STUDIENZENTRUM GERZENSEE
Doctoral Course 2010
Day 3 Study Problems for Macroeconomics

There are three areas where rational expectations has had major effects on macroeconomic modeling. Problem 1 illustrates applications to pricing dynamics, along New Keynesian lines, as will be extensively discussed in week 4 of the class. Problem 2 illustrates applications to the term structure of interest rates.

A virtue of the model solution approach exposited in Lecture 6 is that the underlying economic model can move from one type of behavior to another without the structural equations of the model being altered (only parameters changed) or the solution method being altered. Problem 3 illustrates this idea in the context of a classic area, labor demand, in which the behavior changes from static to dynamic as a result of a change in a parameter. The connection to singularities in model solution methods is explored. Problem 3 illustrates one manner in which dynamics (forward, backward) are introduced into major macroeconomic policy models.

1. *The New Keynesian pricing equation and forecasting.* Some current macroeconomic theories that inflation is determined by a specification of the form

$$\pi_t = (1-\beta\eta)\sum_{j=0}^{\infty}(\beta\eta)^j E_t\psi_{t+j}$$

where ψ_t is real marginal cost. Suppose that real marginal cost is described by

$$\begin{aligned} \psi_t &= hs_t \\ s_t &= Ms_{t-1} + Ge_t \end{aligned}$$

where e_t is serially uncorrelated.

(a) Show how to forecast ψ_{t+k} given this model.

Answer: The forecast takes the form

$$\begin{aligned} E_t\psi_{t+j} &= hE_t s_{t+j} \\ &= hM^j s_t \end{aligned}$$

(b) What is the rational expectations solution that links π_t to s_t?

Answer: The solution takes the form

$$\begin{aligned} \pi_t &= (1-\beta\eta)\sum_{j=0}^{\infty}(\beta\eta)^j E_t\psi_{t+j} \\ &= (1-\beta\eta)\sum_{j=0}^{\infty}(\beta\eta)^j hM^j s_t \\ &= (1-\beta\eta)h[\sum_{j=0}^{\infty}(\beta\eta)^j M^j]s_t \\ &= (1-\beta\eta)h[I-(\beta\eta)M]^{-1}s_t \end{aligned}$$

2. *Rational expectations theory of the term structure.* Suppose that the long term interest rate R_t^L is related to the short-term interest rate by

$$R_t^L = \theta E_t R_{t+1}^L + (1-\theta)R_t$$

with $0 < \theta < 1$.

(a) Given that the long rate is not predetermined, derive the rational expectations solution for this variable.

Answer: Using recursive forward substitution, we can obtain

$$\begin{aligned}
R_t^L &= \theta E_t R_{t+1}^L + (1-\theta)R_t \\
&= \theta E_t[\theta E_{t+1} R_{t+2}^L + (1-\theta)R_{t+1}] + (1-\theta)R_t \\
&= \theta^2 E_t R_{t+2}^L + (1-\theta)[R_t + \theta E_t R_{t+1}] \\
&= \ldots \\
&= (1-\theta)\sum_{j=0}^{\infty} \theta^j E_t R_{t+j}
\end{aligned}$$

(b) Now suppose that $R_t = \rho R_{t-1} + e_t$, where e_t is an iid random variable with mean zero. Derive a rational expectations solution that links the long rate to the short rate, i.e., determine the coefficient π in

$$R_t^L = \pi R_t$$

(It is not necessary to set this up as an undertermined coefficients problem, but it is possible to do so).

Answer: Using recursive forward substitution, we can obtain

$$E_t R_{t+k} = \rho^k R_t$$

$$\begin{aligned}
R_t^L &= (1-\theta\sum_{j=0}^{\infty} \theta^j E_t R_{t+j} \\
&= (1-\theta)[\sum_{j=0}^{\infty} \theta^j \rho^j]R_t \\
&= \frac{1-\theta}{1-\theta\rho} R_t
\end{aligned}$$

(c) Now suppose that

$$R_t - R_{t-1} = \gamma(R_{t-1} - R_{t-2}) + e_t$$

Using an undetermined coefficients representation, show that

$$R_t^L - R_t = \phi(R_t - R_{t-1})$$

and determine the value of ϕ.

Answer: Suppose that the long rate takes the hypothesized form. Then,

$$\begin{aligned} R_t^L &= \theta E_t R_{t+1}^L + (1-\theta) R_t \\ &= \theta E_t[R_{t+1} + \phi(R_{t+1} - R_t)] + (1-\theta)R_t \\ &= \theta E_t[R_t + (1+\phi)(R_{t+1} - R_t)] + (1-\theta)R_t \end{aligned}$$

and the short-rate equation implies that

$$E_t(R_{t+1} - R_t) = \gamma(R_t - R_{t-1})$$

so that

$$R_t^L - R_t = \theta[(1+\phi)\gamma(R_t - R_{t-1})]$$

so that the rational expectations restriction is that

$$\begin{aligned} \phi &= \theta(1+\phi)\gamma \\ &= \frac{\theta\gamma}{1+\theta\gamma} \end{aligned}$$

(d) Discuss the conditions on ρ and γ under which the two solutions are the same. What is the effect of a change in the short rate on the long rate in this setting? Why?

Answer: If $\rho = 1$ and $\gamma = 0$, then the short-term interest rate is given by

$$R_t = R_{t-1} + e_t$$

in each case. Under this "random walk" specification of the the short term rates,

$$E_t R_{t+k} = R_t$$

for all k, so that the long-rate solution is

$$R_t^L - R_t$$

That is, the long rate is specified to be a weighted average of expected future short-term interest rates, with weights that sum to one ($(1-\theta)\sum_{j=0}^{\infty}\theta^j = \frac{1-\theta}{1-\theta} = 1$).

3. *Dynamic Labor Demand.* Consider a firm maximizing the present discounted value of output y_t less its wage bill $w_t n_t$ and adjustment costs $\frac{d}{2}(n_t - n_{t-1})^2$. (The firm's price is fixed or normalized out of the problem).

$$E_t \sum_{t=0}^{\infty} b^t \left[y_t - w_t n_t - \frac{d}{2}(n_t - n_{t-1})^2 \right]$$

Also assume that $y_t = (a_t + f_0)n_t - \frac{f_1}{2}n_t^2$.

(a) Derive the stochastic Euler equation,

$$dbE_t n_{t+1} - [f_1 + d(1+b)]n_t + dn_{t-1} = w_t - a_t - f_0$$

using any optimization method that you find suitable.

(b) If Y_t is a two element vector containing n_t and one other variable, then this specification can be placed in the form

$$AE_t Y_{t+1} = BY_t + Cx_t$$

where $x_t = w_t - a_t - f_0$ What is the other variable? What are the matrices A, B, C?

(c) If $d \neq 0$, is A singular? What are the roots of $|A\text{z-}B| = 0$?

(d) If $d = 0$, is A singular? What are the roots of $|A\text{z-}B| = 0$?

(e) Supposing that d is not zero, show that one of the roots of $|Az - B| = 0$ satisfies the restriction $0 < \mu_1 < 1$ and the other satisfies $\mu_2 > 1/b$. What this condition imply about about whether the dynamic labor demand model has a unique stable rational expectations solution.

(f) Suppose $w_t = \rho_w w_{t-1} + e_{wt}$ and $a_t = \rho_a a_{t-1} + e_{at}$. Suppose that a researcher conjectures a solution of the form

$$n_t = \mu_1 n_{t-1} + \gamma_w w_t + \gamma_a a_t$$

Do you think that this is a good guess? Why or why not?

(g) What restrictions must be satisfied by the coefficients if this conjecture is to result in a unique stable rational expectations model? What restrictions are violated if it is not?