STUDIENZENTRUM GERZENSEE
Doctoral Course

Introductory Study Problem #1: A two period model of optimal consumption over time

Suppose that a household has a utility function depending on consumption (c) in two periods,

$$U = \frac{1}{1-\sigma} c_0^{1-\sigma} + \beta \frac{1}{1-\sigma} c_1^{1-\sigma}$$

with $\sigma > 0$ and $0 < \beta < 1$. Suppose also that the two-period budget constraint takes the form

$$c_0 + \frac{1}{1+r} c_1 = y_0 + \frac{1}{1+r} y_1$$

where y is income and r is the real interest rate.

(1-a) Discuss how this is similar to the standard "two good" problem for a consumer with endowments. What is the numeraire? What are the endowments? What is the relative price and how does it depend on the interest rate?

(1-b) Irving Fisher argued in his *Theory of Interest* that the growth rate of consumption would depend on a gap between "market and personal discount rates" and on a measure of the willingness of an individual to substitute over time. Work out "Fisher's rule" for the growth rate of consumption, identifying each of his channels with an aspect of the two period model.

(1-c) Suppose that utility is held constant at a level $\overline{U}$. What small changes in consumption in the two periods are compatible with this requirement?

(1-d) Suppose that the small changes in consumptions are consistent with Fisher's rule AND with your answer to (1-c). Why does this correspond to the "pure substitution effect" of an interest rate change on the consumptions c_0 and c_1?

(1-e) Using the budget constraint and Fisher's rule, calculate the optimal values of the consumptions c_0 and c_1. What is the effect of the interest rate on these values?

(1-f) Suppose that it is initially optimal for the individual to choose the consumptions $c_0 = y_0$ and $c_1 = y_1$. (This requires a particular interest rate). Show that the response in part (e) corresponds to the response in part (d). Draw a diagram which illustrates why this is the case. What general microeconomic principle does this reflect?

(1-g) Now suppose that $y_1 = 0$. Show that if $\sigma = 1$, there is no effect of the interest rate on c_0. (This is interpretable as exactly offsetting of substitution and income effects of the interest rate change).

(Note: the utility function is not well-defined at $\sigma = 1$, but if it is written as

$$U = \frac{1}{1-\sigma}(c_0^{1-\sigma} - 1) + \beta \frac{1}{1-\sigma}(c_1^{1-\sigma} - 1)$$

then we can use L'Hopital's rule to argue that the $\lim_{\sigma \to 1}\{\frac{1}{1-\sigma}(c_0^{1-\sigma} - 1)\} = \log(c)$. Thus, we are considering $u(c) = \log(c)$ when $\sigma = 1$).