STUDIENZENTRUM GERZENSEE
Doctoral Course 2010

Day 2 Study Problems for Macroeconomics

Macroeconomists use dynamic programming in three different ways, illustrated in these problems and in the Macro-Lab example. First, as in problem 1, DP is used to derive restrictions on outcomes, for example those of a household choosing consumption and labor supply over time. These can be used for analytical or computational purposes. Second, as in problem 2, DP is used to explicitly determine decision rules and the value function, although this approach works out only in a small number of special cases. This problem also illustrates the convergence of finite horizon problem decision rules and value functions to the infinite horizon values. Third, DP is used – together with a particular approximation technique – to determine numerical forms of decision rules and value functions.

1. *Labor supply and choice over time*

Consider a household that is maximizing its lifetime utility,

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t, l_t^*, l_t)$$

where c_t is consumption, l_t is leisure and

$$l_{t+1}^* = \rho l_t^* + \theta l_t$$

depends on current and past leisure ($l_t = 1 - n_t$, where n_t is market work). Suppose further that the household is optimizing within a regime of sequential markets, with its wealth evolving according to

$$a_{t+1} = (1 + r_{t,t+1})[a_t + w_t n_t - c_t + \pi_t]$$

(a) What economic considerations might lead l_t^* to enter in the utility function?

(b) Which variables are controlled state variables from the point of household? What choices affect the motion of these controlled states?

(c) Supposing that $1 + r_{t,t+1}, w_t$, and π_t are all functions of exogenous state variables ς_t, write a Bellman equation suitable for this problem. Append each of the dynamic equations to form a Lagrangian, with a different multiplier attached to each.

(d) Find the first-order conditions that describe the efficient choices.

(e) Use the envelope theorem to determine how the first derivatives of the value function depend on economic variables, including the multipliers from part (c).

2. *A simple dynamic programming model of capital accumulation*

Consider the following economy. Individuals have preferences

$$U = \sum_{t=0}^{\infty} \beta^t \log(c_t)$$

and a constraint of the form

$$k_{t+1} = ak_t^\eta - c_t$$

(a) Write the Bellman equation for this economy.

(b) Find the FOC(s) that must be satisfied for an optimal consumption and capital plan;

(c) Show that the following consumption policy

$$c_t = \theta a k_t^\eta$$

is consistent with the condition(s) that you produced in (b);

(d) Derive the value function by substituting the optimal consumption policy into the objective. Show that it takes the form

$$v = \gamma + \phi \log(k_t)$$

and determine the values of γ and ϕ.

(e) Show that the optimal policy maximizes

$$\log(c_t) + \beta v(k_{t+1})$$

where $v(k_{t+1})$ is the value function derive in part (d).

(f) Now suppose that there is an arbitrary value function of the form

$$v_{n-1}(k_{t+1}) = \gamma_n + \phi_{n-1} \log(k_t)$$

which appears on the right hand side of the Bellman equation. Show that

$$v_n(k_t) = \max\{\log(c_t) + \beta v_{n-1}(k_{t+1})\}$$

takes the form

$$v_n(k_{t+1}) = \gamma_n + \phi_n \log(k_t)$$

and determine the coefficients γ_n and ϕ_n.

(g) Show that the sequence of coefficients $\{\gamma_n\}$ and $\{\phi_n\}$ generated in this manner converges to the γ and ϕ values that you determined in part (d) from any initial γ_0 and ϕ_0.