$$E_t S_{t+j} = M^j S_t$$
$$+E_t\{M^{j-1}Ge_{t+1} + ...$$
$$+ MGe_{t+j-1} + Ge_{t+j}\}$$
$$= M^j S_t$$

Forecasting other variables

- We know that Y is related to S by a simple expression, so we get forecasts very easily

$$Y_{t+j} = \Pi S_{t+j}$$

$$\Rightarrow E_t Y_{t+j} = \Pi E_t S_{t+j} = \Pi M^j S_t$$

Structure above abstracts from constant terms (normal values) but these are easy to add, using either of two mathematically equivalent approaches

APPROACH 1: "previous system is deviations from normal values"

$$Y_t - \overline{Y} = \Pi(S_t - \overline{S}) \qquad S_t - \overline{S} = M(S_{t-1} - \overline{S}) + Ge_t$$

APPROACH 2: "add constant terms to every equation, then work out normal values"

$$Y_t = A + \Pi S_t \qquad S_t = B + MS_{t-1} + Ge_t$$

$$\overline{S} = (I - M)^{-1} B; \quad \overline{Y} = A + \Pi \overline{S}$$

Putting the system to work

- Empirical models (like VARs)
- Linear rational expectations models
  - Related theoretical device
  - Solution of LRE model occurs in state space form

Appendix

1. Analytical framework for dynamic linear system
2. Eigenvector-Eigenvalue decompOsition of M
3. Link to nature of solution to second order difference equation (eg Samuelson model)

1. Analytical question

- Recursive solution takes form

$$
\begin{aligned}
S_t = {} & M^t S_0 \\
& + M^{t-1} G e_1 \\
& + M^{t-2} G e_2 + \dots \\
& + M G e_{t-1} + G e_t
\end{aligned}
$$

- What is behavior of M raised to the j^{th} power?

Key properties from linear algebra

- Any matrix M which has distinct (not repeated) roots can be decomposed into $M=P\mu P^{-1}$, where P is a matrix of eigenvectors (characteristic vectors) and μ is a diagonal matrix *with the eigenvalues on the diagonal* and P^{-1} is the inverse of P so that $PP^{-1}=I$.
- Further, $M^2 =(P\mu P^{-1})(P\mu P^{-1}) = (P\mu^2 P^{-1})$ and more generally $M^j = (P\mu^j P^{-1})$.
- Since μ is diagonal, μ^j is a diagonal matrix with the eigenvalues raised to the jth power on the diagonal.

Samuelson model w/o g

$$M = \begin{bmatrix} \alpha(1+\beta) & -\alpha\beta \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} m_1 & m_2 \\ 1 & 0 \end{bmatrix}$$

Eigenvectors and Eigenvalues

$$P = \begin{bmatrix} \mu_1 & \mu_2 \\ 1 & 1 \end{bmatrix} \qquad \mathrm{P}^{-1} = \frac{1}{\mu_1 - \mu_2} \begin{bmatrix} 1 & -\mu_2 \\ -1 & \mu_1 \end{bmatrix}$$

$$\mu = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}$$

$$\mu_1 = \frac{1}{2}[m_1 + \sqrt{m_1^2 + 4m_2}] \qquad \mu_1 = \frac{1}{2}[m_1 - \sqrt{m_1^2 + 4m_2}]$$

Powers of M

$$M^j = (P\mu^j P^{-1})$$
$$= \frac{1}{\mu_1 - \mu_2}\begin{bmatrix} \mu_1^{j+1} - \mu_2^{j+1} & -\mu_2\mu_1^{j+1} + \mu_1\mu_2^{j+1} \\ \mu_1^j - \mu_2^j & -\mu_2\mu_1^j + \mu_1\mu_2^j \end{bmatrix}$$

$$y_{t+j} = [1\ \ 0]M^j S_t$$
$$= \frac{\mu_1^{j+1} - \mu_2^{j+1}}{\mu_1 - \mu_2} y_t + \frac{-\mu_2\mu_1^{j+1} + \mu_1\mu_2^{j+1}}{\mu_1 - \mu_2} y_{t-1}$$

Solution

- Hence, the solution of the difference equation is of the form described in the main lecture,

$$y_t = \theta_1\mu_1^t + \theta_2\mu_2^t$$

$$\theta_1 = (\mu_1 y_0 - \mu_1\mu_2 y_{-1})/(\mu_1 - \mu_2)$$
$$\theta_1 = (-\mu_2 y_0 + \mu_1\mu_2 y_{-1})/(\mu_1 - \mu_2)$$