STUDIENZENTRUM GERZENSEE
Doctoral Course 2010
Day 1 Study Problems for Macroeconomics

Objective of problem: We have discussed optimization over time for individual decision-makers and for a social planner. We now consider a revenue-maximizing government's optimal money creation decisions, when it is faced with a forward-looking demand for money. This revenue maximization differs in important ways from the others that we have faced, due to forward-looking constraints. In week 3 of the class, we will study recursive methods for optimal policy design in such settings, but now we begin with a simple and direct approach.

Problem #2: The Revenue from Money Creation. Suppose that a government seeks to maximize the revenue that it derives from inflation, within a model with purely flexible prices (P_t is the price level at date t).. In real terms, the government's revenue at date t is

$$z_t = m_t - \frac{1}{\pi_t} m_{t-1} \tag{1}$$

where π_t is the inflation rate – defined as $\pi_t = P_t/P_{t-1}$– and m_t is the amount of real balances held by agents in period t.[1]

The demand for real money balances is given by

$$m_t = \beta f(\pi_{t+1}) \tag{2}$$

which is assumed to be positive, but declining in the inflation rate ($f(\pi) \geq 0$ for all $\pi \geq 0$ and $f_\pi < 0$). The parameter β satisfies $0 < \beta < 1$. In some of the analysis below, it is also assumed that the money demand function takes the particular functional form

$$m_t = \kappa \pi_{t+1}^{-\alpha} \tag{3}$$

with κ and α being positive parameters.

(a) What inflation rate maximizes steady-state rate revenue,

$$m(1 - \frac{1}{\pi}) \tag{4}$$

subject to the particular money demand function $m = \kappa\pi^{-\alpha}$? How does this revenue-maximixing inflation rate depend on κ and α?

Answer: To find the revenue-maximizing inflation rate, we can use a number of strategies. In view of the approach to dynamic analysis below, it is most transparent to form the Lagrangian,

$$L = m(1 - \frac{1}{\pi}) + \phi(\kappa\pi^{-\alpha} - m)$$

and to maximize it with respect to π, m while minimizing it with respect to ϕ. For an optimum, it follows that

$$\begin{aligned} m &: \quad (1 - \frac{1}{\pi}) + \phi(-1) = 0 \\ \pi &: \quad \frac{1}{\pi^2} m - \alpha\phi\kappa^{-\alpha-1} = 0 \\ \phi &: \quad \kappa\pi^{-\alpha} - m = 0 \end{aligned}$$

[1] This revenue may be derived as follows. First, the nominal money stock in period t is M_t and the newly printed money in period t is $M_t - M_{t-1}$. The real value of this newly printed money is $(M_t - M_{t-1})/P_t$, with P_t being the price level. Hence, the real revenue is as specified in the body of the question, if $\pi_t = P_t/P_{t-1}$.

Hence, we can see that

$$\frac{1}{\pi} = 1 - \phi$$

from the first condition and that

$$\frac{1}{\pi} = \alpha\phi$$

from the second two conditions. Hence, it follows that

$$\begin{aligned} \phi &= \frac{1}{1+\alpha} \\ \pi &= \frac{1+\alpha}{\alpha} \end{aligned}$$

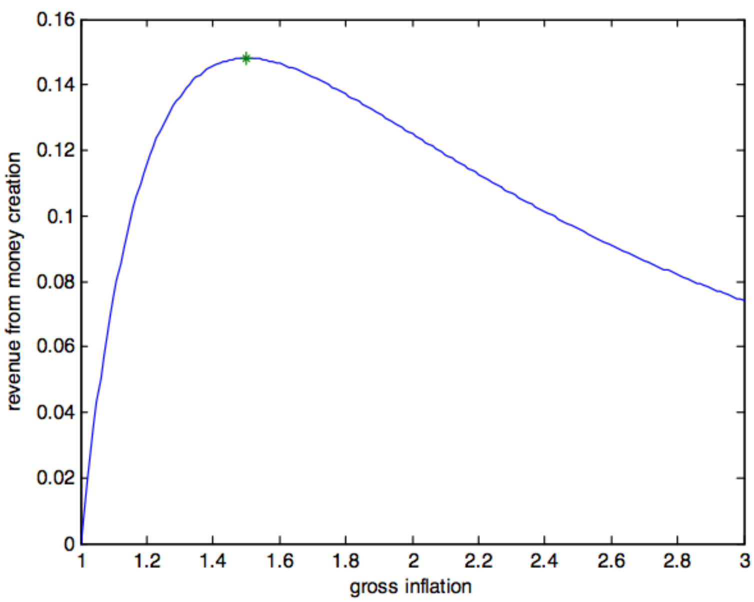
so that the multiplier is less than one and the gross inflation rate is greater than one.

This optimal inflation rate could alternatively be derived by maximizing the revenue

$$z = (1 - \frac{1}{\pi})\kappa\pi^{-\alpha}$$

Note that the optimal inflation rate does not depend on κ, but that it is decreasing in α. This is analogous to the optimal price charged by a monopoly firm facing a constant elasticity demand curve, which does not depend on the scale of demand but depends inversely on the elasticity of demand.

This figure is drawn under the assumption that $\alpha = 2$, so that the optimal inflation rate is 50% ($\pi = 1.5$).



(b) Consider next a government that maximizes the present discounted value of its revenue,

$$\sum_{t=0}^{\infty} \beta^t z_t \tag{5}$$

taking as given the real value of money balances in existence prior to its assuming power, i.e., taking as given

$$m_{-1}$$

Assume further that the government can commit to a series of inflation rates at dates $t = 0, 1, 2, ...$ Form a dynamic Lagrangian for the government's revenue maximization problem, treating the money demand function as an inequality of the form

$$m_t \leq \beta f(\pi_{t+1}) \tag{6}$$

$t = 0, 1, 2, ...$ That is: assume that the government can pick a "tax base" for the inflation tax which is no larger than the real balances that individuals are willing to hold. Call the multiplier on this constraint ϕ_t.

Answer: The dynamic Lagrangian takes the form

$$L = \sum_{t=0}^{\infty} \beta^t (m_t - \frac{1}{\pi_t} m_{t-1}) + \sum_{t=0}^{\infty} \beta^t \phi_t [\beta \kappa \pi_{t+1}^{-\alpha} - m_t]$$

(c) Find the first-order conditions for optimal choice of m_t, π_t, and ϕ_t for all dates $t = 0, 1, 2, ...$ Record these as follows

$$\begin{aligned}
m_0 &: \quad 0 = -\phi_0 + \frac{\partial z_0}{\partial m_0} + \beta \frac{\partial z_1}{\partial m_0} \\
& \quad -\phi_0 + 1 + \beta(-\frac{1}{\pi_1}) \\
\pi_0 &: \quad 0 = \frac{1}{\pi_0^2} m_{-1} \\
\phi_0 &: \quad 0 = \kappa \pi_1^{-\alpha} - m_0
\end{aligned}$$

for $t > 0$

$$\begin{aligned}
m_t &: \quad 0 = -\phi_t + \frac{\partial z_t}{\partial m_t} + \beta \frac{\partial z_{t+1}}{\partial m_t} \\
& \quad -\phi_t + 1 + \beta(-\frac{1}{\pi_{t+1}}) \\
\pi_t &: \quad 0 = \frac{1}{\pi_t^2} m_{t-1} + \frac{1}{\beta} \phi_{t-1} (-\alpha) \frac{m_{t-1}}{\pi_t} \\
\phi_t &: \quad 0 = \kappa \pi_1^{-\alpha} - m_t
\end{aligned}$$

(d) Using the economics of the problem and the first-order conditions, explain why the government has sharply different inflation incentives at date $t = 0$ and date $t > 0$.

Answer: At date t=0, when the policy problem "starts" perhaps because there is a new government, individuals are assumed to have held m_{-1} in the prior period.

The optimal policy involves driving $\pi_0 \to \infty$, with this initial inflation surge corresponding to a "capital levy" on cash held (so that m_{-1} is fully transferred to the government).

This high inflation does not come about in the static model, because money demand declines with π, yielding the revenue curve plotted above.

(e) Work out the stationary level of the revenue from money creation. How does the optimizing government's revenue compare to the answer from part (a)? Why?

Answer: We can solve the system of equations for $t > 0$ as follows. The first implies that

$$\frac{1}{\pi_{t+1}} = \frac{1}{\beta}(1 - \phi_t)$$

and the latter two imply

$$\frac{1}{\pi_{t+1}} = \frac{\alpha}{\beta}\phi_{t-1}$$

Combining these expressions at date t implies that

$$\phi_t \quad = \tfrac{1}{1+\alpha}$$

and

$$\pi_t \quad = \beta\tfrac{1+\alpha}{\alpha}$$

That is, the optimizing government chooses a lower rate of inflation when it solves the dynamic problem. This is because it recognizes that it will face a lower demand for money at t if it raises inflation at t+1 so as to partially confiscate future real balances. Discounting thus lowers the effective reward to (anticipated) inflation.

Given that inflation is lower, we can also say that revenue is lower given that $\pi = \frac{1+\alpha}{\alpha}$ maximized flow revenue. The exception is in the initial period, when revenue is $z_0 = m_{-1}$.