STUDIENZENTRUM GERZENSEE
Doctoral Course 2010
Day 2 Study Problems for Macroeconomics

Macroeconomists use dynamic programming in three different ways, illustrated in these problems and in the Macro-Lab example. First, as in problem 1, DP is used to derive restrictions on outcomes, for example those of a household choosing consumption and labor supply over time. These can be used for analytical or computational purposes. Second, as in problem 2, DP is used to explicitly determine decision rules and the value function, although this approach works out only in a small number of special cases. This problem also illustrates the convergence of finite horizon problem decision rules and value functions to the infinite horizon values. Third, DP is used – together with a particular approximation technique – to determine numerical forms of decision rules and value functions.

1. *Labor supply and consumption choice over time*

Consider a household that is maximizing its lifetime utility,

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t, l_t^*, l_t)$$

where c_t is consumption, l_t is leisure and

$$l_{t+1}^* = \rho l_t^* + \theta l_t$$

depends on current and past leisure ($l_t = 1 - n_t$, where n_t is market work). Suppose further that the household is optimizing within a regime of sequential markets, with its wealth evolving according to

$$a_{t+1} = (1 + r_t)[a_t + w_t n_t - c_t + \pi_t]$$

(a) What economic considerations might lead l_t^* to enter in the utility function?

Answer: This specification says that a particular function (distributed lag) of past leisure enters the utility function. Economic reasons might include exhaustion, health and various kinds of household capital.

(b) Which variables are controlled state variables from the point of household? What choices affect the motion of these controlled states?

Answer: the controlled state variables are financial assets (a) and the variable (l^*) The motion of these state variables is influenced by leisure (work) and consumption.

(c) Supposing that $1 + r_t, w_t$, and π_t are all functions of exogenous state variables ς_t, write a Bellman equation suitable for this problem. Append each of the dynamic equations to form a Lagrangian, with a different multiplier attached to each.

Answer: The Bellman equation is

$$\begin{aligned} V(a_t, l_t^*, \varsigma_t) &= \max\{u(c_t, l_t^*, l_t) + \beta E_t V(a_{t+1}, l_{t+1}^*, \varsigma_{t+1})\} \\ s.t. a_{t+1} &= (1 + r(\varsigma_t))[a_t + w(\varsigma_t)(1 - l_t) - c_t + \pi(\varsigma_t)) \\ l_{t+1}^* &= \rho l_t + \theta l_t \end{aligned}$$

The associated Lagrangian is

$$\begin{aligned} L_t = \; & \{u(c_t, l_t^*, l_t) + \beta E_t V(a_{t+1}, l_{t+1}^*, \varsigma_{t+1})\} \\ & + \lambda_t[(1 + r(\varsigma_t))[a_t + w(\varsigma_t)n_t - c_t + \pi(\varsigma_t)] - a_{t+1}] \\ & + \phi_t[\rho l_t^* + \theta l_t - l_{t+1}^*] \end{aligned}$$

(d) Find the first-order conditions that describe the efficient choices.

Answer: The FOCs are

$$
\begin{aligned}
c_t &: \quad \frac{\partial u(c_t, l_t^*, l_t)}{\partial c_t} - \lambda_t = 0 \\
l_t &: \quad \frac{\partial u(c_t, l_t^*, l_t)}{\partial l_t} - \lambda_t w(\varsigma_t) + \phi_t \theta = 0 \\
l_{t+1}^* &: \quad -\phi_t + E_t \frac{\partial v(a_{t+1}, l_{t+1}^*, \varsigma_{t+1})}{\partial l_t^*} = 0 \\
a_{t+1} &: \quad -\lambda_t + E_t \frac{\partial v(a_{t+1}, l_{t+1}^*, \varsigma_{t+1})}{\partial a_{t+1}} = 0
\end{aligned}
$$

(e) Use the envelope theorem to determine how the first derivatives of the value function depend on economic variables, including the multipliers from part (c).

Answer:

$$
\begin{aligned}
\frac{\partial v(a_t, l_t^*, \varsigma_t)}{\partial a_t} &= (1 + r(\varsigma_t))\lambda_t \\
\frac{\partial v(a_t, l_t^*, \varsigma_t)}{\partial l_t^*} &= \frac{\partial u((c_t, l_t^*, l_t)}{\partial l_t^*} + \phi_t \rho
\end{aligned}
$$

2. *A simple dynamic programming model of capital accumulation* Consider the following economy. Individuals have preferences

$$U = \sum_{t=0}^{\infty} \beta^t \log(c_t)$$

and a constraint of the form

$$k_{t+1} = ak_t^{\alpha} - c_t$$

(a) Write the Bellman equation for this economy.

Answer: The Bellman equation is

$$v(k_t) = \max\{\log(c_t) + \beta v(k_{t+1})\}$$

where the maximization takes place subject to $k_{t+1} = ak_t^{\alpha} - c_t$.

(b) Find the FOC(s) that must be satisfied for an optimal consumption and capital plan;

Answer: The FOCs are variously written depending on the details of the maximization, but always imply that

$$\frac{1}{c_t} = \beta(\frac{1}{c_{t+1}} \alpha a k_{t+1}^{\alpha-1})$$

(c) Show that the following consumption policy

$$c_t = \phi a k_t^{\alpha}$$

is consistent with the condition(s) that you produced in (b).

Answer: The efficiency condition implies that

$$\frac{y_t}{c_t} = \beta\alpha \frac{y_{t+1}}{c_{t+1}} \frac{y_t}{k_{t+1}})$$

so

$$\frac{1}{\phi} = \beta\alpha \frac{1}{\phi} \frac{1}{1-\phi}$$

which implies that

$$\phi = (1 - \beta\alpha)$$

(d) Derive the value function by substituting the optimal consumption policy into the objective. Show that it takes the form

$$v = \gamma + \theta \log(k_t)$$

and determine the values of γ and θ.

Answer – Approach 1: The first approach is to derive a convenient form of the difference equation for capital and the consumption policy, which exresses each as a deviation from a stationary level.

We know from above that

$$\log(c_t) = \log(\phi a) + \alpha \log(k_t)$$

and that

$$\log(k_{t+1}) = \log((1 - \phi)a) + \alpha \log(k_t)$$

Let $\log(k) = \frac{1}{1-\alpha} \log((1 - \phi)a)$ so that

$$\boxed{\log(k_{t+1}/k) = \alpha \log(k_t/k)}$$

and let $\log(c) = \log(\phi a) + \frac{\alpha}{1-\alpha} \log((1-\phi)a)$, so that

$$\log(c_t/c) = \alpha \log(k_t/k)$$

Then, using the solution to the difference equation, we can develop a solution for the value function,

$$\begin{aligned} v &= \sum_{t=0}^{\infty} \beta^t [\log(c_t/c) + \log(c)] \\ &= \frac{1}{1-\beta} \log(c) + \sum_{t=0}^{\infty} \beta^t [\alpha \log(k_t/k)] \\ &= \frac{1}{1-\beta} \log(c) + \sum_{t=0}^{\infty} (\beta\alpha)^t [\alpha \log(k_0/k)] \\ &= \frac{1}{1-\beta} \log(c) + \frac{\alpha}{1-\beta\alpha} [\log(k_0/k)] \end{aligned}$$

Thus, $\theta = \frac{\alpha}{1-\beta\alpha}$ and $\gamma = \frac{1}{1-\beta} \log(c) - \frac{\alpha}{1-\beta\alpha}[\log(k)]$

Answer – Approach 2: The second approach is to derive restrictions that the coefficients γ and θ must satisfy, solving these for the necessary values. We conjecture

$$v = \gamma + \theta \log(k)$$

The Bellman equation implies that

$$v(k_t) = \log(c_t) + \beta[\gamma + \theta \log(k_{t+1})]$$

or

$$\begin{aligned} \gamma + \theta \log(k_t) &= \log(\phi a) + \alpha \log(k_t) \\ &\quad + \beta\gamma + \beta\theta \log((1-\phi)a) + \beta\alpha\theta \log(k_t) \end{aligned}$$

So that

$$\gamma = \beta\gamma + \log(\phi a) + \beta\theta \log((1-\phi)a)$$

and

$$\theta = \alpha + \beta\alpha\theta$$

so that we can solve for the solutions as

$$\theta = \frac{\alpha}{1-\beta\alpha}$$

and

$$\gamma = \frac{\log(\phi a) + \beta\theta \log((1-\phi)a)}{1-\beta}$$

(e) Show that the optimal policy maximizes

$$\log(c_t) + \beta v(k_{t+1})$$

where $v(k_{t+1})$ is the value function derived in part (d).

Answer: Maximizing

$$\log(c_t) + \beta\gamma + \beta\theta \log(k_{t+1})$$

subject to the constraint is the same as maximizing

$$\log(c_t) + \beta\gamma + \beta\theta \log(a k_t^{\alpha} - c_t)$$

with respect to c_t. The FOC leads to

$$\frac{1}{c_t} = \beta\theta \frac{1}{ak_t^\alpha - c_t}$$

or

$$c_t = \frac{1}{1+\beta\theta} ak_t^\alpha = \frac{1}{1+\beta\frac{\alpha}{1-\beta\alpha}} ak_t^\alpha = (1-\beta\alpha)ak_t^\alpha = \phi ak_t^\alpha$$

(f) Now suppose that there is an arbitrary value function of the form

$$v_{n-1}(k_{t+1}) = \gamma_n + \theta_{n-1}\log(k_t)$$

which appears on the right hand side of the Bellman equation. Show that

$$v_n(k_t) = \max\{\log(c_t) + \beta v_{n-1}(k_{t+1})\}$$

takes the form

$$v_n(k_t) = \gamma_n + \theta_n \log(k_t)$$

and determine the coefficients γ_n and θ_n.

Answer: We can proceed as in the analysis above, except that leave the decision rules as

$$\begin{aligned} c_t &= \frac{1}{1+\beta\theta_{n-1}} ak_t^\alpha \\ k_{t+1} &= \frac{\beta\theta_{n-1}}{1+\beta\theta_{n-1}} ak_t^\alpha \end{aligned}$$

so that

$$\begin{aligned} v_n(k_t) &= \gamma_n + \theta_n \log(k_t) \\ &= \log(c_t) + \beta[\gamma_{n-1} + \theta_{n-1}\log(k_{t+1})] \\ &= (\log(\frac{a}{1+\beta\theta_{n-1}})) + \alpha\log(k_t)) \\ &\quad +\beta\gamma_{n-1} + \beta\alpha\theta_{n-1}\log(k_t) + \beta\theta_{n-1}\log(\frac{a\beta\theta_{n-1}}{1+\beta\theta_{n-1}}) \end{aligned}$$

Hence, we can see that

$$\theta_n = (1+\beta\theta_{n-1})\alpha$$

and there is a more complicatied expression for γ_n,

$$\gamma_n = \beta\gamma_{n-1} + \log(\frac{a}{1+\beta\theta_{n-1}})) + \beta\theta_{n-1}\log(\frac{a\beta\theta_{n-1}}{1+\beta\theta_{n-1}})$$

(g) As n increases to infinity, show that the sequence of coefficients $\{\gamma_n\}$ and $\{\theta_n\}$ generated in this manner converges to the γ and θ values that you determined in part (b) from any initial γ_0 and θ_0.

Answer: The expression above is a difference equation in θ_n, which can be written as

$$\theta_n - \theta = \beta\alpha(\theta_{n-1} - \theta)$$

with $\theta = 1/(1-\beta\alpha)$. From any positive θ_0 this difference equation will converge to θ so long as $\beta\alpha < 1$.
The value of γ_n can similarly be shown to converge to γ. That is, we see that

$$\gamma_n = \beta\gamma_{n-1} + f_{n-1}$$

so that if f is constant, then γ converges to

$$\gamma = \frac{f}{1-\beta}$$

Since θ converges to $\alpha/(1-\beta\alpha)$,

$$\begin{aligned} f_{n-1} &= \log(\frac{a}{1+\beta\theta_{n-1}})) + \beta\theta_{n-1}\log(\frac{a\beta\theta_{n-1}}{1+\beta\theta_{n-1}}) \\ &= \log(\phi_{n-1}a) + \beta\theta_{n-1}\log((1-\phi_{n-1})a) \end{aligned}$$

it follows that $\phi_n = \frac{a}{1+\beta\theta_{n-1}} = 1-\alpha\beta$ and γ converges to

$$\frac{\log(\phi a) + \beta\theta\log((1-\phi)a)}{1-\beta}$$

which is our result from above. Hence, the initial starting form of the value function, θ_0, γ_0 has no bearing on the ultimate form of the value function.