Chapter 2 INFINITE-HORIZON AND OVERLAPPING-GENERATIONS MODELS

This chapter investigates two models that resemble the Solow model but in which the dynamics of economic aggregates are determined by decisions at the microeconomic level. Both models continue to take the growth rates of labor and knowledge as given. But the models derive the evolution of the capital stock from the interaction of maximizing households and firms in competitive markets. As a result, the saving rate is no longer exogenous, and it need not be constant.

The first model is conceptually the simplest. Competitive firms rent capital and hire labor to produce and sell output, and a fixed number of infinitely lived households supply labor, hold capital, consume, and save. This model, which was developed by Ramsey (1928), Cass (1965), and Koopmans (1965), avoids all market imperfections and all issues raised by heterogeneous households and links among generations. It therefore provides a natural benchmark case.

The second model is the overlapping-generations model developed by Diamond (1965). The key difference between the Diamond model and the Ramsey–Cass–Koopmans model is that the Diamond model assumes that there is continual entry of new households into the economy. As we will see, this seemingly small difference has important consequences.

Part A The Ramsey–Cass–Koopmans Model

2.1 Assumptions

Firms

There are a large number of identical firms. Each has access to the production function $Y = F(K, AL)$, which satisfies the same assumptions as

in Chapter 1. The firms hire workers and rent capital in competitive factor markets, and sell their output in a competitive output market. Firms take A as given; as in the Solow model, A grows exogenously at rate g. The firms maximize profits. They are owned by the households, so any profits they earn accrue to the households.

Households

There are also a large number of identical households. The size of each household grows at rate n. Each member of the household supplies 1 unit of labor at every point in time. In addition, the household rents whatever capital it owns to firms. It has initial capital holdings of $K(0)/H$, where $K(0)$ is the initial amount of capital in the economy and H is the number of households. For simplicity, there is no depreciation. The household divides its income (from the labor and capital it supplies and, potentially, from the profits it receives from firms) at each point in time between consumption and saving so as to maximize its lifetime utility.

The household's utility function takes the form

$$U = \int_{t=0}^{\infty} e^{-\rho t} u(C(t)) \frac{L(t)}{H}\, dt. \tag{2.1}$$

$C(t)$ is the consumption of each member of the household at time t. $u(\bullet)$ is the *instantaneous utility function*, which gives each member's utility at a given date. $L(t)$ is the total population of the economy; $L(t)/H$ is therefore the number of members of the household. Thus $u(C(t))L(t)/H$ is the household's total instantaneous utility at t. Finally, ρ is the discount rate; the greater is ρ, the less the household values future consumption relative to current consumption.[1]

The instantaneous utility function takes the form

$$u(C(t)) = \frac{C(t)^{1-\theta}}{1-\theta}, \qquad \theta > 0, \qquad \rho - n - (1-\theta)g > 0. \tag{2.2}$$

This functional form is needed for the economy to converge to a balanced growth path. It is known as *constant-relative-risk-aversion* (or *CRRA*) utility. The reason for the name is that the coefficient of relative risk aversion (which is defined as $-Cu''(C)/u'(C)$) for this utility function is θ, and thus is independent of C.

Since there is no uncertainty in this model, the household's attitude toward risk is not directly relevant. But θ also determines the household's

[1] One can also write utility as $\int_{t=0}^{\infty} e^{-\rho' t} u(C(t))\, dt$, where $\rho' \equiv \rho - n$. Since $L(t) = L(0)e^{nt}$, this expression equals the expression in equation (2.1) divided by $L(0)/H$, and thus has the same implications for behavior.

willingness to shift consumption between different periods. When θ is smaller, marginal utility falls more slowly as consumption rises, and so the household is more willing to allow its consumption to vary over time. If θ is close to zero, for example, utility is almost linear in C, and so the household is willing to accept large swings in consumption to take advantage of small differences between the discount rate and the rate of return on saving. Specifically, one can show that the elasticity of substitution between consumption at any two points in time is $1/\theta$.[2]

Three additional features of the instantaneous utility function are worth mentioning. First, $C^{1-\theta}$ is increasing in C if $\theta < 1$ but decreasing if $\theta > 1$; dividing $C^{1-\theta}$ by $1-\theta$ thus ensures that the marginal utility of consumption is positive regardless of the value of θ. Second, in the special case of $\theta \to 1$, the instantaneous utility function simplifies to $\ln C$; this is often a useful case to consider.[3] And third, the assumption that $\rho - n - (1-\theta)g > 0$ ensures that lifetime utility does not diverge: if this condition does not hold, the household can attain infinite lifetime utility, and its maximization problem does not have a well-defined solution.[4]

2.2 The Behavior of Households and Firms

Firms

Firms' behavior is relatively simple. At each point in time they employ the stocks of labor and capital, pay them their marginal products, and sell the resulting output. Because the production function has constant returns and the economy is competitive, firms earn zero profits.

As described in Chapter 1, the marginal product of capital, $\partial F(K,AL)/\partial K$, is $f'(k)$, where $f(\bullet)$ is the intensive form of the production function. Because markets are competitive, capital earns its marginal product. And because there is no depreciation, the real rate of return on capital equals its earnings per unit time. Thus the real interest rate at time t is

$$r(t) = f'(k(t)). \tag{2.3}$$

Labor's marginal product is $\partial F(K,AL)/\partial L$, which equals $A\partial F(K,AL)/\partial AL$. In terms of $f(\bullet)$, this is $A[f(k) - kf'(k)]$.[5] Thus the real wage at t is

$$W(t) = A(t)[f(k(t)) - k(t)f'(k(t))]. \tag{2.4}$$

The wage per unit of *effective* labor is therefore

$$w(t) = f(k(t)) - k(t)f'(k(t)). \tag{2.5}$$

Households' Budget Constraint

The representative household takes the paths of r and w as given. Its budget constraint is that the present value of its lifetime consumption cannot exceed its initial wealth plus the present value of its lifetime labor income. To write the budget constraint formally, we need to account for the fact that r may vary over time. To do this, define $R(t)$ as $\int_{\tau=0}^{t} r(\tau)\,d\tau$. One unit of the output good invested at time 0 yields $e^{R(t)}$ units of the good at t; equivalently, the value of 1 unit of output at time t in terms of output at time 0 is $e^{-R(t)}$. For example, if r is constant at some level $\overline{r}$, $R(t)$ is simply $\overline{r}t$ and the present value of 1 unit of output at t is $e^{-\overline{r}t}$. More generally, $e^{R(t)}$ shows the effects of continuously compounding interest over the period $[0,t]$.

Since the household has $L(t)/H$ members, its labor income at t is $W(t)L(t)/H$, and its consumption expenditures are $C(t)L(t)/H$. Its initial wealth is $1/H$ of total wealth at time 0, or $K(0)/H$. The household's budget constraint is therefore

$$\int_{t=0}^{\infty} e^{-R(t)}C(t)\frac{L(t)}{H}\,dt \le \frac{K(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)}W(t)\frac{L(t)}{H}\,dt. \tag{2.6}$$

In many cases, it is difficult to find the integrals in this expression. Fortunately, we can express the budget constraint in terms of the limiting behavior of the household's capital holdings; and even when it is not possible to compute the integrals in (2.6), it is often possible to describe the limiting behavior of the economy. To see how the budget constraint can be rewritten in this way, first bring all the terms of (2.6) over to the same side and combine the two integrals; this gives us

$$\frac{K(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)}[W(t) - C(t)]\frac{L(t)}{H}\,dt \ge 0. \tag{2.7}$$

We can write the integral from $t = 0$ to $t = \infty$ as a limit. Thus (2.7) is equivalent to

$$\lim_{s\to\infty}\left[\frac{K(0)}{H} + \int_{t=0}^{s} e^{-R(t)}[W(t) - C(t)]\frac{L(t)}{H}\,dt\right] \ge 0. \tag{2.8}$$

[2] See Problem 2.2.

[3] To see this, first subtract $1/(1-\theta)$ from the utility function; since this changes utility by a constant, it does not affect behavior. Then take the limit as θ approaches 1; this requires using l'Hôpital's rule. The result is $\ln C$.

[4] Phelps (1966a) discusses how growth models can be analyzed when households can obtain infinite utility.

[5] See Problem 1.9.

Now note that the household's capital holdings at time s are

$$\frac{K(s)}{H} = e^{R(s)}\frac{K(0)}{H} + \int_{t=0}^{s} e^{R(s)-R(t)}[W(t) - C(t)]\frac{L(t)}{H}\,dt. \tag{2.9}$$

To understand (2.9), note that $e^{R(s)}K(0)/H$ is the contribution of the household's initial wealth to its wealth at s. The household's saving at t is $[W(t) - C(t)]L(t)/H$ (which may be negative); $e^{R(s)-R(t)}$ shows how the value of that saving changes from t to s.

The expression in (2.9) is $e^{R(s)}$ times the expression in brackets in (2.8). Thus we can write the budget constraint as simply

$$\lim_{s\to\infty} e^{-R(s)}\frac{K(s)}{H} \geq 0. \tag{2.10}$$

Expressed in this form, the budget constraint states that the present value of the household's asset holdings cannot be negative in the limit.

Equation (2.10) is known as the *no-Ponzi-game condition.* A Ponzi game is a scheme in which someone issues debt and rolls it over forever. That is, the issuer always obtains the funds to pay off debt when it comes due by issuing new debt. Such a scheme allows the issuer to have a present value of lifetime consumption that exceeds the present value of his or her lifetime resources. By imposing the budget constraint (2.6) or (2.10), we are ruling out such schemes.[6]

Households' Maximization Problem

The representative household wants to maximize its lifetime utility subject to its budget constraint. As in the Solow model, it is easier to work with variables normalized by the quantity of effective labor. To do this, we need to express both the objective function and the budget constraint in terms of consumption and labor income per unit of effective labor.

[6] This analysis sweeps a subtlety under the rug: we have assumed rather than shown that households must satisfy the no-Ponzi-game condition. Because there are a finite number of households in the model, the assumption that Ponzi games are not feasible is correct. A household can run a Ponzi game only if at least one other household has a present value of lifetime consumption that is strictly less than the present value of its lifetime wealth. Since the marginal utility of consumption is always positive, no household will accept this. But in models with infinitely many households, such as the overlapping-generations model of Part B of this chapter, Ponzi games are possible in some situations. We return to this point in Section 11.1.

We start with the objective function. Define $c(t)$ to be consumption per unit of effective labor. Thus $C(t)$, consumption per worker, equals $A(t)c(t)$. The household's instantaneous utility, (2.2), is therefore

$$\begin{aligned}\frac{C(t)^{1-\theta}}{1-\theta} &= \frac{[A(t)c(t)]^{1-\theta}}{1-\theta}\\ &= \frac{[A(0)e^{gt}]^{1-\theta}c(t)^{1-\theta}}{1-\theta}\\ &= A(0)^{1-\theta}e^{(1-\theta)gt}\frac{c(t)^{1-\theta}}{1-\theta}.\end{aligned} \tag{2.11}$$

Substituting (2.11) and the fact that $L(t) = L(0)e^{nt}$ into the household's objective function, (2.1)–(2.2), yields

$$\begin{aligned}U &= \int_{t=0}^{\infty} e^{-\rho t}\frac{C(t)^{1-\theta}}{1-\theta}\frac{L(t)}{H}\,dt\\ &= \int_{t=0}^{\infty} e^{-\rho t}\left[A(0)^{1-\theta}e^{(1-\theta)gt}\frac{c(t)^{1-\theta}}{1-\theta}\right]\frac{L(0)e^{nt}}{H}\,dt\\ &= A(0)^{1-\theta}\frac{L(0)}{H}\int_{t=0}^{\infty} e^{-\rho t}e^{(1-\theta)gt}e^{nt}\frac{c(t)^{1-\theta}}{1-\theta}\,dt\\ &\equiv B\int_{t=0}^{\infty} e^{-\beta t}\frac{c(t)^{1-\theta}}{1-\theta}\,dt,\end{aligned} \tag{2.12}$$

where $B \equiv A(0)^{1-\theta}L(0)/H$ and $\beta \equiv \rho - n - (1-\theta)g$. From (2.2), β is assumed to be positive.

Now consider the budget constraint, (2.6). The household's total consumption at t, $C(t)L(t)/H$, equals consumption per unit of effective labor, $c(t)$, times the household's quantity of effective labor, $A(t)L(t)/H$. Similarly, its total labor income at t equals the wage per unit of effective labor, $w(t)$, times $A(t)L(t)/H$. And its initial capital holdings are capital per unit of effective labor at time 0, $k(0)$, times $A(0)L(0)/H$. Thus we can rewrite (2.6) as

$$\int_{t=0}^{\infty} e^{-R(t)}c(t)\frac{A(t)L(t)}{H}\,dt \leq k(0)\frac{A(0)L(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)}w(t)\frac{A(t)L(t)}{H}\,dt. \tag{2.13}$$

$A(t)L(t)$ equals $A(0)L(0)e^{(n+g)t}$. Substituting this fact into (2.13) and dividing both sides by $A(0)L(0)/H$ yields

$$\int_{t=0}^{\infty} e^{-R(t)}c(t)e^{(n+g)t}\,dt \leq k(0) + \int_{t=0}^{\infty} e^{-R(t)}w(t)e^{(n+g)t}\,dt. \tag{2.14}$$

Finally, because $K(s)$ is proportional to $k(s)e^{(n+g)s}$, we can rewrite the no-Ponzi-game version of the budget constraint, (2.10), as

$$\lim_{s\to\infty} e^{-R(s)}e^{(n+g)s}k(s) \geq 0. \tag{2.15}$$

Household Behavior

The household's problem is to choose the path of $c(t)$ to maximize lifetime utility, (2.12), subject to the budget constraint, (2.14). Although this involves choosing c at each instant of time (rather than choosing a finite set of variables, as in standard maximization problems), conventional maximization techniques can be used. Since the marginal utility ofconsumption is always positive, the household satisfies its budget constraint with equality. We can therefore use the objective function, (2.12), and the budget constraint, (2.14), to set up the Lagrangian:

$$\begin{aligned} \mathcal{L} = {} & B\int_{t=0}^{\infty} e^{-\beta t}\frac{c(t)^{1-\theta}}{1-\theta}\,dt \\ & + \lambda\left[k(0) + \int_{t=0}^{\infty} e^{-R(t)}e^{(n+g)t}w(t)\,dt - \int_{t=0}^{\infty} e^{-R(t)}e^{(n+g)t}c(t)\,dt\right]. \end{aligned} \tag{2.16}$$

The household chooses c at each point in time; that is, it chooses infinitely many $c(t)$'s. The first-order condition for an individual $c(t)$ is[7]

$$Be^{-\beta t}c(t)^{-\theta} = \lambda e^{-R(t)}e^{(n+g)t}. \tag{2.17}$$

The household's behavior is characterized by (2.17) and the budget constraint, (2.14).

[7] This step is slightly informal; the difficulty is that the terms in (2.17) are of order dt in (2.16); that is, they make an infinitesimal contribution to the Lagrangian. There are various ways of addressing this issue more formally than simply "canceling" the dt's (which is what we do in [2.17]). For example, we can model the household as choosing consumption over the finite intervals $[0,\Delta t)$, $[\Delta t, 2\Delta t)$, $[2\Delta t, 3\Delta t)$, ..., with its consumption required to be constant within each interval, and then take the limit as Δt approaches zero. This also yields (2.17). Another possibility is to use the *calculus of variations* (see n. 13, at the end of Section 2.4). In this particular application, however, the calculus-of-variations approach simplifies to the approach we have used here. That is, here the calculus of variations merely provides a formal justification for canceling the dt's in (2.17).

To see what (2.17) implies for the behavior of consumption, first take logs of both sides:

$$\begin{aligned} \ln B - \beta t - \theta \ln c(t) &= \ln\lambda - R(t) + (n+g)t \\ &= \ln\lambda - \int_{\tau=0}^{t} r(\tau)\,d\tau + (n+g)t, \end{aligned} \tag{2.18}$$

where the second line uses the definition of $R(t)$ as $\int_{\tau=0}^{t} r(\tau)\,d\tau$. Now note that since the two sides of (2.18) are equal for every t, the derivatives of the two sides with respect to t must be the same. This condition is

$$-\beta - \theta\frac{\dot{c}(t)}{c(t)} = -r(t) + (n+g), \tag{2.19}$$

where we have once again used the fact that the time derivative of the log of a variable equals its growth rate. Solving (2.19) for $\dot{c}(t)/c(t)$ yields

$$\begin{aligned} \frac{\dot{c}(t)}{c(t)} &= \frac{r(t) - n - g - \beta}{\theta} \\ &= \frac{r(t) - \rho - \theta g}{\theta}, \end{aligned} \tag{2.20}$$

where the second line uses the definition of β as $\rho - n - (1-\theta)g$.

To interpret (2.20), note that since $C(t)$ (consumption per worker, rather than consumption per unit of effective labor) equals $c(t)A(t)$, the growth rate of C is given by

$$\begin{aligned} \frac{\dot{C}(t)}{C(t)} &= \frac{\dot{A}(t)}{A(t)} + \frac{\dot{c}(t)}{c(t)} \\ &= g + \frac{r(t) - \rho - \theta g}{\theta} \\ &= \frac{r(t) - \rho}{\theta}, \end{aligned} \tag{2.21}$$

where the second line uses (2.20). This condition states that consumption per worker is rising if the real return exceeds the rate at which the household discounts future consumption, and is falling if the reverse holds. The smaller is θ—the less marginal utility changes as consumption changes—the larger are the changes in consumption in response to differences between the real interest rate and the discount rate.

Equation (2.20) is known as the *Euler equation* for this maximization problem. A more intuitive way of deriving (2.20) is to think of the household's consumption at two consecutive moments in time.[8] Specifically,

[8] The intuition for the Euler equation is considerably easier if time is discrete rather than continuous. See Section 2.9.

imagine the household reducing c at some date t by a small (formally, infinitesimal) amount Δc, investing this additional saving for a short (again, infinitesimal) period of time Δt, and then consuming the proceeds at time $t+\Delta t$; assume that when it does this, the household leaves consumption and capital holdings at all times other than t and $t+\Delta t$ unchanged. If the household is optimizing, the marginal impact of this change on lifetime utility must be zero. From (2.12), the marginal utility of $c(t)$ is $Be^{-\beta t}c(t)^{-\theta}$. Thus the change has a utility cost of $Be^{-\beta t}c(t)^{-\theta}\Delta c$. Since the instantaneous rate of return is $r(t)$, c at time $t+\Delta t$ can be increased by $e^{[r(t)-n-g]\Delta t}\Delta c$. Similarly, since c is growing at rate $\dot{c}(t)/c(t)$, we can write $c(t+\Delta t)$ as $c(t)e^{[\dot{c}(t)/c(t)]\Delta t}$; thus the marginal utility of $c(t+\Delta t)$ is $Be^{-\beta(t+\Delta t)}c(t+\Delta t)^{-\theta} = Be^{-\beta(t+\Delta t)}[c(t)e^{[\dot{c}(t)/c(t)]\Delta t}]^{-\theta}$. Thus for the path of consumption to be utility-maximizing, it must satisfy

$$Be^{-\beta t}c(t)^{-\theta}\Delta c = Be^{-\beta(t+\Delta t)}[c(t)e^{[\dot{c}(t)/c(t)]\Delta t}]^{-\theta}e^{[r(t)-n-g]\Delta t}\Delta c. \tag{2.22}$$

Dividing by $Be^{-\beta t}c(t)^{-\theta}\Delta c$ and taking logs yields

$$-\beta\,\Delta t - \theta\frac{\dot{c}(t)}{c(t)}\Delta t + [r(t)-n-g]\,\Delta t = 0. \tag{2.23}$$

Finally, dividing by Δt and rearranging yields the Euler equation in (2.20).

Intuitively, the Euler equation describes how c must behave over time given $c(0)$: if c does not evolve according to (2.20), the household can rearrange its consumption in a way that raises lifetime utility without changing the present value of its lifetime spending. The choice of $c(0)$ is then determined by the requirement that the present value of lifetime consumption over the resulting path equals initial wealth plus the present value of future earnings. When $c(0)$ is chosen too low, consumption spending along the path satisfying (2.20) does not exhaust lifetime wealth, and so a higher path is possible; when $c(0)$ is set too high, consumption spending more than uses up lifetime wealth, and so the path is not feasible.[9]

2.3 The Dynamics of the Economy

The most convenient way to describe the behavior of the economy is in terms of the evolution of c and k.

[9] Formally, equation (2.20) implies that $c(t)=c(0)e^{[R(t)-(\rho+\theta g)t]/\theta}$, which implies that $e^{-R(t)}e^{(n+g)t}c(t) = c(0)e^{[(1-\theta)R(t)+(\theta n-\rho)t]/\theta}$. Thus $c(0)$ is determined by the fact that $c(0)\int_{t=0}^{\infty} e^{[(1-\theta)R(t)+(\theta n-\rho)t]/\theta}\,dt$ must equal the right-hand side of the budget constraint, (2.14).

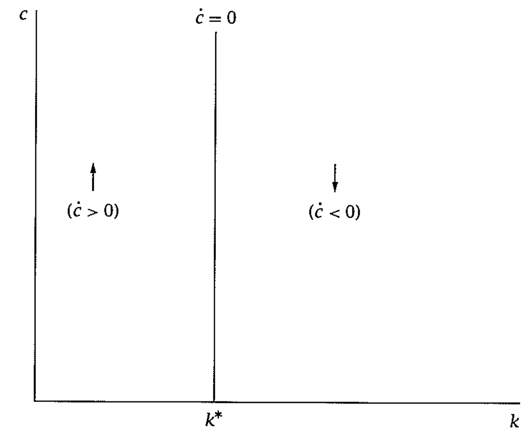


FIGURE 2.1 The dynamics of c

The Dynamics of c

Since all households are the same, equation (2.20) describes the evolution of c not just for a single household but for the economy as a whole. Since $r(t) = f'(k(t))$, we can rewrite (2.20) as

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t))-\rho-\theta g}{\theta}. \tag{2.24}$$

Thus $\dot{c}$ is zero when $f'(k)$ equals $\rho+\theta g$. Let k^* denote this level of k. When k exceeds k^*, $f'(k)$ is less than $\rho+\theta g$, and so $\dot{c}$ is negative; when k is less than k^*, $\dot{c}$ is positive.

This information is summarized in Figure 2.1. The arrows show the direction of motion of c. Thus c is rising if $k<k^*$ and falling if $k>k^*$. The $\dot{c}=0$ line at $k=k^*$ indicates that c is constant for this value of k.[10]

The Dynamics of k

As in the Solow model, $\dot{k}$ equals actual investment minus break-even investment. Since we are assuming that there is no depreciation, break-even

[10] Note that (2.24) implies that $\dot{c}$ also equals zero when c is zero. That is, $\dot{c}$ is also zero along the horizontal axis of the diagram. But in equilibrium c is never zero, and so this is not relevant to the analysis of the model.

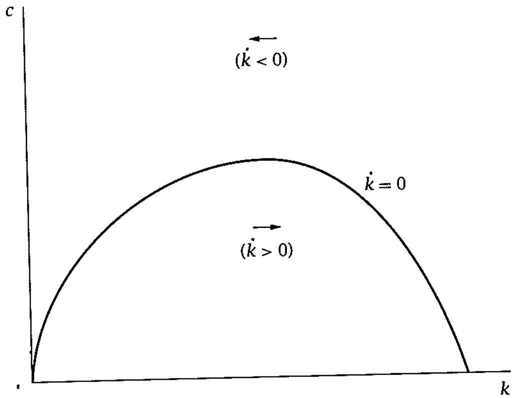


FIGURE 2.2 The dynamics of k

investment is $(n + g)k$. Actual investment is output minus consumption, $f(k) - c$. Thus,

$$\dot{k}(t) = f(k(t)) - c(t) - (n + g)k(t). \tag{2.25}$$

For a given k, the level of c that implies $\dot{k} = 0$ is given by $f(k) - (n + g)k$; in terms of Figure 1.6 (in Chapter 1), $\dot{k}$ is zero when consumption equals the difference between the actual output and break-even investment lines. This value of c is increasing in k until $f'(k) = n + g$ (the golden-rule level of k) and is then decreasing. When c exceeds the level that yields $\dot{k} = 0$, k is falling; when c is less than this level, k is rising. For k sufficiently large, break-even investment exceeds total output, and so $\dot{k}$ is negative for all positive values of c. This information is summarized in Figure 2.2; the arrows show the direction of motion of k.

The Phase Diagram

Figure 2.3 combines the information in Figures 2.1 and 2.2. The arrows now show the directions of motion of both c and k. To the left of the $\dot{c} = 0$ locus and above the $\dot{k} = 0$ locus, for example, $\dot{c}$ is positive and $\dot{k}$ negative. Thus c is rising and k falling, and so the arrows point up and to the left. The arrows in the other sections of the diagram are based on similar reasoning. On the $\dot{c} = 0$ and $\dot{k} = 0$ curves, only one of c and k is changing. On the $\dot{c} = 0$ line above the $\dot{k} = 0$ locus, for example, c is constant and k is falling; thus

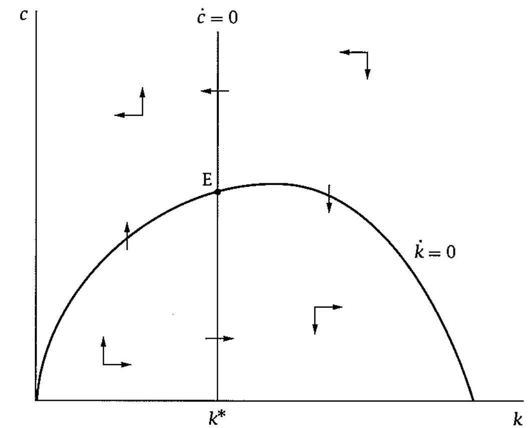


FIGURE 2.3 The dynamics of c and k

the arrow points to the left. Finally, at Point E both $\dot{c}$ and $\dot{k}$ are zero; thus there is no movement from this point.[11]

Figure 2.3 is drawn with k^* (the level of k that implies $\dot{c} = 0$) less than the golden-rule level of k (the value of k associated with the peak of the $\dot{k} = 0$ locus). To see that this must be the case, recall that k^* is defined by $f'(k^*) = \rho + \theta g$, and that the golden-rule k is defined by $f'(k_{GR}) = n + g$. Since $f''(k)$ is negative, k^* is less than k_{GR} if and only if $\rho + \theta g$ is greater than $n + g$. This is equivalent to $\rho - n - (1 - \theta)g > 0$, which we have assumed to hold so that lifetime utility does not diverge (see [2.2]). Thus k^* is to the left of the peak of the $\dot{k} = 0$ curve.

The Initial Value of c

Figure 2.3 shows how c and k must evolve over time to satisfy households' intertemporal optimization condition (equation [2.24]) and the equation

[11] There are two other points where c and k are constant. The first is the origin: if the economy starts with no capital and no consumption, it remains there. The second is the point where the $\dot{k} = 0$ curve crosses the horizontal axis. Here all of output is being used to hold k constant, so $c = 0$ and $f(k) = (n + g)k$. Since having consumption change from zero to any positive amount violates households' intertemporal optimization condition, (2.24), if the economy is at this point it must remain there to satisfy (2.24) and (2.25). As we will see shortly, however, the economy is never at this point.

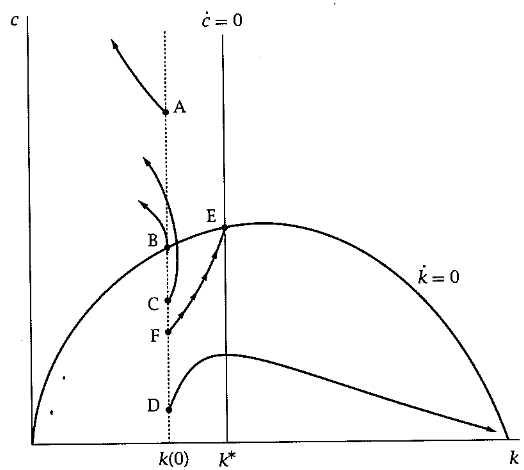


FIGURE 2.4 **The behavior of c and k for various initial values of c**

relating the change in k to output and consumption (equation [2.25]) *given initial values of c and k.* The initial value of k is given; but the initial value of c must be determined.

This issue is addressed in Figure 2.4. For concreteness, $k(0)$ is assumed to be less than k^*. The figure shows the trajectory of c and k for various assumptions concerning the initial level of c. If $c(0)$ is above the $\dot{k} = 0$ curve, at a point like A, then $\dot{c}$ is positive and $\dot{k}$ negative; thus the economy moves continually up and to the left in the diagram. If $c(0)$ is such that $\dot{k}$ is initially zero (Point B), the economy begins by moving directly up in (k, c) space; thereafter $\dot{c}$ is positive and $\dot{k}$ negative, and so the economy again moves up and to the left. If the economy begins slightly below the $\dot{k} = 0$ locus (Point C), $\dot{k}$ is initially positive but small (since $\dot{k}$ is a continuous function of c), and $\dot{c}$ is again positive. Thus in this case the economy initially moves up and slightly to the right; after it crosses the $\dot{k} = 0$ locus, however, $\dot{k}$ becomes negative and once again the economy is on a path of rising c and falling k.

Point D shows a case of very low initial consumption. Here $\dot{c}$ and $\dot{k}$ are both initially positive. From (2.24), $\dot{c}$ is proportional to c; when c is small, $\dot{c}$ is therefore small. Thus c remains low, and so the economy eventually crosses the $\dot{c} = 0$ line. After this point, $\dot{c}$ becomes negative, and $\dot{k}$ remains positive. Thus the economy moves down and to the right.

$\dot{c}$ and $\dot{k}$ are continuous functions of c and k. Thus there is some critical point between Points C and D—Point F in the diagram—such that at that

level of initial c, the economy converges to the stable point, Point E. For any level of consumption above this critical level, the $\dot{k} = 0$ curve is crossed before the $\dot{c} = 0$ line is reached, and so the economy ends up on a path of perpetually rising consumption and falling capital. And if consumption is less than the critical level, the $\dot{c} = 0$ locus is reached first, and so the economy embarks on a path of falling consumption and rising capital. But if consumption is just equal to the critical level, the economy converges to the point where both c and k are constant.

All these various trajectories satisfy equations (2.24) and (2.25). But we have not yet imposed the requirement that households satisfy their budget constraint, nor have we imposed the requirement that the economy's capital stock not be negative. These conditions determine which of the trajectories in fact describes the behavior of the economy.

If the economy starts at some point above F, c is high and rising. As a result, the equation of motion for k, (2.25), implies that k eventually reaches zero. For (2.24) and (2.25) to continue to be satisfied, c must continue to rise and k must become negative. But this cannot occur. Since output is zero when k is zero, c must drop to zero. This means that households are not satisfying their intertemporal optimization condition, (2.24). We can therefore rule out such paths.

To rule out paths starting below F, we use the budget constraint expressed in terms of the limiting behavior of capital holdings, equation (2.15): $\lim_{s\to\infty} e^{-R(s)}e^{(n+g)s}k(s) \geq 0$. If the economy starts at a point like D, eventually k exceeds the golden-rule capital stock. After that time, the real interest rate, $f'(k)$, is less than $n + g$, so $e^{-R(s)}e^{(n+g)s}$ is rising. Since k is also rising, $e^{-R(s)}e^{(n+g)s}k(s)$ diverges. Thus $\lim_{s\to\infty} e^{-R(s)}e^{(n+g)s}k(s)$ is infinity. From the derivation of (2.15), we know that this is equivalent to the statement that the present value of households' lifetime income is infinitely larger than the present value of their lifetime consumption. Thus each household can afford to raise its consumption at each point in time, and so can attain higher utility. That is, households are not maximizing their utility. Hence, such a path cannot be an equilibrium.

Finally, if the economy begins at Point F, k converges to k^*, and so r converges to $f'(k^*) = \rho + \theta g$. Thus eventually $e^{-R(s)}e^{(n+g)s}$ is falling at rate $\rho - n - (1-\theta)g = \beta > 0$, and so $\lim_{s\to\infty} e^{-R(s)}e^{(n+g)s}k(s)$ is zero. Thus the path beginning at F, and only this path, is possible.

The Saddle Path

Although this discussion has been in terms of a single value of k, the idea is general. For any positive initial level of k, there is a unique initial level of c that is consistent with households' intertemporal optimization, the dynamics of the capital stock, households' budget constraint, and the requirement that k not be negative. The function giving this initial c as a function of k is

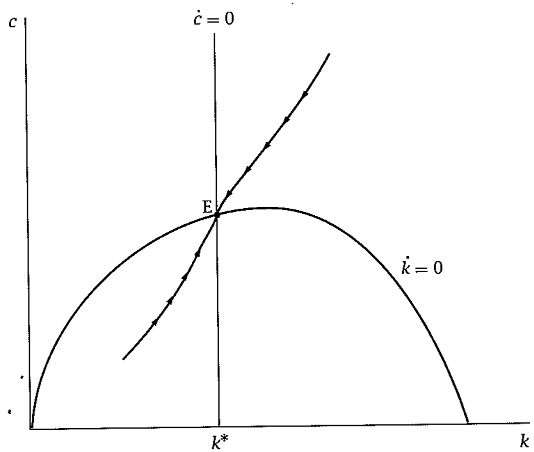


FIGURE 2.5 The saddle path

known as the *saddle path;* it is shown in Figure 2.5. For any starting value for k, the initial c must be the value on the saddle path. The economy then moves along the saddle path to Point E.

2.4 Welfare

A natural question is whether the equilibrium of this economy represents a desirable outcome. The answer to this question is simple. The *first welfare theorem* from microeconomics tells us that if markets are competitive and complete and there are no externalities (and if the number of agents is finite), then the decentralized equilibrium is Pareto-efficient—that is, it is impossible to make anyone better off without making someone else worse off. Since the conditions of the first welfare theorem hold in our model, the equilibrium must be Pareto-efficient. And since all households have the same utility, this means that the decentralized equilibrium produces the highest possible utility among allocations that treat all households in the same way.

To see this more clearly, consider the problem facing a social planner who can dictate the division of output between consumption and investment at each date and who wants to maximize the lifetime utility of a representative household. This problem is identical to that of an individual household except that, rather than taking the paths of w and r as given, the planner

takes into account the fact that these are determined by the path of k, which is in turn determined by (2.25).

The intuitive argument involving consumption at consecutive moments used to derive (2.20) or (2.24) applies to the social planner as well: reducing c by Δc at time t and investing the proceeds allows the planner to increase c at time $t + \Delta t$ by $e^{f'(k(t))\Delta t}e^{-(n+g)\Delta t}\Delta c$.[12] Thus $c(t)$ along the path chosen by the planner must satisfy (2.24). And since equation (2.25) giving the evolution of k reflects technology, not preferences, the social planner must obey it as well. Finally, as with households' optimization problem, paths that require that the capital stock becomes negative can be ruled out on the grounds that they are not feasible, and paths that cause consumption to approach zero can be ruled out on the grounds that they do not maximize households' utility.

In short, the solution to the social planner's problem is for the initial value of c to be given by the value on the saddle path, and for c and k to then move along the saddle path. That is, the competitive equilibrium maximizes the welfare of the representative household.[13]

2.5 The Balanced Growth Path

Properties of the Balanced Growth Path

The behavior of the economy once it has converged to Point E is identical to that of the Solow economy on the balanced growth path. Capital, output, and consumption per unit of effective labor are constant. Since y and c are constant, the saving rate, $(y - c)/y$, is also constant. The total capital stock, total output, and total consumption grow at rate $n + g$. And capital per worker, output per worker, and consumption per worker grow at rate g.

Thus the central implications of the Solow model concerning the driving forces of economic growth do not hinge on its assumption of a constant saving rate. Even when saving is endogenous, growth in the effectiveness of

[12] Note that this change does affect r and w over the (brief) interval from t to $t+\Delta t$. r falls by $f''(k)$ times the change in k, while w rises by $-f''(k)k$ times the change in k. But the effect of these changes on total income (per unit of effective labor), which is given by the change in w plus k times the change in r, is zero. That is, since capital is paid its marginal product, total payments to labor and to previously existing capital remain equal to the previous level of output (again per unit of effective labor). This is just a specific instance of the general result that the *pecuniary externalities*—externalities operating through prices—balance in the aggregate under competition.

[13] A formal solution to the planner's problem involves the use of the calculus of variations. For a formal statement and solution of the problem, see Blanchard and Fischer (1989, pp. 38–43). For an introduction to the calculus of variations, see Section 8.2; Kamien and Schwartz (1991); Dixit (1990, Chapter 10); or Obstfeld (1992).

labor remains the only source of persistent growth in output per worker. And since the production function is the same as in the Solow model, one can repeat the calculations of Section 1.6 demonstrating that significant differences in output per worker can arise from differences in capital per worker only if the differences in capital per worker, and in rates of return to capital, are enormous.

The Balanced Growth Path and the Golden-Rule Level of Capital

The only notable difference between the balanced growth paths of the Solow and Ramsey–Cass–Koopmans models is that a balanced growth path with a capital stock above the golden-rule level is not possible in the Ramsey–Cass–Koopmans model. In the Solow model, a sufficiently high saving rate causes the economy to reach a balanced growth path with the property that there are feasible alternatives that involve higher consumption at every moment. In the Ramsey–Cass–Koopmans model, in contrast, saving is derived from the behavior of households whose utility depends on their consumption, and there are no externalities. As a result, it cannot be an equilibrium for the economy to follow a path where higher consumption can be attained in every period; if the economy were on such a path, households would reduce their saving and take advantage of this opportunity.

This can be seen in the phase diagram. Consider again Figure 2.5. If the initial capital stock exceeds the golden-rule level (that is, if $k(0)$ is greater than the k associated with the peak of the $\dot{k}=0$ locus), initial consumption is above the level needed to keep k constant; thus $\dot{k}$ is negative. k gradually approaches k^*, which is below the golden-rule level.

Finally, the fact that k^* is less than the golden-rule capital stock implies that the economy does not converge to the balanced growth path that yields the maximum sustainable level of c. The intuition for this result is clearest in the case of g equal to zero, so that there is no long-run growth of consumption and output per worker. In this case, k^* is defined by $f'(k^*)=\rho$ (see [2.24]) and k_{GR} is defined by $f'(k_{GR})=n$, and our assumption that $\rho-n-(1-\theta)g>0$ simplifies to $\rho>n$. Since k^* is less than k_{GR}, an increase in saving starting at $k=k^*$ would cause consumption per worker to eventually rise above its previous level and remain there (see Section 1.4). But because households value present consumption more than future consumption, the benefit of the eventual permanent increase in consumption is bounded. At some point—specifically, when k exceeds k^*—the tradeoff between the temporary short-term sacrifice and the permanent long-term gain is sufficiently unfavorable that accepting it reduces rather than raises lifetime utility. Thus k converges to a value below the golden-rule level. Because k^* is the optimal level of k for the economy to converge to, it is known as the *modified golden-rule* capital stock.

2.6 The Effects of a Fall in the Discount Rate

Consider a Ramsey–Cass–Koopmans economy that is on its balanced growth path, and suppose that there is a fall in ρ, the discount rate. Because ρ is the parameter governing households' preferences between current and future consumption, this change is the closest analogue in this model to a rise in the saving rate in the Solow model.

Since the division of output between consumption and investment is determined by forward-looking households, we must specify whether the change is expected or unexpected. If a change is expected, households may alter their behavior before the change occurs. We therefore focus on the simple case where the change is unexpected. That is, households are optimizing given their belief that their discount rate will not change, and the economy is on the resulting balanced growth path. At some date households suddenly discover that their preferences have changed, and that they now discount future utility at a lower rate than before.[14]

Qualitative Effects

Since the evolution of k is determined by technology rather than preferences, ρ enters the equation for $\dot{c}$ but not the one for $\dot{k}$. Thus only the $\dot{c}=0$ locus is affected. Recall equation (2.24): $\dot{c}(t)/c(t)=[f'(k(t))-\rho-\theta g]/\theta$. Thus the value of k where $\dot{c}$ equals zero is defined by $f'(k^*)=\rho+\theta g$. Since $f''(\bullet)$ is negative, this means that the fall in ρ raises k^*. Thus the $\dot{c}=0$ line shifts to the right. This is shown in Figure 2.6.

At the time of the change in ρ, the value of k—the *stock* of capital per unit of effective labor—is given by the history of the economy, and it cannot change discontinuously. In particular, k at the time of the change equals the value of k^* on the old balanced growth path. In contrast, c—the *rate* at which households are consuming—can jump at the time of the shock.

Given our analysis of the dynamics of the economy, it is clear what occurs: at the instant of the change, c jumps down so that the economy is on the new saddle path (Point A in Figure 2.6).[15] Thereafter, c and k rise gradually to their new balanced-growth-path values; these are higher than their values on the original balanced growth path.

Thus the effects of a fall in the discount rate are similar to the effects of a rise in the saving rate in the Solow model with a capital stock below the

[14] See Section 2.7 and Problems 2.10 and 2.11 for examples of how to analyze anticipated changes.

[15] Since we are assuming that the change is unexpected, the discontinuous change in c does not imply that households are not optimizing. Their original behavior is optimal given their beliefs; the fall in c is the optimal response to the new information that ρ is lower.

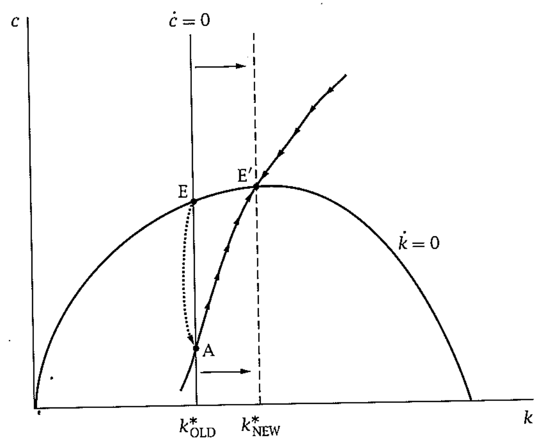


FIGURE 2.6 The effects of a fall in the discount rate

golden-rule level. In both cases, k rises gradually to a new higher level, and in both c initially falls but then rises to a level above the one it started at. Thus, just as with a permanent rise in the saving rate in the Solow model, the permanent fall in the discount rate produces temporary increases in the growth rates of capital per worker and output per worker. The only difference between the two experiments is that, in the case of the fall in ρ, in general the fraction of output that is saved is not constant during the adjustment process.

The Rate of Adjustment and the Slope of the Saddle Path

Equations (2.24) and (2.25) describe $\dot{c}(t)$ and $\dot{k}(t)$ as functions of $k(t)$ and $c(t)$. A fruitful way to analyze their quantitative implications for the dynamics of the economy is to replace these nonlinear equations with linear approximations around the balanced growth path. Thus we begin by taking first-order Taylor approximations to (2.24) and (2.25) around $k = k^*$, $c = c^*$. That is, we write

$$\dot{c} \simeq \frac{\partial \dot{c}}{\partial k}[k - k^*] + \frac{\partial \dot{c}}{\partial c}[c - c^*], \tag{2.26}$$

$$\dot{k} \simeq \frac{\partial \dot{k}}{\partial k}[k - k^*] + \frac{\partial \dot{k}}{\partial c}[c - c^*], \tag{2.27}$$

where $\partial \dot{c}/\partial k$, $\partial \dot{c}/\partial c$, $\partial \dot{k}/\partial k$, and $\partial \dot{k}/\partial c$ are all evaluated at $k = k^*$, $c = c^*$. Our strategy will be to treat (2.26) and (2.27) as exact and analyze the dynamics of the resulting system.[16]

It helps to define $\tilde{c} = c - c^*$ and $\tilde{k} = k - k^*$. Since c^* and k^* are both constant, $\dot{\tilde{c}}$ equals $\dot{c}$, and $\dot{\tilde{k}}$ equals $\dot{k}$. We can therefore rewrite (2.26) and (2.27) as

$$\dot{\tilde{c}} \simeq \frac{\partial \dot{c}}{\partial k}\tilde{k} + \frac{\partial \dot{c}}{\partial c}\tilde{c}, \tag{2.28}$$

$$\dot{\tilde{k}} \simeq \frac{\partial \dot{k}}{\partial k}\tilde{k} + \frac{\partial \dot{k}}{\partial c}\tilde{c}. \tag{2.29}$$

(Again, the derivatives are all evaluated at $k = k^*$, $c = c^*$.) Recall that $\dot{c} = \{[f'(k) - \rho - \theta g]/\theta\}c$ (equation [2.24]). Using this expression to compute the derivatives in (2.28) and evaluating them at $k = k^*$, $c = c^*$ gives us

$$\dot{\tilde{c}} \simeq \frac{f''(k^*)c^*}{\theta}\tilde{k}. \tag{2.30}$$

Similarly, (2.25) states that $\dot{k} = f(k) - c - (n + g)k$. We can use this to find the derivatives in (2.29); this yields

$$\begin{aligned} \dot{\tilde{k}} &\simeq [f'(k^*) - (n + g)]\tilde{k} - \tilde{c} \\ &= [(\rho + \theta g) - (n + g)]\tilde{k} - \tilde{c} \\ &= \beta\tilde{k} - \tilde{c}, \end{aligned} \tag{2.31}$$

where the second line uses the fact that (2.24) implies that $f'(k^*) = \rho + \theta g$ and the third line uses the definition of β as $\rho - n - (1 - \theta)g$. Dividing both sides of (2.30) by $\tilde{c}$ and both sides of (2.31) by $\tilde{k}$ yields expressions for the growth rates of $\tilde{c}$ and $\tilde{k}$:

$$\frac{\dot{\tilde{c}}}{\tilde{c}} \simeq \frac{f''(k^*)c^*}{\theta}\frac{\tilde{k}}{\tilde{c}}, \tag{2.32}$$

$$\frac{\dot{\tilde{k}}}{\tilde{k}} \simeq \beta - \frac{\tilde{c}}{\tilde{k}}. \tag{2.33}$$

Equations (2.32) and (2.33) imply that the growth rates of $\tilde{c}$ and $\tilde{k}$ depend only on the ratio of $\tilde{c}$ and $\tilde{k}$. Given this, consider what happens if the values of $\tilde{c}$ and $\tilde{k}$ are such that $\tilde{c}$ and $\tilde{k}$ are falling at the same rate (that is, if they imply $\dot{\tilde{c}}/\tilde{c} = \dot{\tilde{k}}/\tilde{k}$). This implies that the ratio of $\tilde{c}$ to $\tilde{k}$ is not changing, and thus that their growth rates are also not changing. That is, if $c - c^*$ and

[16] For a more formal introduction to the analysis of systems of differential equations (such as [2.26]–[2.27]), see Simon and Blume (1994, Chapter 25).

$k - k^*$ are initially falling at the same rate, they continue to fall at that rate. In terms of the diagram, from a point where $\tilde{c}$ and $\tilde{k}$ are falling at equal rates, the economy moves along a straight line to (k^*, c^*), with the distance from (k^*, c^*) falling at a constant rate.

Let μ denote $\dot{\tilde{c}}/\tilde{c}$. Equation (2.32) implies

$$\frac{\tilde{c}}{\tilde{k}} = \frac{f''(k^*)c^*}{\theta}\frac{1}{\mu}. \tag{2.34}$$

From (2.33), the condition that $\dot{\tilde{k}}/\tilde{k}$ equals $\dot{\tilde{c}}/\tilde{c}$ is thus

$$\mu = \beta - \frac{f''(k^*)c^*}{\theta}\frac{1}{\mu}, \tag{2.35}$$

or

$$\mu^2 - \beta\mu + \frac{f''(k^*)c^*}{\theta} = 0. \tag{2.36}$$

This is a quadratic equation in μ. The solutions are

$$\mu = \frac{\beta \pm [\beta^2 - 4f''(k^*)c^*/\theta]^{1/2}}{2}. \tag{2.37}$$

Let μ_1 and μ_2 denote these two values of μ.

If μ is positive, then $\tilde{c}$ and $\tilde{k}$ are growing; that is, instead of moving along a straight line toward (k^*, c^*), the economy is moving on a straight line away from (k^*, c^*). Thus if the economy is to converge to (k^*, c^*), then μ must be negative. Inspection of (2.37) shows that only one of the μ's, namely $\{\beta - [\beta^2 - 4f''(k^*)c^*/\theta]^{1/2}\}/2$, is negative. Let μ_1 denote this value of μ. Equation (2.34) (with $\mu = \mu_1$) then tells us how $\tilde{c}$ must be related to $\tilde{k}$ for both to be falling at rate μ_1.

Figure 2.7 shows the line along which the economy converges smoothly to (k^*, c^*); it is labeled AA. This is the saddle path of the linearized system. The figure also shows the line along which the economy moves directly away from (k^*, c^*); it is labeled BB. If the initial values of $c(0)$ and $k(0)$ lay along this line, (2.32) and (2.33) would imply that $\tilde{c}$ and $\tilde{k}$ would grow steadily at rate μ_2.[17] Since $f''(\bullet)$ is negative, (2.34) implies that the relation between $\tilde{c}$ and $\tilde{k}$ has the opposite sign from μ. Thus the saddle path AA is positively sloped, and the BB line is negatively sloped.

Thus if we linearize the equations for $\dot{c}$ and $\dot{k}$, we can characterize the dynamics of the economy in terms of the model's parameters. At time 0, c must equal $c^* + [f''(k^*)c^*/(\theta\mu_1)](k - k^*)$. Thereafter, c and k converge to their balanced-growth-path values at rate μ_1. That is, $k(t) = k^* + e^{\mu_1 t}[k(0) - k^*]$ and $c(t) = c^* + e^{\mu_1 t}[c(0) - c^*]$.

[17] Of course, it is not possible for the initial value of (k, c) to lie along the BB line. As we saw in Section 2.3, if it did, either k would eventually become negative or households would accumulate infinite wealth.

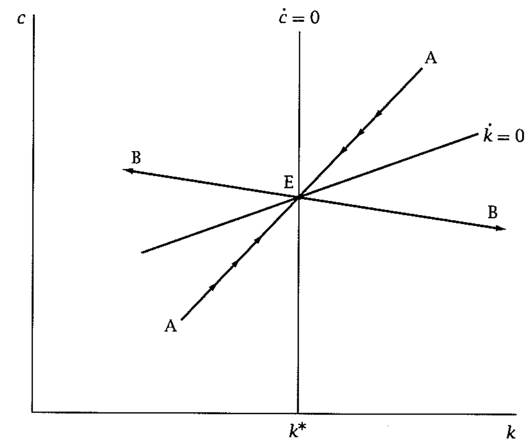


FIGURE 2.7 The linearized phase diagram

The Speed of Adjustment

To understand the implications of (2.37) for the speed of convergence to the balanced growth path, consider our usual example of Cobb–Douglas production, $f(k) = k^\alpha$. This implies $f''(k^*) = \alpha(\alpha - 1)k^{*\alpha-2}$. Since consumption on the balanced growth path equals output minus break-even investment, consumption per unit of effective labor, c^*, equals $k^{*\alpha} - (n+g)k^*$. Thus in this case we can write the expression for μ_1 as

$$\mu_1 = \frac{1}{2}\left(\beta - \left\{\beta^2 - \frac{4}{\theta}\alpha(\alpha - 1)k^{*\alpha-2}[k^{*\alpha} - (n+g)k^*]\right\}^{1/2}\right). \tag{2.38}$$

Recall that on the balanced growth path, $f'(k)$ equals $\rho + \theta g$ (see [2.24]). For the Cobb-Douglas case, this is equivalent to $\alpha k^{*\alpha-1} = \rho + \theta g$, or $k^* = [(\rho + \theta g)/\alpha]^{1/(\alpha-1)}$. Substituting this into (2.38) and doing some uninteresting algebraic manipulations yields

$$\mu_1 = \frac{1}{2}\left(\beta - \left\{\beta^2 + \frac{4}{\theta}\frac{1-\alpha}{\alpha}(\rho + \theta g)[\rho + \theta g - \alpha(n+g)]\right\}^{1/2}\right). \tag{2.39}$$

Equation (2.39) expresses the rate of adjustment in terms of the underlying parameters of the model.

To get a feel for the magnitudes involved, suppose $\alpha = \frac{1}{3}$, $\rho = 4\%$, $n = 2\%$, $g = 1\%$, and $\theta = 1$. One can show that these parameter values imply that on the balanced growth path, the real interest rate is 5 percent and the saving rate 20 percent. And since β is defined as $\rho - n - (1-\theta)g$, they imply $\beta = 2\%$. Equation (2.38) or (2.39) then implies $\mu_1 \simeq -5.4\%$. Thus adjustment is quite rapid in this case; for comparison, the Solow model with the same values of α, n, and g (and as here, no depreciation) implies an adjustment speed of 2 percent per year (see equation [1.31]). The reason for the difference is that in this example, the saving rate is greater than s^* when k is less than k^* and less than s^* when k is greater than k^*. In the Solow model, in contrast, s is constant by assumption.

2.7 The Effects of Government Purchases

Thus far, we have left government out of our model. Yet modern economies devote their resources not just to investment and private consumption but also to public uses. In the United States, for example, about 20 percent of total output is purchased by the government; in many other countries the figure is considerably higher. It is thus natural to extend our model to include a government sector.

Adding Government to the Model

Assume that the government buys output at rate $G(t)$ per unit of effective labor per unit time. Government purchases are assumed not to affect utility from private consumption; this can occur if the government devotes the goods to some activity that does not affect utility at all, or if utility equals the sum of utility from private consumption and utility from government-provided goods. Similarly, the purchases are assumed not to affect future output; that is, they are devoted to public consumption rather than public investment. The purchases are financed by lump-sum taxes of amount $G(t)$ per unit of effective labor per unit time; thus the government always runs a balanced budget. Consideration of deficit finance is postponed to Chapter 11. We will see there, however, that in this model the government's choice between tax and deficit finance has no impact on any important variables. Thus the assumption that the purchases are financed with current taxes only serves to simplify the presentation.

Investment is now the difference between output and the sum of private consumption and government purchases. Thus the equation of motion for k, (2.25), becomes

$$\dot{k}(t) = f(k(t)) - c(t) - G(t) - (n+g)k(t). \qquad (2.40)$$

A higher value of G shifts the $\dot{k} = 0$ locus down: the more goods that are purchased by the government, the fewer that can be purchased privately if k is to be held constant.

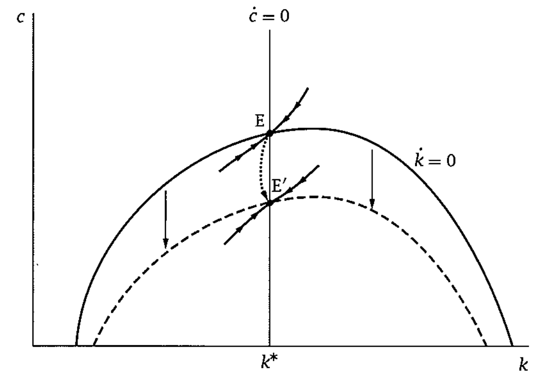


FIGURE 2.8 The effects of a permanent increase in government purchases

By assumption, households' preferences ([2.1]–[2.2] or [2.12]) are unchanged. Since the Euler equation ([2.20] or [2.24]) is derived from households' preferences without imposing their lifetime budget constraint, this condition continues to hold as before. The taxes that finance the government's purchases affect households' budget constraint, however. Specifically, (2.14) becomes

$$\int_{t=0}^{\infty} e^{-R(t)} c(t) e^{(n+g)t}\, dt \leq k(0) + \int_{t=0}^{\infty} e^{-R(t)} [w(t) - G(t)] e^{(n+g)t}\, dt. \qquad (2.41)$$

Reasoning parallel to that used before shows that this implies the same expression as before for the limiting behavior of k (equation [2.15]).

The Effects of Permanent and Temporary Changes in Government Purchases

To see the implications of the model, suppose that the economy is on a balanced growth path with $G(t)$ constant at some level G_L, and that there is an unexpected, permanent increase in G to G_H. From (2.40), the $\dot{k} = 0$ locus shifts down by the amount of the increase in G. Since government purchases do not affect the Euler equation, the $\dot{c} = 0$ locus is unaffected. This is shown in Figure 2.8.[18]

[18] We assume that G_H is not so large that $\dot{k}$ is negative when $c = 0$. That is, the intersection of the new $\dot{k} = 0$ locus with the $\dot{c} = 0$ line is assumed to occur at a positive level of c. If it does not, the government's policy is not feasible. Even if c is always zero, $\dot{k}$ is negative, and eventually the economy's output per unit of effective labor is less than G_H.

We know that in response to such a change, c must jump so that the economy is on its new saddle path. If not, then as before, either capital would become negative at some point or households would accumulate infinite wealth. In this case, the adjustment takes a simple form: c falls by the amount of the increase in G, and the economy is immediately on its new balanced growth path. Intuitively, the permanent increases in government purchases and taxes reduce households' lifetime wealth. And because the increases in purchases and taxes are permanent, there is no scope for households to raise their utility by adjusting the time pattern of their consumption. Thus the size of the immediate fall in consumption is equal to the full amount of the increase in government purchases, and the capital stock and the real interest rate are unaffected.

An older approach to modeling consumption behavior assumes that consumption depends only on current disposable income and that it moves less than one-for-one with disposable income. Recall, for example, that the Solow model assumes that consumption is simply fraction $1-s$ of current income. With that approach, consumption falls by less than the amount of the increase in government purchases. As a result, the rise in government purchases crowds out investment, and so the capital stock starts to fall and the real interest rate starts to rise. Our analysis shows that those results rest critically on the assumption that households follow mechanical rules: with intertemporal optimization, a permanent increase in government purchases does not cause crowding out.

A more complicated case is provided by an unanticipated increase in G that is expected to be temporary. For simplicity, assume that the terminal date is known with certainty. In this case, c does not fall by the full amount of the increase in G, $G_H - G_L$. To see this, note that if it did, consumption would jump up discontinuously at the time that government purchases returned to G_L; thus marginal utility would fall discontinuously. But since the return of G to G_L is anticipated, the discontinuity in marginal utility would also be anticipated, which cannot be optimal for households.

During the period of time that government purchases are high, $\dot{k}$ is governed by the capital-accumulation equation, (2.40), with $G = G_H$; after G returns to G_L, it is governed by (2.40) with $G = G_L$. The Euler equation, (2.24), determines the dynamics of c throughout, and c cannot change discontinuously at the time that G returns to G_L. These facts determine what happens at the time of the increase in G: c must jump to the value such that the dynamics implied by (2.40) with $G = G_H$ (and by [2.24]) bring the economy to the old saddle path at the time that G returns to its initial level. Thereafter, the economy moves along that saddle path to the old balanced growth path.[19]

[19] As in the previous example, because the initial change in G is unexpected, the discontinuities in consumption and marginal utility at that point do not mean that households are not behaving optimally. See n. 15.

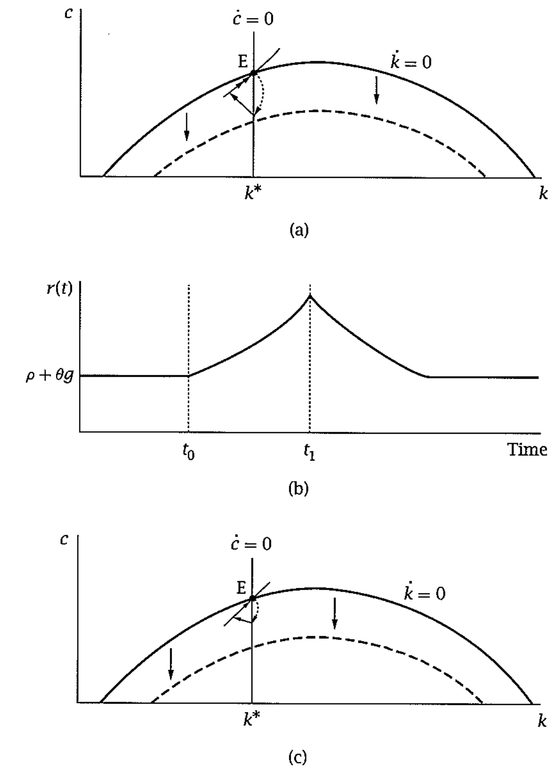


FIGURE 2.9 The effects of a temporary increase in government purchases

This is depicted in Figure 2.9. Panel (a) shows a case where the increase in G is relatively long-lasting. In this case c falls by most of the amount of the increase in G. As the time of the return of G to G_L approaches, however, households increase their consumption and decrease their capital holdings in anticipation of the fall in G.

Since $r = f'(k)$, we can deduce the behavior of r from the behavior of k. Thus r rises gradually during the period that government spending is high and then gradually returns to its initial value. This is shown in Panel (b); t_0 denotes the time of the increase in G, and t_1 the time of its return to its initial value.

Finally, Panel (c) shows the case of a short-lived rise in G. Here households change their consumption relatively little, choosing instead to pay for most of the temporarily higher taxes out of their savings. Because government purchases are high for only a short period, the effects on the capital stock and the real interest rate are small.

Note that once again allowing for forward-looking behavior yields insights we would not get from the older approach of assuming that consumption depends only on current disposable income. With that approach, the duration of a change in government purchases is irrelevant. But the idea that households do not look ahead and put some weight on the likely future path of government purchases and taxes is implausible.

Empirical Application: Wars and Real Interest Rates

This analysis suggests that temporarily high government purchases cause real interest rates to rise, whereas permanently high purchases do not. Intuitively, when the government's purchases are high only temporarily, households expect their consumption to be greater in the future than it is in the present. To make them willing to accept this, the real interest rate must be high. When the government's purchases are permanently high, on the other hand, households' current consumption is low, and they expect it to remain low. Thus in this case, no movement in real interest rates is needed for households to accept their current low consumption.

A natural example of a period of temporarily high government purchases is a war. Thus our analysis predicts that real interest rates are high during wars. Barro (1987) tests this prediction by examining military spending and interest rates in the United Kingdom from 1729 to 1918. The most significant complication he faces is that, instead of having data on short-term real interest rates, he has data only on long-term nominal interest rates. Long-term interest rates should be, loosely speaking, a weighted average of expected short-term interest rates.[20] Thus, since our analysis implies that temporary increases in government purchases raise the short-term rate over an extended period, it also implies that they raise the long-term rate. Similarly, since the analysis implies that permanent increases never change the short-term rate, it predicts that they do not affect the long-term rate. In addition, the real interest rate equals the nominal rate minus expected inflation; thus the nominal rate should be corrected for changes in expected inflation. Barro does not find any evidence, however, of systematic changes in expected inflation in his sample period; thus the data are at least consistent with the view that movements in nominal rates represent changes in real rates.

[20] See Section 10.2.

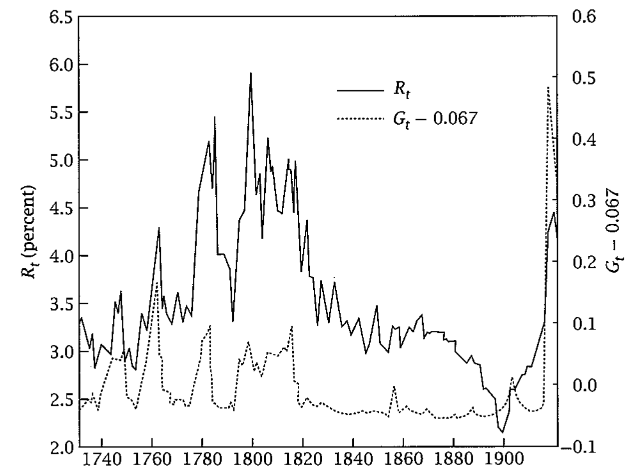


FIGURE 2.10 Temporary military spending and the long-term interest rate in the United Kingdom (from Barro, 1987; used with permission)

Figure 2.10 plots British military spending as a share of GNP (relative to the mean of this series for the full sample) and the long-term interest rate. The spikes in the military spending series correspond to wars; for example, the spike around 1760 reflects the Seven Years' War, and the spike around 1780 corresponds to the American Revolution. The figure suggests that the interest rate is indeed higher during periods of temporarily high government purchases.

To test this formally, Barro estimates a process for the military purchases series and uses it to construct estimates of the temporary component of military spending. Not surprisingly in light of the figure, the estimated temporary component differs little from the raw series.[21] Barro then regresses the long-term interest rate on this estimate of temporary military spending. Because the residuals are serially correlated, he includes a first-order serial correlation correction. The results are

$$\begin{aligned} R_t = \underset{(0.27)}{3.54} + \underset{(0.7)}{2.6}\,\tilde{G}_t, \qquad \lambda = \underset{(0.03)}{0.91} \\ R^2 = 0.89, \qquad \text{s.e.e.} = 0.248, \qquad \text{D.W.} = 2.1. \end{aligned} \tag{2.42}$$

[21] Since there is little permanent variation in military spending, the data cannot be used to investigate the effects of permanent changes in government purchases on interest rates.

Here R_t is the long-term nominal interest rate, $\tilde{G}_t$ is the estimated value of temporary military spending as a fraction of GNP, λ is the first-order autoregressive parameter of the residual, and the numbers in parentheses are standard errors. Thus there is a statistically significant link between temporary military spending and interest rates. The results are even stronger when World War I is excluded: stopping the sample period in 1914 raises the coefficient on $\tilde{G}_t$ to 6.1 (and the standard error to 1.3). Barro argues that the comparatively small rise in the interest rate given the tremendous rise in military spending in World War I may have occurred because the government imposed price controls and used a variety of nonmarket means of allocating resources. If this is right, the results for the shorter sample may provide a better estimate of the impact of government purchases on interest rates in a market economy.

Thus the evidence from the United Kingdom supports the predictions of the theory. The success of the theory is not universal, however. In particular, for the United States real interest rates appear to have been, if anything, generally lower during wars than in other periods (Barro, 1993, pp. 321–322). The reasons for this anomalous behavior are not well understood. Thus the theory does not provide a full account of how real interest rates respond to changes in government purchases.

Part B The Diamond Model

2.8 Assumptions

We now turn to the Diamond overlapping-generations model. The central difference between the Diamond model and the Ramsey-Cass-Koopmans model is that there is turnover in the population: new individuals are continually being born, and old individuals are continually dying.

With turnover, it turns out to be simpler to assume that time is discrete rather than continuous. That is, the variables of the model are defined for $t = 0, 1, 2, \ldots$ rather than for all values of $t \geq 0$. To further simplify the analysis, the model assumes that each individual lives for only two periods. It is the general assumption of turnover in the population, however, and not the specific assumptions of discrete time and two-period lifetimes, that is crucial to the model's results.[22]

[22] See Problem 2.14 for a discrete-time version of the Solow model. Blanchard (1985) develops a tractable continuous-time model in which the extent of the departure from the infinite-horizon benchmark is governed by a continuous parameter. Weil (1989a) considers a variant of Blanchard's model where new households enter the economy but existing households do not leave. He shows that the arrival of new households is sufficient to generate most of the main results of the Diamond and Blanchard models. Finally, Auerbach and Kotlikoff (1987) use simulations to investigate a much more realistic overlapping-generations model.

L_t individuals are born in period t. As before, population grows at rate n; thus $L_t = (1+n)L_{t-1}$. Since individuals live for two periods, at time t there are L_t individuals in the first period of their lives and $L_{t-1} = L_t/(1+n)$ individuals in their second periods. Each individual supplies 1 unit of labor when he or she is young and divides the resulting labor income between first-period consumption and saving. In the second period, the individual simply consumes the saving and any interest he or she earns.

Let C_{1t} and C_{2t} denote the consumption in period t of young and old individuals. Thus the utility of an individual born at t, denoted U_t, depends on C_{1t} and C_{2t+1}. We again assume constant-relative-risk-aversion utility:

$$U_t = \frac{C_{1t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_{2t+1}^{1-\theta}}{1-\theta}, \qquad \theta > 0, \qquad \rho > -1. \tag{2.43}$$

As before, this functional form is needed for balanced growth. Because lifetimes are finite, we no longer have to assume $\rho > n + (1-\theta)g$ to ensure that lifetime utility does not diverge. If $\rho > 0$, individuals place greater weight on first-period than second-period consumption; if $\rho < 0$, the situation is reversed. The assumption $\rho > -1$ ensures that the weight on second-period consumption is positive.

Production is described by the same assumptions as before. There are many firms, each with the production function $Y_t = F(K_t, A_t L_t)$. $F(\bullet)$ again has constant returns to scale and satisfies the Inada conditions, and A again grows at exogenous rate g (so $A_t = [1+g]A_{t-1}$). Markets are competitive; thus labor and capital earn their marginal products, and firms earn zero profits. As in the first part of the chapter, there is no depreciation. The real interest rate and the wage per unit of effective labor are therefore given as before by $r_t = f'(k_t)$ and $w_t = f(k_t) - k_t f'(k_t)$. Finally, there is some initial capital stock K_0 that is owned equally by all old individuals.

Thus, in period 0 the capital owned by the old and the labor supplied by the young are combined to produce output. Capital and labor are paid their marginal products. The old consume both their capital income and their existing wealth; they then die and exit the model. The young divide their labor income, $w_t A_t$, between consumption and saving. They carry their saving forward to the next period; thus the capital stock in period $t+1$, K_{t+1}, equals the number of young individuals in period t, L_t, times each of these individuals' saving, $w_t A_t - C_{1t}$. This capital is combined with the labor supplied by the next generation of young individuals, and the process continues.

2.9 Household Behavior

The second-period consumption of an individual born at t is

$$C_{2t+1} = (1 + r_{t+1})(w_t A_t - C_{1t}). \tag{2.44}$$

Dividing both sides of this expression by $1+r_{t+1}$ and bringing C_{1t} over to the left-hand side yields the individual's budget constraint:

$$C_{1t}+\frac{1}{1+r_{t+1}}C_{2t+1}=A_tw_t. \tag{2.45}$$

This condition states that the present value of lifetime consumption equals initial wealth (which is zero) plus the present value of lifetime labor income (which is A_tw_t).

The individual maximizes utility, (2.43), subject to the budget constraint, (2.45). We will consider two ways of solving this maximization problem. The first is to proceed along the lines of the intuitive derivation of the Euler equation for the Ramsey model in (2.22)–(2.23). Because the Diamond model is in discrete time, the intuitive derivation of the Euler equation is much easier here than in the Ramsey model. Specifically, imagine the individual decreasing C_{1t} by a small (formally, infinitesimal) amount ΔC and then using the additional saving and capital income to raise C_{2t+1} by $(1+r_{t+1})\Delta C$. This change does not affect the present value of the individual's lifetime consumption stream. Thus if the individual is optimizing, the utility cost and benefit of the change must be equal. If the cost is less than the benefit, the individual can increase lifetime utility by making the change. And if the cost exceeds the benefit, the individual can increase utility by making the opposite change.

The marginal contributions of C_{1t} and C_{2t+1} to lifetime utility are $C_{1t}^{-\theta}$ and $[1/(1+\rho)]C_{2t+1}^{-\theta}$, respectively. Thus as we let ΔC approach 0, the utility cost of the change approaches $C_{1t}^{-\theta}\Delta C$ and the utility benefit approaches $[1/(1+\rho)]C_{2t+1}^{-\theta}(1+r_{t+1})\Delta C$. As just described, these are equal when the individual is optimizing. Thus optimization requires

$$C_{1t}^{-\theta}\Delta C=\frac{1}{1+\rho}C_{2t+1}^{-\theta}(1+r_{t+1})\,\Delta C. \tag{2.46}$$

Canceling the ΔC's and multiplying both sides by C_{2t+1}^{θ} gives us

$$\frac{C_{2t+1}^{\theta}}{C_{1t}^{\theta}}=\frac{1+r_{t+1}}{1+\rho}, \tag{2.47}$$

or

$$\frac{C_{2t+1}}{C_{1t}}=\left(\frac{1+r_{t+1}}{1+\rho}\right)^{1/\theta}. \tag{2.48}$$

This condition and the budget constraint describe the individual's behavior.

Expression (2.48) is analogous to equation (2.21) in the Ramsey model. It implies that whether an individual's consumption is increasing or decreasing over time depends on whether the real rate of return is greater or less than the discount rate. θ again determines how much individuals' consumption varies in response to differences between r and ρ.

The second way to solve the individual's maximization problem is to set up the Lagrangian:

$$\mathcal{L}=\frac{C_{1t}^{1-\theta}}{1-\theta}+\frac{1}{1+\rho}\frac{C_{2t+1}^{1-\theta}}{1-\theta}+\lambda\left[A_tw_t-\left(C_{1t}+\frac{1}{1+r_{t+1}}C_{2t+1}\right)\right]. \tag{2.49}$$

The first-order conditions for C_{1t} and C_{2t+1} are

$$C_{1t}^{-\theta}=\lambda, \tag{2.50}$$

$$\frac{1}{1+\rho}C_{2t+1}^{-\theta}=\frac{1}{1+r_{t+1}}\lambda. \tag{2.51}$$

Substituting the first equation into the second yields

$$\frac{1}{1+\rho}C_{2t+1}^{-\theta}=\frac{1}{1+r_{t+1}}C_{1t}^{-\theta}. \tag{2.52}$$

This can be rearranged to obtain (2.48). As before, this condition and the budget constraint characterize utility-maximizing behavior.

We can use the Euler equation and the budget constraint to express C_{1t} in terms of labor income and the real interest rate. Specifically, multiplying both sides of (2.48) by C_{1t} and substituting into the budget constraint gives

$$C_{1t}+\frac{(1+r_{t+1})^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta}}C_{1t}=A_tw_t. \tag{2.53}$$

This implies

$$C_{1t}=\frac{(1+\rho)^{1/\theta}}{(1+\rho)^{1/\theta}+(1+r_{t+1})^{(1-\theta)/\theta}}A_tw_t. \tag{2.54}$$

Equation (2.54) shows that the interest rate determines the fraction of income the individual consumes in the first period. If we let $s(r)$ denote the fraction of income saved, (2.54) implies

$$s(r)=\frac{(1+r)^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta}+(1+r)^{(1-\theta)/\theta}}. \tag{2.55}$$

We can therefore rewrite (2.54) as

$$C_{1t}=[1-s(r_{t+1})]A_tw_t. \tag{2.56}$$

Equation (2.55) implies that young individuals' saving is increasing in r if and only if $(1+r)^{(1-\theta)/\theta}$ is increasing in r. The derivative of $(1+r)^{(1-\theta)/\theta}$ with respect to r is $[(1-\theta)/\theta](1+r)^{(1-2\theta)/\theta}$. Thus s is increasing in r if θ is less than 1, and decreasing if θ is greater than 1. Intuitively, a rise in r has both an income and a substitution effect. The fact that the trade-off between consumption in the two periods has become more favorable for second-period consumption tends to increase saving (the substitution effect), but the fact that a given amount of saving yields more second-period consumption tends to decrease saving (the income effect). When individuals are very willing to substitute consumption between the two periods to take advantage of rate-of-return incentives (that is, when θ is low), the substitution effect dominates. When individuals have strong preferences for similar levels of consumption in the two periods (that is, when θ is high), the income effect dominates. And in the special case of $\theta = 1$ (logarithmic utility), the two effects balance, and young individuals' saving rate is independent of r.

2.10 The Dynamics of the Economy

The Equation of Motion of k

As in the infinite-horizon model, we can aggregate individuals' behavior to characterize the dynamics of the economy. As described above, the capital stock in period $t+1$ is the amount saved by young individuals in period t. Thus,

$$K_{t+1} = s(r_{t+1})L_t A_t w_t. \tag{2.57}$$

Note that because saving in period t depends on labor income that period and on the return on capital that savers expect the next period, it is w in period t and r in period $t+1$ that enter the expression for the capital stock in period $t+1$.

Dividing both sides of (2.57) by $L_{t+1}A_{t+1}$ gives us an expression for $K_{t+1}/(A_{t+1}L_{t+1})$, capital per unit of effective labor:

$$k_{t+1} = \frac{1}{(1+n)(1+g)} s(r_{t+1})w_t. \tag{2.58}$$

We can then substitute for r_{t+1} and w_t to obtain

$$k_{t+1} = \frac{1}{(1+n)(1+g)} s(f'(k_{t+1}))[f(k_t) - k_t f'(k_t)]. \tag{2.59}$$

The Evolution of k

Equation (2.59) implicitly defines k_{t+1} as a function of k_t. (It defines k_{t+1} only implicitly because k_{t+1} appears on the right-hand side as well as the left-hand side.) It therefore determines how k evolves over time given its initial value. A value of k_t such that $k_{t+1} = k_t$ satisfies (2.59) is a balanced-growth-path value of k: once k reaches that value, it remains there. We therefore want to know whether there is a balanced-growth-path value (or values) of k, and whether k converges to such a value if it does not begin at one.

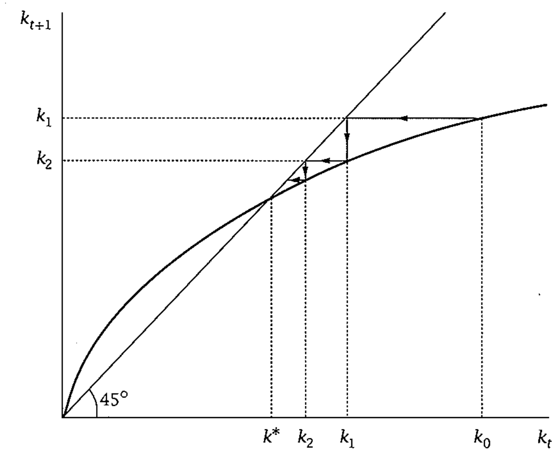


FIGURE 2.11 The dynamics of k

To answer these questions, we need to describe how k_{t+1} depends on k_t. Unfortunately, we can say relatively little about this for the general case. We therefore begin by considering the case of logarithmic utility and Cobb–Douglas production. With these assumptions, (2.59) takes a particularly simple form. We then briefly discuss what occurs when these assumptions are relaxed.

Logarithmic Utility and Cobb–Douglas Production

When θ is 1, the fraction of labor income saved is $1/(2+\rho)$ (see equation [2.55]). And when production is Cobb–Douglas, $f(k)$ is k^α and w is $(1-\alpha)k^\alpha$. Equation (2.59) therefore becomes

$$k_{t+1} = \frac{1}{(1+n)(1+g)} \frac{1}{2+\rho}(1-\alpha)k_t^\alpha. \tag{2.60}$$

Figure 2.11 shows k_{t+1} as a function of k_t. A point where the k_{t+1} function intersects the 45-degree line is a point where k_{t+1} equals k_t. In the case we are considering, k_{t+1} equals k_t at $k_t = 0$; it rises above k_t when k_t is small; and it then crosses the 45-degree line and remains below. There is thus a unique balanced-growth-path level of k (aside from $k = 0$), which is denoted k^*.

k^* is globally stable: wherever k starts (other than at 0), it converges to k^*. Suppose, for example, that the initial value of k, k_0, is greater than k^*.

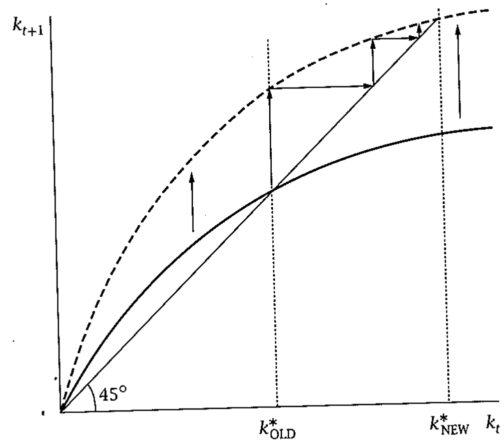


FIGURE 2.12 The effects of a fall in the discount rate

Because k_{t+1} is less than k_t when k_t exceeds k^*, k_1 is less than k_0. And because k_0 exceeds k^* and k_{t+1} is increasing in k_t, k_1 is larger than k^*. Thus k_1 is between k^* and k_0: k moves partway toward k^*. This process is repeated each period, and so k converges smoothly to k^*. A similar analysis applies when k_0 is less than k^*.

These dynamics are shown by the arrows in Figure 2.11. Given k_0, the height of the k_{t+1} function shows k_1 on the vertical axis. To find k_2, we first need to find k_1 on the horizontal axis; to do this, we move across to the 45-degree line. The height of the k_{t+1} function at this point then shows k_2, and so on.

The properties of the economy once it has converged to its balanced growth path are the same as those of the Solow and Ramsey economies on their balanced growth paths: the saving rate is constant, output per worker is growing at rate g, the capital-output ratio is constant, and so on.

To see how the economy responds to shocks, consider our usual example of a fall in the discount rate, ρ, when the economy is initially on its balanced growth path. The fall in the discount rate causes the young to save a greater fraction of their labor income. Thus the k_{t+1} function shifts up. This is depicted in Figure 2.12. The upward shift of the k_{t+1} function increases k^*, the value of k on the balanced growth path. As the figure shows, k rises monotonically from the old value of k^* to the new one.

Thus the effects of a fall in the discount rate in the Diamond model in the case we are considering are similar to its effects in the Ramsey–Cass–Koopmans model, and to the effects of a rise in the saving rate in the Solow model. The change shifts the paths over time of output and capital per worker permanently up, but it leads only to temporary increases in the growth rates of these variables.

The Speed of Convergence

Once again, we may be interested in the model's quantitative as well as qualitative implications. In the special case we are considering, we can solve for the balanced-growth-path values of k and y. Equation (2.60) gives k_{t+1} as a function of k_t. The economy is on its balanced growth path when these two are equal. That is, k^* is defined by

$$k^* = \frac{1}{(1+n)(1+g)}\frac{1}{2+\rho}(1-\alpha)k^{*\alpha}. \tag{2.61}$$

Solving this expression for k^* yields

$$k^* = \left[\frac{1-\alpha}{(1+n)(1+g)(2+\rho)}\right]^{1/(1-\alpha)}. \tag{2.62}$$

Since y equals k^α, this implies

$$y^* = \left[\frac{1-\alpha}{(1+n)(1+g)(2+\rho)}\right]^{\alpha/(1-\alpha)}. \tag{2.63}$$

This expression shows how the model's parameters affect output per unit of effective labor on the balanced growth path. If we want to, we can choose values for the parameters and obtain quantitative predictions about the long-run effects of various developments.[23]

We can also find how quickly the economy converges to the balanced growth path. To do this, we again linearize around the balanced growth path. That is, we replace the equation of motion for k, (2.60), with a first-order approximation around $k = k^*$. We know that when k_t equals k^*, k_{t+1} also equals k^*. Thus,

$$k_{t+1} \simeq k^* + \left(\left.\frac{dk_{t+1}}{dk_t}\right|_{k_t=k^*}\right)(k_t - k^*). \tag{2.64}$$

Let λ denote dk_{t+1}/dk_t evaluated at $k_t = k^*$. With this definition, we can rewrite (2.64) as $k_{t+1} - k^* \simeq \lambda(k_t - k^*)$. This implies

$$k_t - k^* \simeq \lambda^t(k_0 - k^*), \tag{2.65}$$

where k_0 is the initial value of k.

[23] In choosing parameter values, it is important to keep in mind that individuals are assumed to live for only two periods. Thus, for example, n should be thought of as population growth not over a year, but over half a lifetime.

The convergence to the balanced growth path is determined by λ. If λ is between 0 and 1, the system converges smoothly. If λ is between -1 and 0, there are damped oscillations toward k^*: k alternates between being greater and less than k^*, but each period it gets closer. If λ is greater than 1, the system explodes. Finally, if λ is less than -1, there are explosive oscillations.

To find λ, we return to (2.60): $k_{t+1} = (1-\alpha)k_t^\alpha/[(1+n)(1+g)(2+\rho)]$. Thus,

$$\begin{aligned}\lambda \equiv \left.\frac{dk_{t+1}}{dk_t}\right|_{k_t=k^*} &= \alpha\frac{1-\alpha}{(1+n)(1+g)(2+\rho)}k^{*\alpha-1}\\ &= \alpha\frac{1-\alpha}{(1+n)(1+g)(2+\rho)}\left[\frac{1-\alpha}{(1+n)(1+g)(2+\rho)}\right]^{(\alpha-1)/(1-\alpha)}\\ &= \alpha,\end{aligned} \tag{2.66}$$

where the second line uses equation (2.62) to substitute for k^*. That is, λ is simply α, capital's share.

Since α is between 0 and 1, this analysis implies that k converges smoothly to k^*. If α is one-third, for example, k moves two-thirds of the way toward k^* each period.[24]

The rate of convergence in the Diamond model differs from that in the Solow model (and in a discrete-time version of the Solow model—see Problem 2.14). The reason is that although the saving of the young is a constant fraction of their income and their income is a constant fraction of total income, the dissaving of the old is not a constant fraction of total income. The dissaving of the old as a fraction of output is $K_t/F(K_t, A_tL_t)$, or $k_t/f(k_t)$. The fact that there are diminishing returns to capital implies that this ratio is increasing in k. Since this term enters negatively into saving, it follows that total saving as a fraction of output is a decreasing function of k. Thus total saving as a fraction of output is above its balanced-growth-path value when $k < k^*$, and is below when $k > k^*$. As a result, convergence is more rapid than in the Solow model.

The General Case

Let us now relax the assumptions of logarithmic utility and Cobb–Douglas production. It turns out that, despite the simplicity of the model, a wide range of behaviors of the economy are possible. Rather than attempting a comprehensive analysis, we simply discuss some of the more interesting cases.

[24] Recall, however, that each period in the model corresponds to half of a person's lifetime.

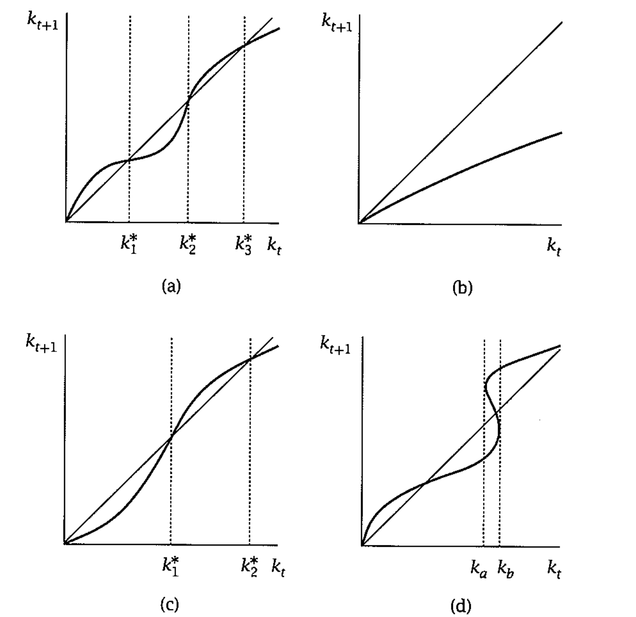


FIGURE 2.13 Various possibilities for the relationship between k_t and k_{t+1}

To understand the possibilities intuitively, it is helpful to rewrite the equation of motion, (2.59), as

$$k_{t+1} = \frac{1}{(1+n)(1+g)}s(f'(k_{t+1}))\frac{f(k_t)-k_tf'(k_t)}{f(k_t)}f(k_t). \tag{2.67}$$

Equation (2.67) expresses capital per unit of effective labor in period $t+1$ as the product of four terms. From right to left, those four terms are the following: output per unit of effective labor at t, the fraction of that output that is paid to labor, the fraction of that labor income that is saved, and the ratio of the amount of effective labor in period t to the amount in period $t+1$.

Figure 2.13 shows some possible forms for the relation between k_{t+1} and k_t other than the well-behaved case shown in Figure 2.11. Panel (a) shows a case with multiple values of k^*. In the case shown, k_1^* and k_3^* are stable: if k starts slightly away from one of these points, it converges to that level. k_2^* is unstable (as is $k = 0$). If k starts slightly below k_2^*, then k_{t+1} is less than k_t each period, and so k converges to k_1^*. If k begins slightly above k_2^*, it converges to k_3^*.

To understand the possibility of multiple values of k^*, note that since output per unit of capital is lower when k is higher (capital has a diminishing marginal product), for there to be two k^*'s the saving of the young as a fraction of total output must be higher at the higher k^*. When the fraction of output going to labor and the fraction of labor income saved are constant, the saving of the young is a constant fraction of total output, and so multiple k^*'s are not possible. This is what occurs with Cobb–Douglas production and logarithmic utility. But if labor's share is greater at higher levels of k (which occurs if $f(\bullet)$ is more sharply curved than in the Cobb–Douglas case) or if workers save a greater fraction of their income when the rate of return is lower (which occurs if $\theta > 1$), or both, there may be more than one level of k at which saving reproduces the existing capital stock.

Panel (b) shows a case in which k_{t+1} is always less than k_t, and in which k therefore converges to zero regardless of its initial value. What is needed for this to occur is for either labor's share or the fraction of labor income saved (or both) to approach zero as k approaches zero.

Panel (c) shows a case in which k converges to zero if its initial value is sufficiently low, but to a strictly positive level if its initial value is sufficiently high. Specifically, if $k_0 < k_1^*$, then k approaches zero; if $k_0 > k_1^*$, then k converges to k_2^*.

Finally, Panel (d) shows a case in which k_{t+1} is not uniquely determined by k_t: when k_t is between k_a and k_b, there are three possible values of k_{t+1}. This can happen if saving is a decreasing function of the interest rate. When saving is decreasing in r, saving is high if individuals expect a high value of k_{t+1} and therefore expect r to be low, and is low when individuals expect a low value of k_{t+1}. If saving is sufficiently responsive to r, and if r is sufficiently responsive to k, there can be more than one value of k_{t+1} that is consistent with a given k_t. Thus the path of the economy is indeterminate: equation (2.59) (or [2.67]) does not fully determine how k evolves over time given its initial value. This raises the possibility that *self-fulfilling prophecies* and *sunspots* can affect the behavior of the economy and that the economy can exhibit fluctuations even though there are no exogenous disturbances. Depending on precisely what is assumed, various dynamics are possible.[25]

Thus assuming that there are overlapping generations rather than infinitely lived households has potentially important implications for the dynamics of the economy: for example, sustained growth may not be possible, or it may depend on initial conditions.

At the same time, the model does no better than the Solow and Ramsey models at answering our basic questions about growth. Because of the Inada conditions, k_{t+1} must be less than k_t for k_t sufficiently large. Specifically,

[25] These issues are briefly discussed further in Section 6.7.

since the saving of the young cannot exceed the economy's total output, k_{t+1} cannot be greater than $f(k_t)/[(1+n)(1+g)]$. And because the marginal product of capital approaches zero as k becomes large, this must eventually be less than k_t. The fact that k_{t+1} is eventually less than k_t implies that unbounded growth of k is not possible. Thus, once again, growth in the effectiveness of labor is the only potential source of long-run growth in output per worker. Because of the possibility of multiple k^*'s, the model does imply that otherwise identical economies can converge to different balanced growth paths simply because of differences in their initial conditions. But, as in the Solow and Ramsey models, we can account for quantitatively large differences in output per worker in this way only by positing immense differences in capital per worker and in rates of return.

2.11 The Possibility of Dynamic Inefficiency

The one major difference between the balanced growth paths of the Diamond and Ramsey–Cass–Koopmans models involves welfare. We saw that the equilibrium of the Ramsey–Cass–Koopmans model maximizes the welfare of the representative household. In the Diamond model, individuals born at different times attain different levels of utility, and so the appropriate way to evaluate social welfare is not clear. If we specify welfare as some weighted sum of the utilities of different generations, there is no reason to expect the decentralized equilibrium to maximize welfare, since the weights we assign to the different generations are arbitrary.

A minimal criterion for efficiency, however, is that the equilibrium be Pareto-efficient. It turns out that the equilibrium of the Diamond model need not satisfy even this standard. In particular, the capital stock on the balanced growth path of the Diamond model may exceed the golden-rule level, so that a permanent increase in consumption is possible.

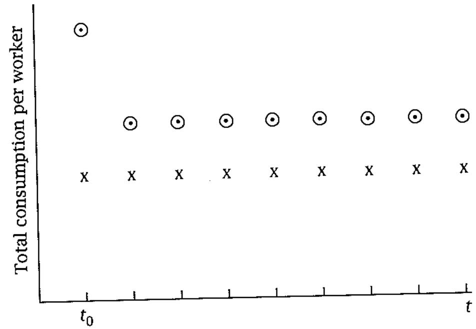
To see this possibility as simply as possible, assume that utility is logarithmic, production is Cobb–Douglas, and g is zero. With $g = 0$, equation (2.62) for the value of k on the balanced growth path simplifies to

$$k^* = \left[\frac{1}{1+n}\frac{1}{2+\rho}(1-\alpha)\right]^{1/(1-\alpha)}. \tag{2.68}$$

Thus the marginal product of capital on the balanced growth path, $\alpha k^{*\alpha-1}$, is

$$f'(k^*) = \frac{\alpha}{1-\alpha}(1+n)(2+\rho). \tag{2.69}$$

The golden-rule capital stock is defined by $f'(k_{GR}) = n$. $f'(k^*)$ can be either more or less than $f'(k_{GR})$. In particular, for α sufficiently small, $f'(k^*)$ is



X maintaining k at $k^* > k_{GR}$

⊙ reducing k to k_{GR} in period t_0

FIGURE 2.14 How reducing k to the golden-rule level affects the path of consumption per worker

less than $f'(k_{GR})$—the capital stock on the balanced growth path exceeds the golden-rule level.

To see why it is inefficient for k^* to exceed k_{GR}, imagine introducing a social planner into a Diamond economy that is on its balanced growth path with $k^* > k_{GR}$. If the planner does nothing to alter k, the amount of output per worker available each period for consumption is output, $f(k^*)$, minus the new investment needed to maintain k at k^*, nk^*. This is shown by the crosses in Figure 2.14. Suppose instead, however, that in some period, period t_0, the planner allocates more resources to consumption and fewer to saving than usual, so that capital per worker the next period is k_{GR}, and that thereafter he or she maintains k at k_{GR}. Under this plan, the resources per worker available for consumption in period t_0 are $f(k^*)+(k^*-k_{GR})-nk_{GR}$. In each subsequent period, the output per worker available for consumption is $f(k_{GR})-nk_{GR}$. Since k_{GR} maximizes $f(k)-nk$, $f(k_{GR})-nk_{GR}$ exceeds $f(k^*)-nk^*$. And since k^* is greater than k_{GR}, $f(k^*)+(k^*-k_{GR})-nk_{GR}$ is even larger than $f(k_{GR})-nk_{GR}$. The path of total consumption under this policy is shown by the circles in Figure 2.14. As the figure shows, this policy makes more resources available for consumption in every period than the policy of maintaining k at k^*. The planner can therefore allocate consumption between the young and the old each period to make every generation better off.

Thus the equilibrium of the Diamond model can be Pareto-inefficient. This may seem puzzling: given that markets are competitive and there are no externalities, how can the usual result that equilibria are Pareto-efficient

fail? The reason is that the standard result assumes not only competition and an absence of externalities, but also a finite number of agents. Specifically, the possibility of inefficiency in the Diamond model stems from the fact that the infinity of generations gives the planner a means of providing for the consumption of the old that is not available to the market. If individuals in the market economy want to consume in old age, their only choice is to hold capital, even if its rate of return is low. The planner, however, need not have the consumption of the old determined by the capital stock and its rate of return. Instead, he or she can divide the resources available for consumption between the young and old in any manner. The planner can take, for example, 1 unit of labor income from each young person and transfer it to the old. Since there are $1+n$ young people for each old person, this increases the consumption of each old person by $1+n$ units. The planner can prevent this change from making anyone worse off by requiring the next generation of young to do the same thing in the following period, and then continuing this process every period. If the marginal product of capital is less than n—that is, if the capital stock exceeds the golden-rule level—this way of transferring resources between youth and old age is more efficient than saving, and so the planner can improve on the decentralized allocation.

Because this type of inefficiency differs from conventional sources of inefficiency, and because it stems from the intertemporal structure of the economy, it is known as *dynamic inefficiency*.[26]

Empirical Application: Are Modern Economies Dynamically Efficient?

The Diamond model shows that it is possible for a decentralized economy to accumulate capital beyond the golden-rule level, and thus to produce an allocation that is Pareto-inefficient. Given that capital accumulation in actual economies is not dictated by social planners, this raises the issue of whether actual economies might be dynamically inefficient. If they were, there would be important implications for public policy: the great concern about low rates of saving would be entirely misplaced, and it would be easy to increase both present and future consumption.

This issue is addressed by Abel, Mankiw, Summers, and Zeckhauser (1989). They start by observing that at first glance, dynamic inefficiency appears to be a possibility for the United States and other major economies. A balanced growth path is dynamically inefficient if the real rate of return, $f'(k^*)-\delta$, is less than the growth rate of the economy. A straightforward measure of the real rate of return is the real interest rate on short-term government debt. Abel et al. report that in the United States over the

[26] Problem 2.19 investigates the sources of dynamic inefficiency further.

period 1926–1986, this interest rate averaged only a few tenths of a percent, much less than the average growth rate of the economy. Similar findings hold for other major industrialized countries. Thus the real interest rate is less than the golden-rule level, suggesting that these economies have overaccumulated capital.

As Abel et al. point out, however, there is a problem with this argument. In a world of certainty, all interest rates must be equal; thus there is no ambiguity in what is meant by "the" rate of return. But if there is uncertainty, different assets can have different expected returns. Suppose, for example, we assess dynamic efficiency by examining the marginal product of capital net of depreciation instead of the return on a fairly safe asset. If capital earns its marginal product, the net marginal product can be estimated as the ratio of overall capital income minus total depreciation to the value of the capital stock. For the United States, this ratio is about 10 percent, which is much greater than the economy's growth rate. Thus using this approach, we would conclude that the U.S. economy is dynamically efficient. Our simple theoretical model, in which the marginal product of capital and the safe interest rate are the same, provides no guidance concerning which of these contradictory conclusions is correct.

Abel et al. therefore tackle the issue of how to assess dynamic efficiency in a world of uncertainty. Their principal theoretical result is that under uncertainty, a sufficient condition for dynamic efficiency is that net capital income exceed investment. For the balanced growth path of an economy with certainty, this condition is the same as the usual comparison of the real interest rate with the economy's growth rate. In this case, net capital income is the real interest rate times the stock of capital, and investment is the growth rate of the economy times the stock of capital. Thus capital income exceeds investment if and only if the real interest rate exceeds the economy's growth rate. But Abel et al. show that under uncertainty these two conditions are not equivalent, and that it is the comparison of capital income and investment that provides the correct way of judging whether there is dynamic efficiency. Intuitively, a capital sector that is on net making resources available by producing more output than it is using for new investment is contributing to consumption, whereas one that is using more resources than it is producing is not.

Abel et al.'s principal empirical result is that the condition for dynamic efficiency seems to be satisfied in practice. They measure capital income as national income minus employees' compensation and the part of the income of the self-employed that appears to represent labor income;[27] investment is taken directly from the national income accounts. They find that for the period 1929–1985, capital income consistently exceeds investment in the

[27] They argue that adjusting these figures to account for land income and monopoly rents does not change the basic results.

United States and in the six other major industrialized countries they consider. Even in Japan, where investment is remarkably high, the profit rate is so great that the returns to capital comfortably exceed investment. Thus, although decentralized economies can produce dynamically inefficient outcomes in principle, they do not appear to in practice.

2.12 Government in the Diamond Model

As in the infinite-horizon model, it is natural to ask what happens in the Diamond model if we introduce a government that makes purchases and levies taxes. For simplicity, we focus on the case of logarithmic utility and Cobb–Douglas production.

Let G_t denote the government's purchases of goods per unit of effective labor in period t. We again assume that it finances those purchases by lump-sum taxes on the young.

When the government finances its purchases entirely with taxes, workers' after-tax income in period t is $(1-\alpha)k_t^\alpha - G_t$ rather than $(1-\alpha)k_t^\alpha$. The equation of motion for k, equation (2.60), therefore becomes

$$k_{t+1} = \frac{1}{(1+n)(1+g)}\frac{1}{2+\rho}[(1-\alpha)k_t^\alpha - G_t]. \tag{2.70}$$

A higher G_t therefore reduces k_{t+1} for a given k_t.

To see the effects of government purchases, suppose that the economy is on a balanced growth path with G constant, and that G increases permanently. From (2.70), this shifts the k_{t+1} function down; this is shown in Figure 2.15. The downward shift of the k_{t+1} function reduces k^*. Thus—in contrast to what occurs in the infinite-horizon model—higher government purchases lead to a lower capital stock and a higher real interest rate. Intuitively, since individuals live for two periods, they reduce their first-period consumption less than one-for-one with the increase in G. But since taxes are levied only in the first period of life, this means that their saving falls. As usual, the economy moves smoothly from the initial balanced growth path to the new one.

As a second example, consider a temporary increase in government purchases from G_L to G_H, again with the economy initially on its balanced growth path. The dynamics of k are thus described by (2.70) with $G = G_H$ during the period that government purchases are high and by (2.70) with $G = G_L$ before and after. That is, the fact that individuals know that government purchases will return to G_L does not affect the behavior of the economy during the time that purchases are high. The saving of the young—and hence next period's capital stock—is determined by their after-tax labor income, which is determined by the current capital stock and by the government's current purchases. Thus during the time that government purchases

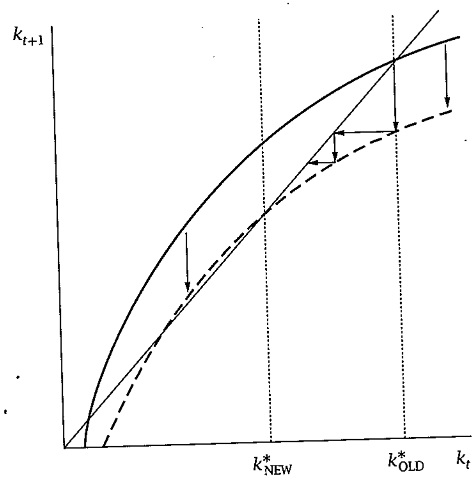


FIGURE 2.15 The effects of a permanent increase in government purchases

are high, k gradually falls and r gradually increases. Once G returns to G_L, k rises gradually back to its initial level.[28]

Problems

2.1. Consider N firms each with the constant-returns-to-scale production function $Y = F(K, AL)$, or (using the intensive form) $Y = ALf(k)$. Assume $f'(\bullet) > 0$, $f''(\bullet) < 0$. Assume that all firms can hire labor at wage wA and rent capital at cost r, and that all firms have the same value of A.

(*a*) Consider the problem of a firm trying to produce Y units of output at minimum cost. Show that the cost-minimizing level of k is uniquely defined and is independent of Y, and that all firms therefore choose the same value of k.

(*b*) Show that the total output of the N cost-minimizing firms equals the output that a single firm with the same production function has if it uses all the labor and capital used by the N firms.

[28] The result that future values of G do not affect the current behavior of the economy does not depend on the assumption of logarithmic utility. Without logarithmic utility, the saving of the current period's young depends on the rate of return as well as on after-tax labor income. But the rate of return is determined by the next period's capital-labor ratio, which is not affected by government purchases in that period.

2.2. **The elasticity of substitution with constant-relative-risk-aversion utility.** Consider an individual who lives for two periods and whose utility is given by equation (2.43). Let P_1 and P_2 denote the prices of consumption in the two periods, and let W denote the value of the individual's lifetime income; thus the budget constraint is $P_1C_1 + P_2C_2 = W$.

(*a*) What are the individual's utility-maximizing choices of C_1 and C_2, given P_1, P_2, and W?

(*b*) The elasticity of substitution between consumption in the two periods is $-[(P_1/P_2)/(C_1/C_2)][\partial(C_1/C_2)/\partial(P_1/P_2)]$, or $-\partial \ln(C_1/C_2)/\partial \ln(P_1/P_2)$. Show that with the utility function (2.43), the elasticity of substitution between C_1 and C_2 is $1/\theta$.

2.3. (*a*) Suppose it is known in advance that at some time t_0 the government will confiscate half of whatever wealth each household holds at that time. Does consumption change discontinuously at time t_0? If so, why (and what is the condition relating consumption immediately before t_0 to consumption immediately after)? If not, why not?

(*b*) Suppose it is known in advance that at t_0 the government will confiscate from each household an amount of wealth equal to half of the wealth of the average household at that time. Does consumption change discontinuously at time t_0? If so, why (and what is the condition relating consumption immediately before t_0 to consumption immediately after)? If not, why not?

2.4. Assume that the instantaneous utility function $u(C)$ in equation (2.1) is $\ln C$. Consider the problem of a household maximizing (2.1) subject to (2.6). Find an expression for C at each time as a function of initial wealth plus the present value of labor income, the path of $r(t)$, and the parameters of the utility function.

2.5. Consider a household with utility given by (2.1)–(2.2). Assume that the real interest rate is constant, and let W denote the household's initial wealth plus the present value of its lifetime labor income (the right-hand side of [2.6]). Find the utility-maximizing path of C, given r, W, and the parameters of the utility function.

2.6. **The productivity slowdown and saving.** Consider a Ramsey–Cass–Koopmans economy that is on its balanced growth path, and suppose there is a permanent fall in g.

(*a*) How, if at all, does this affect the $\dot{k} = 0$ curve?

(*b*) How, if at all, does this affect the $\dot{c} = 0$ curve?

(*c*) What happens to c at the time of the change?

(*d*) Find an expression for the impact of a marginal change in g on the fraction of output that is saved on the balanced growth path. Can one tell whether this expression is positive or negative?

(*e*) For the case where the production function is Cobb-Douglas, $f(k) = k^{\alpha}$, rewrite your answer to part (*d*) in terms of ρ, n, g, θ, and α. (Hint: Use the fact that $f'(k^*) = \rho + \theta g$.)

2.7. Describe how each of the following affects the $\dot{c} = 0$ and $\dot{k} = 0$ curves in Figure 2.5, and thus how they affect the balanced-growth-path values of c and k:

(*a*) A rise in θ.

(*b*) A downward shift of the production function.

(*c*) A change in the rate of depreciation from the value of zero assumed in the text to some positive level.

2.8. Derive an expression analogous to (2.39) for the case of a positive depreciation rate.

2.9. Capital taxation in the Ramsey–Cass–Koopmans model. Consider a Ramsey–Cass–Koopmans economy that is on its balanced growth path. Suppose that at some time, which we will call time 0, the government switches to a policy of taxing investment income at rate τ. Thus the real interest rate that households face is now given by $r(t) = (1-\tau)f'(k(t))$. Assume that the government returns the revenue it collects from this tax through lump-sum transfers. Finally, assume that this change in tax policy is unanticipated.

(*a*) How does the tax affect the $\dot{c} = 0$ locus? The $\dot{k} = 0$ locus?

(*b*) How does the economy respond to the adoption of the tax at time 0? What are the dynamics after time 0?

(*c*) How do the values of c and k on the new balanced growth path compare with their values on the old balanced growth path?

(*d*) (This is based on Barro, Mankiw, and Sala-i-Martin, 1995.) Suppose there are many economies like this one. Workers' preferences are the same in each country, but the tax rates on investment income may vary across countries. Assume that each country is on its balanced growth path.

(*i*) Show that the saving rate on the balanced growth path, $(y^* - c^*)/y^*$, is decreasing in τ.

(*ii*) Do citizens in low-τ, high-k^*, high-saving countries have any incentive to invest in low-saving countries? Why or why not?

(*e*) Does your answer to part (*c*) imply that a policy of *subsidizing* investment (that is, making $\tau < 0$), and raising the revenue for this subsidy through lump-sum taxes, increases welfare? Why or why not?

(*f*) How, if at all, do the answers to parts (*a*) and (*b*) change if the government does not rebate the revenue from the tax but instead uses it to make government purchases?

2.10. Using the phase diagram to analyze the impact of an anticipated change. Consider the policy described in Problem 2.9, but suppose that instead of announcing and implementing the tax at time 0, the government announces at time 0 that at some later time, time t_1, investment income will begin to be taxed at rate τ.

(*a*) Draw the phase diagram showing the dynamics of c and k after time t_1.

(*b*) Can c change discontinuously at time t_1? Why or why not?

(*c*) Draw the phase diagram showing the dynamics of c and k before t_1.

(*d*) In light of your answers to parts (*a*), (*b*), and (*c*), what must c do at time 0?

(*e*) Summarize your results by sketching the paths of c and k as functions of time.

2.11. Using the phase diagram to analyze the impact of unanticipated and anticipated temporary changes. Analyze the following two variations on Problem 2.10:

(*a*) At time 0, the government announces that it will tax investment income at rate τ from time 0 until some later date t_1; thereafter investment income will again be untaxed.

(*b*) At time 0, the government announces that from time t_1 to some later time t_2, it will tax investment income at rate τ; before t_1 and after t_2, investment income will not be taxed.

2.12. The analysis of government policies in the Ramsey–Cass–Koopmans model in the text assumes that government purchases do not affect utility from private consumption. The opposite extreme is that government purchases and private consumption are perfect substitutes. Specifically, suppose that the utility function (2.12) is modified to be

$$U = B \int_{t=0}^{\infty} e^{-\beta t} \frac{[c(t) + G(t)]^{1-\theta}}{1-\theta}\, dt.$$

If the economy is initially on its balanced growth path and if households' preferences are given by U, what are the effects of a temporary increase in government purchases on the paths of consumption, capital, and the interest rate?

2.13. Consider the Diamond model with logarithmic utility and Cobb-Douglas production. Describe how each of the following affects k_{t+1} as a function of k_t:

(*a*) A rise in n.

(*b*) A downward shift of the production function (that is, $f(k)$ takes the form Bk^{α}, and B falls).

(*c*) A rise in α.

2.14. A discrete-time version of the Solow model. Suppose $Y_t = F(K_t, A_tL_t)$, with $F(\bullet)$ having constant returns to scale and the intensive form of the production function satisfying the Inada conditions. Suppose also that $A_{t+1} = (1+g)A_t$, $L_{t+1} = (1+n)L_t$, and $K_{t+1} = K_t + sY_t - \delta K_t$.

(*a*) Find an expression for k_{t+1} as a function of k_t.

(*b*) Sketch k_{t+1} as a function of k_t. Does the economy have a balanced growth path? If the initial level of k differs from the value on the balanced growth path, does the economy converge to the balanced growth path?

(c) Find an expression for consumption per unit of effective labor on the balanced growth path as a function of the balanced-growth-path value of k. What is the marginal product of capital, $f'(k)$, when k maximizes consumption per unit of effective labor on the balanced growth path?

(d) Assume that the production function is Cobb–Douglas.

(i) What is k_{t+1} as a function of k_t?

(ii) What is k^*, the value of k on the balanced growth path?

(iii) Along the lines of equations (2.64)–(2.66), in the text, linearize the expression in subpart (i) around $k_t = k^*$, and find the rate of convergence of k to k^*.

2.15. Depreciation in the Diamond model and microeconomic foundations for the Solow model. Suppose that in the Diamond model capital depreciates at rate δ, so that $r_t = f'(k_t) - \delta$.

(a) How, if at all, does this change in the model affect equation (2.59) giving k_{t+1} as a function of k_t?

(b) In the special case of logarithmic utility, Cobb–Douglas production, and $\delta = 1$, what is the equation for k_{t+1} as a function of k_t? Compare this with the analogous expression for the discrete-time version of the Solow model with $\delta = 1$ from part (a) of Problem 2.14.

2.16. Social security in the Diamond model. Consider a Diamond economy where g is zero, production is Cobb–Douglas, and utility is logarithmic.

(a) **Pay-as-you-go social security.** Suppose the government taxes each young individual an amount T and uses the proceeds to pay benefits to old individuals; thus each old person receives $(1+n)T$.

(i) How, if at all, does this change affect equation (2.60) giving k_{t+1} as a function of k_t?

(ii) How, if at all, does this change affect the balanced-growth-path value of k?

(iii) If the economy is initially on a balanced growth path that is dynamically efficient, how does a marginal increase in T affect the welfare of current and future generations? What happens if the initial balanced growth path is dynamically inefficient?

(b) **Fully funded social security.** Suppose the government taxes each young person an amount T and uses the proceeds to purchase capital. Individuals born at t therefore receive $(1 + r_{t+1})T$ when they are old.

(i) How, if at all, does this change affect equation (2.60) giving k_{t+1} as a function of k_t?

(ii) How, if at all, does this change affect the balanced-growth-path value of k?

2.17. The basic overlapping-generations model. (This follows Samuelson, 1958, and Allais, 1947.) Suppose, as in the Diamond model, that L_t two-period-lived individuals are born in period t and that $L_t = (1+n)L_{t-1}$. For simplicity, let utility be logarithmic with no discounting: $U_t = \ln(C_{1t}) + \ln(C_{2t+1})$.

The production side of the economy is simpler than in the Diamond model. Each individual born at time t is endowed with A units of the economy's single good. The good can be either consumed or stored. Each unit stored yields $x > 0$ units of the good in the following period.[29]

Finally, assume that in the initial period, period 0, in addition to the L_0 young individuals each endowed with A units of the good, there are $[1/(1+n)]L_0$ individuals who are alive only in period 0. Each of these "old" individuals is endowed with some amount Z of the good; their utility is simply their consumption in the initial period, C_{20}.

(a) Describe the decentralized equilibrium of this economy. (Hint: Given the overlapping-generations structure, will the members of any generation engage in transactions with members of another generation?)

(b) Consider paths where the fraction of agents' endowments that is stored, f_t, is constant over time. What is total consumption (that is, consumption of all the young plus consumption of all the old) per person on such a path as a function of f? If $x < 1+n$, what value of f satisfying $0 \leq f \leq 1$ maximizes consumption per person? Is the decentralized equilibrium Pareto-efficient in this case? If not, how can a social planner raise welfare?

2.18. Stationary monetary equilibria in the Samuelson overlapping-generations model. (Again this follows Samuelson, 1958.) Consider the setup described in Problem 2.17. Assume that $x < 1+n$. Suppose that the old individuals in period 0, in addition to being endowed with Z units of the good, are each endowed with M units of a storable, divisible commodity, which we will call money. Money is not a source of utility.

(a) Consider an individual born at t. Suppose the price of the good in units of money is P_t in t and P_{t+1} in $t+1$. Thus the individual can sell units of endowment for P_t units of money and then use that money to buy P_t/P_{t+1} units of the next generation's endowment the following period. What is the individual's behavior as a function of P_t/P_{t+1}?

(b) Show that there is an equilibrium with $P_{t+1} = P_t/(1+n)$ for all $t \geq 0$ and no storage, and thus that the presence of "money" allows the economy to reach the golden-rule level of storage.

(c) Show that there are also equilibria with $P_{t+1} = P_t/x$ for all $t \geq 0$.

[29] Note that this is the same as the Diamond economy with $g = 0$, $F(K_t, AL_t) = AL_t + xK_t$, and $\delta = 1$. With this production function, since individuals supply 1 unit of labor when they are young, an individual born in t obtains A units of the good. And each unit saved yields $1 + r = 1 + \partial F(K, AL)/\partial K - \delta = 1 + x - 1 = x$ units of second-period consumption.

(*d*) Finally, explain why $P_t = \infty$ for all t (that is, money is worthless) is also an equilibrium. Explain why this is the *only* equilibrium if the economy ends at some date, as in Problem 2.19(*b*) below. (Hint: Reason backward from the last period.)

2.19. The source of dynamic inefficiency. There are two ways in which the Diamond and Samuelson models differ from textbook models. First, markets are incomplete: because individuals cannot trade with individuals who have not been born, some possible transactions are ruled out. Second, because time goes on forever, there are an infinite number of agents. This problem asks you to investigate which of these is the source of the possibility of dynamic inefficiency. For simplicity, it focuses on the Samuelson overlapping-generations model (see the previous two problems), again with log utility and no discounting. To simplify further, it assumes $n = 0$ and $0 < x < 1$.

(*a*) **Incomplete markets.** Suppose we eliminate incomplete markets from the model by allowing all agents to trade in a competitive market "before" the beginning of time. That is, a Walrasian auctioneer calls out prices $Q_0, Q_1, Q_2, \ldots$ for the good at each date. Individuals can then make sales and purchases at these prices given their endowments and their ability to store. The budget constraint of an individual born at t is thus $Q_tC_{1t} + Q_{t+1}C_{2t+1} = Q_t(A-S_t) + Q_{t+1}xS_t$, where S_t (which must satisfy $0 \le S_t \le A$) is the amount the individual stores.

(*i*) Suppose the auctioneer announces $Q_{t+1} = Q_t/x$ for all $t > 0$. Show that in this case individuals are indifferent concerning how much to store, that there is a set of storage decisions such that markets clear at every date, and that this equilibrium is the same as the equilibrium described in part (*a*) of Problem 2.17.

(*ii*) Suppose the auctioneer announces prices that fail to satisfy $Q_{t+1} = Q_t/x$ at some date. Show that at the first date that does not satisfy this condition the market for the good cannot clear, and thus that the proposed price path cannot be an equilibrium.

(*b*) **Infinite duration.** Suppose that the economy ends at some date T. That is, suppose the individuals born at T live only one period (and hence seek to maximize C_{1T}), and that thereafter no individuals are born. Show that the decentralized equilibrium is Pareto-efficient.

(*c*) In light of these answers, is it incomplete markets or infinite duration that is the source of dynamic inefficiency?

2.20. Explosive paths in the Samuelson overlapping-generations model. (Black, 1974; Brock, 1975; Calvo, 1978a.) Consider the setup described in Problem 2.18. Assume that x is zero, and assume that utility is constant-relative-risk-aversion with $\theta < 1$ rather than logarithmic. Finally, assume for simplicity that $n = 0$.

(*a*) What is the behavior of an individual born at t as a function of P_t/P_{t+1}? Show that the amount of his or her endowment that the individual sells for money is an increasing function of P_t/P_{t+1} and approaches zero as this ratio approaches zero.

(*b*) Suppose $P_0/P_1 < 1$. How much of the good are the individuals born in period 0 planning to buy in period 1 from the individuals born then? What must P_1/P_2 be for the individuals born in period 1 to want to supply this amount?

(*c*) Iterating this reasoning forward, what is the qualitative behavior of P_t/P_{t+1} over time? Does this represent an equilibrium path for the economy?

(*d*) Can there be an equilibrium path with $P_0/P_1 > 1$?