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SOME RESULTS ON THE UNIQUENESS OF STEADY STATES IN MULTISECTOR MODELS OF OPTIMUM GROWTH WHEN FUTURE UTILITIES ARE DISCOUNTED*

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1. INTRODUCTION

IT IS WELL KNOWN that in multisector models of optimal growth that optimal paths converge to a unique steady state when future utilities are not discounted. See Gale [**8**], McKenzie [**18**], and Brock [**3**] for results of this type. Then Sutherland [**32**], in the case when future utilities are discounted, produced examples of multiple steady states in Gale's [**8**] model. Thus the qualitative behavior of optimal growth when future utilities are discounted may be quite different than the Ramsey case when future utilities are not discounted. Also Kurz [**11**] and Liviatan and Samuelson [**13**] have produced multiple equilibria in the discounted case by introducing wealth effects and joint production effects respectively.

What we shall do here is give conditions for several multisector models that give uniqueness of steady states. In the case of no joint production and one primary factor of production we use a nonsubstitution theorem to determine relative prices and the choice of technique independently of the utility function. We then assume normality of the utility function and show that there is at most one steady state for that case.

We examine a model with no joint production, no primary factor and no depreciation of capital. In this case we show that summability of the utility function and a type of "normality" condition on each production function yields uniqueness of steady state. New techniques have to be invented to handle this case. The techniques may be independently interesting.

A very general model that handles cases of joint production is examined in Section 3. All that is required here is that the steady state equations can be written in the form $G(k, \rho) = 0$ where ρ is the discount. It turns out that if

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$G(k, 0) = 0$ is satisfied by only one k and if the derivative matrix $\partial G/\partial k$ is nonsingular on the set $M_\rho \equiv \{k \mid G(k, \rho) = 0 \text{ and all } k \geqq 0\}$ then there is only one steady state. This theorem captures[2] most multisector models but the interpretation of the condition that $\partial G/\partial k$ be nonsingular is not clear.

In Section 2.3 we show that steady state equilibria in multisector models of optimum growth under discounting are special cases of static general equilibria when a tax is placed on capital services. This result holds in very general models and allows us to map out a strategy for a uniqueness of steady state theory along the lines of the static uniqueness of general equilibrium theory discussed in Arrow and Hahn [**1**].

2.1. UNIQUENESS OF STEADY STATE IN A SPECIAL CASE

Consider the following model of optimal economic growth.

$$\begin{aligned} &\text{Maximize} \int_0^\infty e^{-\delta t} u(c_1, c_2, \ldots, c_n) dt \\ &\text{s. t.} \qquad c_i + \dot{k}_i = F^i(k_{1i}, \ldots, k_{ni}), \qquad i = 1, 2, \ldots, n \\ (1) \qquad &k_i(0) = k_{i0}, \qquad i = 1, 2, \ldots, n \\ &\sum_{j=1}^n k_{ij} = k_i, \qquad i = 1, 2, \ldots, n \end{aligned}$$

where u, F^i, $i = 1, 2, \ldots, n$ are concave, increasing, and twice continuously differentiable.[3] We will assume that zero input of any capital good implies zero output for all sectors, and marginal utility of consumption of the i-th good is $+\infty$ if the consumption level of the ith good is zero. Think of labor as being institutionally assigned to each sector. Thus we will assume that the F^i are *not* homogeneous of degree one. Let us agree not to worry about the optimum allocation of labor across sectors at this stage. We also assume that capital is infinitely durable and is shiftable at no cost across sectors. Furthermore assume that all steady states are interior.

The interior steady states of problem (1) are described (after going through the usual Pontryagin's Maximal Principle [**13**] exercise) by

$$u_i F^i_j = \delta u_j, \qquad i, j = 1, 2, \ldots, n \tag{2}$$

where $u_i = \partial u/\partial c_i$, $F^i_j = \partial F^i/\partial k_{ji}$. Also

$$c_i = F^i(k_{1i}, \ldots, k_{ni}), \qquad i = 1, 2, \ldots, n \tag{3}$$

$$\sum_{j=1}^n k_{ij} = k_i, \qquad i = 1, 2, \ldots, n. \tag{4}$$

Our task is: Find conditions that make economic sense and that imply (2)–(4)

[2] We require differentiability of the utility and production functions, of course.

[3] This assumption is much too strong for our purposes. Any assumption that yields interior steady states will do.

have at most one solution. The obvious thing to do is go after some kind of normality assumption on consumption and factor inputs in each sector. We will develop two sets of assumptions. Let us assume from the outset that u is summable, i.e. $u(c_1, \ldots, c_n) = \sum_{i=1}^n v_i(c_i)$ where each v_i is concave, increasing, $v_i'(0) = +\infty$. Now let $[F^k_{ij}]$ be the Hessian matrix of F^k. Assume that this is negative definite (it is negative semi-definite by concavity). We may now state assumption 1.

ASSUMPTION 1. $[F^k_{ij}]$ has a negative inverse $k = 1, 2, \ldots, n$.

This is a very strong assumption. One can get by with much less, but a study of this assumption will motivate our weaker assumption. Let us develop some implications.

Consider the problem

$$\text{Maximize } F^i(k_{1i}, \ldots, k_{ni}) \quad \text{s. t. } \sum_{j=1}^n k_{ji}p_{ij} \leqq I \tag{5}$$

where $(p_{i1}, \ldots, p_{in})$ is a vector of prices and I is an income level. For any income level I the necessary conditions generated by (5) are

$$F^i_j = \lambda p_{ij}, \qquad j = 1, 2, \ldots, n \tag{6}$$

where λ is the derivative of the maximum with respect to I. Look upon this as a consumer's maximization problem in the theory of demand. Since F^i is concave, $\lambda(I)$ is decreasing in I. Now put $p_{ij} \equiv u_j/u_i$, $\lambda = \delta$ then (6) becomes (2). This exercise allows us to make an extremely useful observation.[4] Viz., given a vector of goods prices, and given a preassigned level of the marginal utility of income, λ, then there is only one solution $(k_{1i}, \ldots, k_{ni})$ to (6). Thus we get a mapping from $(p_{i1}, \ldots, p_{in}; \lambda)$ to $(k_{1i}, \ldots, k_{ni})$ that solves (6). Suppose we increase p_{ij}, holding λ fixed at the level δ. What happens to $(k_{1i}, \ldots, k_{ni})$ and $F^i(k_{1i}, \ldots, k_{ni})$?

LEMMA 1. *Under* A1 *an increase of one of the p_{ij}, say p_{il}, will decrease all of the k_{ji}, $j = 1, 2, \ldots, n$ provided that λ is held fixed at the level δ.*

PROOF. Just differentiate (6) and use Assumption 1. Recall that λ is held fixed at the level δ. Doing this we obtain

$$(F^i_{js})\left(\frac{\partial k_{si}}{\partial p_{il}}\right) = \delta e_l, \tag{7}$$

[4] Assume that F^i is strictly concave. Then the marginal utility of income is a decreasing function of income. A vector of positive prices generates an expansion path. Marginal utility of income falls on this path as income increases because F^i is strictly concave. Thus to each value of the marginal utility of income there is a unique income level, I. Given I and given a price vector there is just one $(k_{1i}, \ldots, k_{ni})$ that solves (5). This follows from strict concavity. Thus given λ and p_{ij} there is just one solution to (6).

where e_l is the vector of zeroes in all places except the l-th which is unity. Inverting (F^i_{js}) gives the result.

COROLLARY 1. *Under the hypotheses of Lemma* 1, c_i *decreases.*

PROOF. Obvious from the monotony of F^i.

Corollary 1 leads to the following question: When can we say that c_i decreases when p_{il} "say" increases holding $\lambda = \delta$? I.e., when does an increase in price holding marginal utility of income constant lead to a reduction in utility? Things become more natural if we look at the "dual" problem. I.e., consider

$$\text{(8)} \qquad \begin{aligned} &\text{Min} \sum_{j=1}^{n} p_{ij}k_{ij} \\ &\text{s. t. } F^i(k_{1i}, \ldots, k_{ni}) \geqq c\,. \end{aligned}$$

I.e., attain the output c at minimum cost. Introduce a Lagrange multiplier η set $L = \sum_{j=1}^{n} p_{ij}k_{ji} + \eta(F^i(k_{1i}, \ldots, k_{ni}) - c)$ minimize to yield the necessary conditions

$$\text{(9)} \qquad p_{ij} = -\eta F_j^i \qquad j = 1, 2, \ldots, n.$$

Thus $-1/\eta$ plays the role of λ in (5). Marginal cost $= -\eta$ which increases as c increases. Write $M(c, p_{il}, \ldots, p_{in}) = -\eta$. We may now state Assumption 2.

ASSUMPTION 2. $M(c, p_{il}, \ldots, p_{in})$ increases if any component of $(p_{i1}, \ldots, p_{in})$ increase.

This is a kind of "normality" assumption. It says that an increase in factor prices causes the marginal cost curve to rise.

LEMMA 2. *Let* A2 *hold. Let* p_{il} *"say" increase then* c_i *decreases, provided marginal cost is held constant at* $1/\delta$.

PROOF. By Assumption 2 the marginal cost curve must rise but $M[c_i,(p_{il}, \ldots, p_{in})] = 1/\delta$ therefore c_i must fall.

We may now prove our theorem.

THEOREM: Let Assumption 2 hold. Let u be summable then there is at most one solution to equations (2)–(4).

PROOF. We outline the method first. Consider the set of problems

$$\text{(10)} \qquad \begin{aligned} F_j^1 &= \delta(u_j/u_1) \\ F_j^2 &= \delta(u_j/u_2) \\ &\vdots \\ F_j^n &= \delta(u_j/u_n)\,. \end{aligned}$$

Put $p_{ij} \equiv u_j/u_i$. Assume that there are two solutions, $[k_{ji}]$, $[\bar{k}_{ji}]$. We first show (by induction on n that there are two rows of (u_j/u_i) such that "say" row s

nondecreases and row t nonincreases. This then implies that c_s nonincreases, c_t nondecreases. Since u is summable, concave, u_s nondecreases, u_t nonincreases. But u_t/u_s being the (s, t) element of row s nonincreases. This will lead to a contradiction provided that at least one element of row t decreases. This will obtain provided that $(k_{ji}) \neq (\bar{k}_{ji})$. So we have the outline of the proof.

Let us first establish the row property on the matrix $[u_j/u_i]$.

LEMMA 3. *Suppose $(k_{ji}) \neq (\bar{k}_{ji})$ are two solutions to (2)–(4). Then there are rows s, t of $[u_j/u_i]$ such that "say" $u_j/u_s \leqq \bar{u}_j/\bar{u}_s$ $j = 1, 2, \ldots, n$; $u_j/u_t \geqq \bar{u}_j/\bar{u}_t$, $j = 1, 2, \ldots, n$. Also $u_j/u_s \neq \bar{u}_j/\bar{u}_s$ for some j and $u_j/u_t \neq \bar{u}_j/\bar{u}_t$ for some j.*

PROOF. For $n = 2$ the first half of the Lemma is trivial because

$$[u_j/u_i] = \begin{bmatrix} 1 & u_2/u_1 \\ u_1/u_2 & 1 \end{bmatrix}.$$

Put $\lambda = \delta$. Suppose $u_2/u_1 = \bar{u}_2/\bar{u}_1$ then by (6) $k_{11} = \bar{k}_{11}, k_{21} = \bar{k}_{21}$. Thus $c_1 = \bar{c}_1$ by equation (3). Thus $u_1 = \bar{u}_1$ by summability of u. But $u_1 = \bar{u}_1$ implies $u_2 = \bar{u}_2$. Thus $u_1/u_2 = \bar{u}_1/\bar{u}_2$ which implies $(k_{12}, k_{22}) = (\bar{k}_{12}, \bar{k}_{22})$ a contradiction to $(k_{ji}) \neq (\bar{k}_{ji})$. We could have obtained $u_1/u_2 = \bar{u}_1/\bar{u}_2$ by using $u_2/u_1 = \bar{u}_2/\bar{u}_1$ but this method won't generalize to arbitrary n.

Suppose $[u_j/u_i]$ has the row property for $n - 1$. We show it for n. Consider the matrix

$$\begin{bmatrix} 1\ , & u_2/u_1, \ldots, u_n/u_1 \\ u_1/u_2, & 1\ , \ldots, u_n/u_2 \\ \vdots & \\ u_1/u_n, & u_2/u_n, \ldots, 1 \end{bmatrix}.$$

Look at the $(n - 1) \times (n - 1)$ submatrix

$$\begin{bmatrix} 1\ , & u_3/u_2, \ldots, u_{n-1}/u_2, & u_n/u_2 \\ u_2/u_3, & 1\ , \ldots, u_{n-1}/u_3, & u_n/u_3 \\ \vdots & & \\ u_2/u_n, & u_3/u_n, \ldots, u_{n-1}/u_n, & 1 \end{bmatrix}.$$

By induction row s nondecreases and row t nonincreases "say." There are two cases: (1) $u_1/u_s \leqq \bar{u}_1/\bar{u}_s$, (2) $u_1/u_s > \bar{u}_1/\bar{u}_s$. Take care of case (1) first. There are two subcases (1a): $u_1/u_t \leqq \bar{u}_1/\bar{u}_t$, (1b) $u_1/u_t > \bar{u}_1/\bar{u}_t$. If $u_1/u_t \leqq \bar{u}_1/\bar{u}_t$ then because each element, l, of row t nonincreases for $l \geqq 2$ the ratios must nondecrease. Thus $u_t/u_l \leqq \bar{u}_t/\bar{u}_l$. Therefore, $(u_1/u_t)(u_t/u_l) \leqq (\bar{u}_1/\bar{u}_t)(\bar{u}_t/\bar{u}_l)$. Thus $u_1/u_l \leqq \bar{u}_1/\bar{u}_l$ for $l = 2, 3, \ldots, n$. I.e., each element u_l/u_1 of row one nonincreases. Thus we have found a row that nonincreases. We claim that row s nondecreases. Now $u_1/u_s \leqq \bar{u}_1/\bar{u}_s$ because $u_s/u_1 \geqq \bar{u}_s/\bar{u}_1$. The last holds because row one nonincreases. The rest of row s nondecreases by hypothesis. Hence for case (1a) we have shown that there are two rows, one nondecreasing the other nonincreasing.

Consider case (1b): $u_1/u_t > \bar{u}_1/\bar{u}_t$. Here the t-th row nonincreases. By hypothesis row s nondecreases. Thus we are finished with case 1.

Let us go to case 2: $u_1/u_s > \bar{u}_1/\bar{u}_s$. We claim that row 1 nondecreases. To see this note that $u_s/u_1 < \bar{u}_s/\bar{u}_1$ and by induction hypothesis $u_l/u_s \leqq \bar{u}_l/\bar{u}_s$ so that $(u_s/u_1)(u_l/u_s) \leqq (\bar{u}_s/\bar{u}_1)(\bar{u}_l/\bar{u}_s)$. I.e., $u_l/u_1 \leqq \bar{u}_l/\bar{u}_1$. Thus row one nondecreases and the claim is proved.

Now $u_t/u_1 \leqq \bar{u}_t/\bar{u}_1$ so that $u_1/u_t \geqq \bar{u}_1/\bar{u}_t$. I.e., the first element of row t nonincreases. But by induction hypothesis all the other elements of row t nonincreased. Thus row t nonincreases and we are finished with case 2. This ends the proof of the first part of Lemma 3.

Now suppose that $u_j/u_s = \bar{u}_j/\bar{u}_s$, $j = 1, 2, \ldots, n$.[5] Then $c_s = \bar{c}_s$. From this we have $u_j = \bar{u}_j$, $j = 1, 2, \ldots, n$. Therefore $u_j/u_i = \bar{u}_j/\bar{u}_i$ all i, j.

Look at the row of problems (10). No row of prices u_j/u_i changes. Thus $k_{ji} = \bar{k}_{ji}$ all i, j, a contradiction to $[k_{ji}] \neq [\bar{k}_{ji}]$.

REMARK. The row property of the matrix $[u_j/u_i]$ across steady states has nothing to do with summability of u.

The proof of Lemma 3 is hard to follow. Actually it is easier if one develops an "algebra" of inequalities on the matrix $[u_j/u_i]$ across steady states. We do this in the sequel. But let us first finish off the proof of the theorem.

Armed with Lemma 3 we may say that there are two rows of $[u_j/u_i]$ such that each element of s nondecreases and each element of t nonincreases. Furthermore at least one element of s increases and at least one element of t decreases. Thus $c_s > \bar{c}_s$, $c_t < \bar{c}_t$. Since u is summable $u_s < \bar{u}_s$, $u_t > \bar{u}_t$. Thus $u_t/u_s > \bar{u}_t/\bar{u}_s$. I.e. element t of row s decreases, a contradiction to the fact that all elements of row s nondecrease. This ends the proof of the Theorem.

Now let us build some understanding of the row property $[u_j/u_i]$ across steady states. Not only will this be useful for understanding our proof, but also it should be useful for further workers in this area. We introduce an associated matrix $[a_{ij}]$ where

$$a_{ij} = \begin{cases} + & \text{if } u_j/u_i < \bar{u}_j/\bar{u}_i \\ 0 & \text{if } u_j/u_i = \bar{u}_j/\bar{u}_i \\ - & \text{if } u_j/u_i > \bar{u}_j/\bar{u}_i . \end{cases}$$

Introduce the operation o on the symbols a_{ij} by

$$a_{ij} o a_{kl} = \begin{cases} + \text{ if } a_{ij} = + \text{or } 0,\ a_{kl} = + \text{ or } 0 \text{ and at least one is } +. \\ 0 \text{ if both } a_{ij},\ a_{kl} \text{ are equal to zero} \\ - \text{ if } a_{ij} = - \text{or } 0,\ a_{kl} = - \text{or } 0 \text{ and at least one is } -. \\ \text{undefined otherwise.} \end{cases}$$

It is easy to show that $a_{ij} o a_{jk} = a_{ik}$ when the operation o is defined. Also $a_{ij} =$

[5] This statement could use some argument. Use equation (6) with $\lambda = \delta$ and $i = s$ to assert: $u_j/u_s = \bar{u}_j/\bar{u}_s \equiv p_{sj}$ implies $k_{js} = \bar{k}_{js}$, $j = 1, 2, \ldots, n$. But by (3) this implies that $c_s = \bar{c}_s$.

$-a_{ij}$. The meaning of $-a_{ij}$ is $-a_{ji} = +, 0, -$ when $a_{ji} = -, 0, +$ respectively. To discover relations among the symbols a_{ij} the reader should think of a_{ij} as $j - i$. Then $a_{ij} o a_{jk} = (j - i) + (k - j) = k - i = a_{ik}$ and $-(j - i) = -a_{ij} = i - j = a_{ji}$. Also $a_{ij} o(-a_{kj}) = j - i - (j - k) = k - i = a_{ik}$. Use of this symbolism should allow the reader to systematically enumerate the possible $[a_{ij}]$ matrices. In particular the proof of the first part of Lemma 3 (the row property on $[a_{ij}]$) is easier to follow. We discovered the proof, in the first place, by using the $[a_{ij}]$ matrix together with its associated algebra.

2.2 UNIQUENESS OF STEADY STATE IN THE GENERAL CASE OF EXPONENTIALLY GROWING LABOR FORCE, EXPONENTIALLY DECAYING CAPITAL, CONSTANT RETURNS TO SCALE PRODUCTION FUNCTIONS, AND NORMAL UTILITY FUNCTION

In this case the model is

$$\text{Maximize} \int_0^\infty e^{-\rho t} u\left(\frac{C_1}{L}, \frac{C_2}{L}, \ldots, \frac{C_n}{L}\right) dt$$

$$\begin{aligned}
\text{s. t.} \quad & C_i + \dot{K}_i = F_i(L_i, K_{1i}, \ldots, K_{ni}) - \delta_i K_i\,, && i = 1, 2, \ldots, n \\
(1) \quad & \sum_{i=1}^n K_{ji} \leqq K_j\,, && j = 1, 2, \ldots, n \\
& \sum_{i=1}^n L_i \leqq L \\
& L = L_0 e^{gt} \\
& K_i(0) = K_{i0}\,, && i = 1, 2, \ldots, n.
\end{aligned}$$

Here the utility of society at time t is taken to be

$$e^{-\rho t} u\left(\frac{C_1}{L}, \frac{C_2}{L}, \ldots, \frac{C_n}{L}\right),$$

where ρ can be taken to be the difference between the subjective rate of discount and the rate of population growth. We assume that $\rho > 0$. The objective is to maximize the discounted sum of utilities. Controls are total consumption C_i of good i, allocation of capital good j over sectors $i = 1, 2, \ldots, n$ which is denoted by k_{ji} and allocation of available labor to sector $i = 1, 2, \ldots n$ which is denoted by L_i. We will assume that u, F^i are twice continuously differentiable, concave and increasing. The F^i are assumed to be homogeneous of degree 1. Putting (1) into per capita form we get

$$\text{Maximize} \int_0^\infty e^{-\rho t} u(c_1, c_2, \ldots, c_n) dt$$

$$\begin{aligned}
& c_i + \dot{k} = F^i(l_i, k_{1i}, \ldots, k_{ni}) - g k_i - \delta_i k_i\,; && i = 1, 2, \ldots, n \\
(2) \quad & \sum_{i=1}^n k_{ji} \leqq k_j\,, && j = 1, 2, \ldots, n
\end{aligned}$$

$$\sum_{i=1}^{n} l_i \leqq 1$$
$$k_i(0) = k_{i0}\,, \qquad i = 1, 2, \ldots, n$$

where lower case letters denote per capita quantities, e.g., $k_{ji} = K_{ji}/L$. Set up the Hamiltonian

$$\begin{aligned} H = e^{-\rho t}u(c_1, \ldots, c_n) + \sum_{i=1}^{n} q_i[F^i - gk_i - \delta_i k_i - c_i] \\ + \lambda\left(1 - \sum_{i=1}^{n} l_i\right) + \sum_{j=1}^{n} \lambda_j\left[k_j - \sum_{i=1}^{n} k_{ji}\right]. \end{aligned} \tag{3}$$

Apply the Maximum Principle [13] to H; assume that optimum $k_{ji} > 0, l_i > 0$ all i, j so that no multipliers appear for the constraints $k_{ji} \geqq 0, l_i \geqq 0$; listing the output of the Maximum Principle we get

$$\begin{aligned} &e^{-\rho t}u_k - q_k = 0, \quad q_i F_0{}^i - \lambda = 0\,, \\ &q_i F_j{}^i - \lambda_i = 0, \quad \dot{q}_i = -\partial H/\partial k_i = (g + \delta_i)q_i - \lambda_i\,. \end{aligned} \tag{4}$$

It is clear from the context what the range of the indices i, j, k is; so we drop specification of their ranges when the context is clear. Here $u_k \equiv \partial u/\partial c_j$, $F^i_{lj} = \partial F^i/\partial k_{li}\partial k_{ji}$, $F_0{}^i \equiv \lambda F^i/\partial L_i$, $F_j{}^i \equiv \partial F^i/\partial k_{ji}$, etc. Put $p_i \equiv u_i = e^{\rho t}q_i$. Then system (4) may be written

$$\begin{aligned} &\dot{p}_i = (\rho + g + \delta_i)p_i - p_j F_i{}^j, \quad p_j = u_j \\ &c_i + \dot{k}_i = F^i(l_i, k_{1i}, \ldots, k_{ni}) - (g + \delta_i)k_i; \ \sum_i k_{ji} \leqq k_j; \\ &\sum_i l_i \leqq 1; \ p_i F_0{}^i = e^{-\rho t}\lambda \end{aligned} \tag{5}$$

where $\cdot$ over a symbol denotes time derivative. Let us assume that for all steady states all capital is used and all labor is used i.e., $\sum_i k_{ij} = k_j$, $\sum_i l_i = 1$ for all i, j. As is well known this can be obtained from more basic considerations on utility and technology—Inada [10] conditions, marginal utility of starvation $= +\infty$, etc. Let us agree just to spell out at each stage exactly what we need at each point of our argument. Not only does this save space, and avoids repetition of the well known, but also it lays bare the basic structure of our argument.

Equations (5) should look familiar to the reader. They can be viewed as equations determining the price path in a competitive economy with perfect capital markets and short run perfect foresight. To see this put $W_i = p_j F_i{}^j, j = 1, 2, \ldots, n$; $W_0 = p_j F_0{}^j, j = 1, 2, \ldots, n$. Interpret W_i as the rental rate for factor $i, i = 0, 1, 2, \ldots, n$. Then equations (5) become

$$\begin{aligned} &\dot{p}_i/p_i + W_i/p_i - \delta_i = \rho + g, \quad p_i F_0{}^i = W_0, \quad p_j = u_j \\ &c_i + \dot{k}_i = F^i(l_i, k_{1i}, \ldots, k_{ni}) - (g + \delta_i)k_i\,; \ \sum_{i=1}^{n} k_{ji} \leqq k_j\,; \\ &\sum_{i=1}^{n} l_i \leqq 1\,. \end{aligned} \tag{6}$$

Equations (6) are easy to interpret. The first line just says that capital gains plus rent yield minus depreciation equals a common rate of return for all assets. Furthermore labor and capital must be allocated across industries so that marginal value products (in utils) are equated. The reader will note that we are using the notation of Burmeister and Dobell as in their book [**4**]. We shall draw heavily from that book and from an article by Burmeister and Kuga [**5**] from this point on. Let us sketch our proof of uniqueness before we go into the details.[6]

Look at equations (6). Set $\dot{p}_i = 0 = \dot{k}_i, i = 1, 2, \ldots, n$. Notice that constant returns to scale and no joint production allow one to apply the nonsubstitution theorem (Burmeister and Dobell [**4**, (280)]) to determine the relative prices p_j/p_i and the choice of technique $K_{ij}/Y_j \equiv a_{ij}$, $L_j/Y_j \equiv a_{0j}$. Here $Y_j = C_j + \dot{K}_j + \delta_j K_j$. This defines a consumption possibility frontier (CPF). It turns out that steady states are precisely those points where the income consumption curve generated by

$$(7) \qquad \text{Maximize } u(c_1, \ldots, c_n)$$

$$c_1 + c_2 q_2 + \cdots + c_n q_n = I, \qquad q_j \equiv p_j/p_1 = u_j/u_1$$

intersects the CPF. Let us go on to the theorems and proofs.

Put $a_{ij} = K_{ij}/Y_j$, $a_{0j} = L_j/Y_j$, $y_j = Y_j/L$, $k_{ij} = K_{ij}/L$, $i, j = 1, 2, \ldots, n$.

Then the following may be checked; where $a = [a_{ij}]$, $a_0 = [a_{0j}]$, $k = [k_j]$, and $y = [y_j]$.

$$(8) \qquad ay = k, \quad a_0 y = 1$$

Set $\dot{p} = \dot{k} = 0$, Then the first line of equations (6) becomes

$$(9) \qquad W_i/(\delta_i + \rho + g) = p_i, \qquad i = 1, 2, \ldots, n.$$

This says that unit price of the i-th machine, p_i, is equal to the present value of the stream of rentals where the discount is $\delta_i + \rho + g$. The a_{ji} in steady state are determined by

$$(10) \qquad \text{infimum} \sum_{j=0}^{n} W_j a_{ji}$$

$$F^i(a_{0i}, a_{1i}, \ldots, a_{ni}) \geqq 1.$$

Notice that the optimum a_{ji} are homogeneous of degree zero in the vector $W = (W_0, W_1, \ldots, W_n)$. Now $\sum_{j=0}^{n} W_j a_{ji}$ is unit cost of i. In equilibrium price must equal unit cost under constant returns. Thus from (9) we obtain (the infimum will be attained for $W \gg 0$, Here $W \gg 0$ means that all components are positive)

[6] We owe a good deal here to Mr. E. Sieper of the Australian National University. Our original proof was clumsy and needed stronger assumptions. Sieper weakened the assumptions and greatly simplified the proofs by pointing out the relation between equations (6) and classical nonsubstitution theorems.

$$W_i/(\delta_i - \rho + g) = \sum_{j=0}^{n} W_j a_{ji}\,, \qquad i = 1, 2, \ldots, n. \tag{11}$$

Another way of establishing (11) is to notice that Euler's theorem on homogeneous functions applied to $F^i(a_{0i}, \ldots, a_{ni}) = 1$ yields

$$\sum_{j=0}^{n} F_j{}^i a_{ji} = \sum_{j=0}^{n} (W_j/p_i)a_{ji} = 1\,.$$

Thus $p_i = \sum_{j=0}^{n} W_j a_{ji}$ which yields (11). Put $w_i = W_i/W_0, i = 1, 2, \ldots, n$. Since a_{ji} is homogeneous of degree zero, $a_{ij}(W) = a_{ij}(W/W_0) = a_{ij}(w)$. Put $C_i(W_0, W_1, \ldots, W_n)$ equal to the solution of (10). Thus C_i is unit cost of i and it is homogeneous of degree 1 in W. Hence (11) becomes

$$W_i/(\delta_i + \rho + g) = C_i(W_0, \ldots, W_n)\,, \qquad i = 1, 2, \ldots, n\,. \tag{12}$$

We are now ready to apply a version of the standard nonsubstitution theorem contained in Burmeister and Kuga [5]. We write down their assumption.

Assumption 2.2(1). Labor is required, either directly or indirectly, in the production of every good.

Lemma. (Burmeister and Kuga [5, (165)]). If 2.2(1) holds then there is at most one vector of relative prices $\bar{w}$ that solves (12). Here $\bar{w} = (\bar{W}_i/\bar{W}_0)$ i.e., only *relative* prices are unique.

Proof. See [5, (165)].

Since there is at most one set of equilibrium relative prices and since the a_{ij} are homogeneous of degree zero therefore (10) implies that there is at most one technique used in equilibrium. Further note that the choice of technique and the relative prices do not depend on the utility function u. The only subjective parameter they depend upon is ρ. This independence of the choice of technique from the utility function u also turns up in the one sector model [4, (359)]. Let $\bar{a}_0, \bar{a}$ be the choice of technique. From (6) we obtain

$$c = y - (g + \delta)k = y - (g + \delta)\bar{a}y = [I - (g + \delta)\bar{a}]y \tag{13}$$

where $g + \delta$ is the diagonal matrix with $g + \delta_i$ the element in the i-th row and i-th column, and I is the $n \times n$ identity matrix. Suppose for the moment that $[I - (g + \delta)\bar{a}]$ has a positive inverse.[7] Then $y = [I - (g + \delta)\bar{a}]^{-1}c$. Thus the labor constraint $\bar{a}_0 y = 1$ gives us the CPF

$$\bar{a}_0[I - (g + \delta)\bar{a}]^{-1}c = 1\,. \tag{14}$$

Put $\bar{b} = \bar{a}_0[I - (g + \delta)\bar{a}]^{-1}$. One may look upon this vector $\bar{b}$ as a vector of direct plus indirect labor requirements to produce the vector c. Thus relative

[7] Burmeister and Kuga [5] give conditions for this inverse to exist and to be positive. They include assuming that none of $g + \delta_i$ are "too large." See [5] for details. We shall give different conditions for positivity of the inverse in the sequel.

prices of the consumption goods should be given by the "congealed" labor vector $\bar{b}$. But the discount ρ introduces a "distortion." To see this rewrite the price equals unit cost equations (11) using $p_i = W_i/(\rho + g + \delta_i)$ in the form (assuming all $p_l > 0$).

$$\text{(15)} \qquad \bar{s}_j \equiv \overline{(p_j/W_0)} = \bar{a}_{0j} + \bar{a}_{1j}(\rho + g + \delta_1)\bar{s}_1 + \cdots + \bar{a}_{nj}(\rho + g + \delta_n)\bar{s}_n\,, \qquad j = 1, 2, \ldots, n\,.$$

In matrix form this becomes

$$\text{(16)} \qquad \bar{s} = \bar{a}_0 + \bar{s}(\rho + g + \delta)\bar{a}$$

where $\rho + g + \delta$ is the diagonal matrix with $\rho + g + \delta_i$ in the i-th row and i-th column. Inverting (if the inverse exists) we get

$$\text{(17)} \qquad \bar{s} = \bar{a}_0(I - (\rho + g + \delta)\bar{a})^{-1}\,.$$

Notice that this is equal to the $\bar{b}$ vector when $\rho = 0$. Thus ρ introduces a distortion. We shall see later that steady states are solutions to a general equilibrium problem with distortions caused by a tax on capital. It is this distortion that generates nonuniqueness of steady states when $\rho > 0$. The following condition assures unique steady states.

DEFINITION. The utility function u is normal if for all $(p_1, \ldots, p_n) \gg 0$, $I \geqq 0$ the functions $c_i(q, I)$ increase in I for all i. Here the functions $c_i(p, I)$ solve: maximize $u(c)$ s.t. $p \cdot c = I$. The $c_i(p, I)$ are point to point maps because u is strictly concave. Here $p \gg 0$ means that all components are positive, and $c_i(p, I)$ are the demands given prices p and income I.

THEOREM. *If the vector* $\bar{a}_0[I - (g + \delta)\bar{a}]^{-1} \gg 0$ *and u is normal then there exists at most one steady state.*

PROOF. Suppose there are two distinct steady states $c \neq \bar{c}$. Now for some $M, \bar{M}$; $c, \bar{c}$ must solve

$$\text{Maximize } u(x_1, \ldots, x_n); \quad \text{maximize } u(x_1, \ldots, x_n)$$
$$\text{s. t. } \bar{s} \cdot x = M \qquad\qquad \bar{s}x = \bar{M}$$

respectively, where $\bar{s}$ solves (16). Now $M \neq \bar{M}$ otherwise $c = \bar{c}$. Thus either $M < \bar{M}$ or $M > \bar{M}$. Let us consider the case $M < \bar{M}$. The other case is similar. If $M < \bar{M}$ then normality implies that $c \ll \bar{c}$. Thus $\bar{b} \gg 0$ implies $1 = \bar{b}c < \bar{b}\bar{c} = 1$, a contradiction. This ends the proof.

It is interesting to note that the "national income," M, at a steady state, c, is $M = \bar{s}c = \bar{a}_0(I - (\rho + g + \delta)\bar{a})^{-1}c$ is exactly equal to 1 when $\rho = 0$. Thus when $\rho = 0$, c must solve maximize $u(x_1, \ldots, x_n)$ s.t. $\bar{s}x = 1$ which immediately gives unique steady state. If u is not normal and $\rho > 0$ we may construct an example of multiple steady states as in the following diagram.

The curve $0c$ is the income consumption curve determined by u when prices are $\bar{s}$. The parallel lines are perpendicular to the price vector $\bar{s}$. They describe a family of budget lines. The line AB is the CPF. This diagram shows us how to go about constructing examples of multiple steady states and how normality of u rules out multiple steady states. The diagram is due to E. Sieper of the Australian National University. Since it is a fairly straightforward task to put together a function u that generates multiple steady states we leave it.

We might remark that methods that depend on nonsubstitution theorems should allow generalization to pure consumption goods industries of Morishima

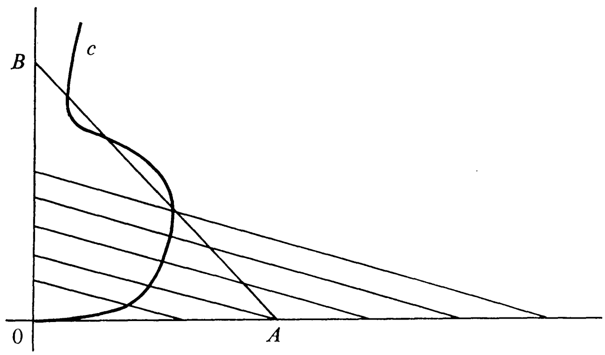

FIGURE 1
MULTIPLE STEADY STATES

[**20**] type and nonjoint production of Mirrlees [**19**] type. Mirrlees' theorem seems to be the most general nonsubstitution theorem available. Any optimum growth model that generates steady state equation that satisfy his nonsubstitution theorem should allow one to assert (following our proof) that normality of u implies unique steady state. At any rate there should be some interesting generalization possible along these lines.

We might mention the following interesting fact pointed out by E. Burmeister. There is a relationship between uniqueness of steady states in multisector models of optimum growth under discounting and the definition of "capital deepening" and "nonparadoxical" behavior discussed in Burmeister and Dobell [**4**, (284–294)] and further investigated by Burmeister and Turnovsky [**6**]. To exposit this relation put $r \equiv \rho + g$. Let $\bar{z} = (\bar{c}, \bar{p}, \bar{k}, \bar{l})$ be a steady state solution of equations 2.2(6). Let $\phi(r)$ equal the set of all steady states with $\rho + g = r$. ϕ is a point to set mapping. Look at $u(\bar{c}(r))$. Suppose that it is possible to apply the implicit function theorem in a neighborhood of $(\bar{c}(r), r)$, for a particular steady state $\bar{c}(r)$ associated with r.[8] Then

[8] This basically follows from an envelope theorem. Since $c_i = F^i(l_i, k_{1i}, \ldots, k_{ni}) - (g + \delta_i)k_i$ therefore

$$dc_i/dr = \sum_{j=0}^{n} F_j{}^i(dk_{ji}/dr) - (g + \delta_i)(\sum_{t=1}^{n} dk_{it}/dr) .$$

We set $l_i = k_{0i}$. Put $p_i = \partial u/\partial c_i$. Then

$$\sum_{i=1}^{n} p_i dc_i/dr = \sum_{i=1}^{n} \sum_{j=0}^{n} p_i F_j{}^i dk_{ji}/dr - \sum_{i=1}^{n} \sum_{t=1}^{n} (g + \delta_i) p_i dk_{it}/dr$$

(*Continued on next page*)

$$\frac{du(\bar{c}(r))}{dr} = \sum_{i=1}^{n} \frac{\partial u}{\partial c_i}\frac{d\bar{c}_i}{dr} = (r-g)\sum_{i=1}^{n}\frac{\partial u}{\partial c_i}\frac{d\bar{k}_i}{dr} = \rho\sum_{i=1}^{n}\frac{\partial u}{\partial c_i}\frac{d\bar{k}_i}{dr}.$$

Thus if the "value" of per capita capital increments,

$$\sum_{i=1}^{m}\frac{\partial u}{\partial c_i}\frac{d\bar{k}_i}{dr}$$

is always negative for all $r \geqq g$, then it should be a theorem that steady states are unique for all $r \geqq g$. Thus the existence of paradoxical behavior of the well known type discussed in Burmeister and Dobell [4, (284–2P4)] is closely tied to inferiority on the demand side. We are exploring these relationships with Burmeister now.

We have said nothing about the *existence* of a steady state. We have little new to add to this problem. However, in order to make this paper more self contained a few words will be said on the matter.

One approach is to apply a version of the nonsubstitution theorem like that contained in Burmeister and Kuga [5] to obtain a vector of factor prices $(W_0, W_1, \ldots, W_n)$ that solves the price equals minimum cost equations (12). Burmeister and Kuga need to assume that the $\rho + g + \delta$ are "small enough" in order to get their existence result. The easiest way to see the need for an assumption like this is to consider the case when (12) is generated by a Leontief system (letting $\eta_i = \rho + g + \delta_i$)

$$W_i/\eta_i = \sum_{i=0}^{m} W_j a_{ji}, \qquad i = 1, 2, \ldots, n. \tag{18}$$

Thus, letting $W = (W_1, \ldots, W_o)$, $\eta = (\eta_1, \ldots, \eta_n)$ we have

$$W = W_0 a_0 [I - \eta a]^{-1}. \tag{19}$$

Hence, some restriction on the size of the η_i is needed to insure that $[I - \eta a]^{-1}$ exists and is nonnegative.

After using the nonsubstitution theorem to produce $(W_0, \ldots, W_n)$ and assuming the usual irreducibility assumptions and necessity of labor assumptions to get $(W_0, \ldots, W_n) \gg 0$ (see [5]) we can put $\bar{s} = a_0[I - \eta a]^{-1}$ and solve for a steady state by finding a point $\bar{c}$ such that the income consumption curve generated by: maximize $u(c)$ s.t., $\bar{s}c = M$ as M varies cuts the CPF:

(*Continued*)

$$= \sum_{j=0}^{n} W_j dk_j/dr - \sum_{j=1}^{n} (g + \delta_j) p_j (dk_j/dr).$$

The latter follows from

$$\sum_{j=1}^{n} k_{0i} = 1,$$

the definition of W_j, and k_j, respectively. Now $W_j = (\rho + \delta_i + g)p_j$ in steady state. Thus

$$\sum_{j=1}^{n} (W_j - (g + \delta_i)p_j) dk_j/dr = \rho \sum_{i=1}^{n} p_i dk_i/dr = \rho \sum_{i=1}^{n} \partial u/\partial c_i \, dk_i/dr.$$

The last equality follows from the definiton of p_i.

$$a_0[I - (g + \delta)a]^{-1} = 1 .$$

There is another approach that is more in the tradition of multisectoral growth theory set by McKenzie [**18**] for example. We outline this approach below. In this approach if the marginal product of k_i gets large as k_i tends to zero then for our purposes one can get by with weaker assumptions on the η_i than Burmeister and Kuga [**5**] and still produce a positive steady state.

Existence theorems have been established by Peleg and Ryder [**22**] and Sutherland [**32**]. Peleg and Ryder's theorem is the most useful for our purposes. We shall just sketch it. It deals with a much more general model. Let

$$T \equiv \{(k, y) \,|\, y_i \leqq F^i(k_{0i}, \ldots, k_{ni}); \sum_{s=1}^{n} k_{js} \leqq k_j , j = 0, 1, \ldots, n; i = 1, 2, \ldots n; \\ k_0 \leqq 1\} \quad \text{where} \quad k \equiv (k_1, \ldots, k_n) .$$

T represents the set of all capital output combinations attainable with one unit of labor. Since the F^i are concave, continuous and there is only one unit of labor it follows that T is compact and convex. We say that T is η-productive if there is (k_p, y_p) in T such that $y_p - \eta k_p \gg 0$. If T were generated by a Leontief system a then η-productivity of T amounts to: There is $k_p, y_p, ay_p = k_p$ such that $y_p - \eta a y_p \gg 0$. But this implies that $[I - \eta a]^{-1}$ exists and is non negative and vice versa. So η-productivity is a natural generalization of the usual conditions on Leontief models. If the marginal product of a factor become large as input of that factor tends to zero and this property holds for all factors then it is quite likely that η-productivity will hold.

We say that $(\bar{c}, \bar{k}, \bar{p}, \bar{y})$ is a *solution* if; (a) $\bar{c} = \bar{y} - (g + \delta)\bar{k}$, (b) $u(\bar{c}) - \bar{p}\bar{c} \geqq u(c) - \bar{p}c$ for all $c \geqq 0$ (c) $\bar{k}, \bar{y}$ solves maximize $\{\bar{p}(y - \eta k) \,|\, (k, y) \text{ in } T\}$. It is easy to see that a solution in this sense is a steady state and vice versa. (b) means that $\bar{p}_i = \partial u/\partial c_i$ when $\bar{c}_i > 0$. (c) amounts to choosing $\bar{k}_i, \bar{k}_{ji}$ so that

$$\sum_{i=1}^{n} [\bar{p}_i F^i(k_{0i}, \ldots, k_{ni}) - \sum_{s=1}^{n} \bar{W}_s k_{si}] \tag{19}$$

is maximum subject to $\sum_{i=1}^{n} k_{0i} \leqq 1$ where $\bar{W}_s \equiv \eta_s \bar{p}_s$. But this defines a steady state for our growth problem when inequalities are allowed.

Peleg and Ryder generate a solution by finding a fixed point of the point to set mapping $h(p, (k, y)) = G(k, y) x f(p)$ which is the Cartesian product of the two sets $G(k, y), f(p)$ which are defined by (set $S = \{p \,|\, p_i \geqq 0, i = 1, 2, \ldots, n; \sum_{i=1}^{n} p_i = 1\}$, $c = y - (g + \delta)k$, $d = b - (g + \delta)a$; (k, y), $(a, b$ in $T)$.

$$\mathrm{G}(k, y) = \{p \,|\, p \text{ is in } S, pc \leqq pd, \text{ for all } d \text{ in } A(c)\} \text{ if } c \geqq 0$$

$$G(k, y) = \{p \,|\, p \text{ is in } S, pc \leqq qc, \text{ for all } q \text{ in } S\} \text{ if } c \ngeqq 0$$

$$f(p) = \{(k, y) \text{ in } T \,|\, p[y - \eta k] \geqq p[b - \eta a] \text{ for all } (a, b) \text{ in } T\}$$

where

$$A(c) = \{d \,|\, d \geqq 0, u(d) \geqq u(c)\} .$$

Peleg and Ryder show that h: $S \times T \longrightarrow S \times T$ satisfies the conditions of Kakutani's fixed point theorem. Let $p, \bar{k}, \bar{y}$ be a fixed point. Peleg and Ryder use η-productivity and the definition of G to show that $p\bar{c} > 0$. Then the definition of G and $p\bar{c} > 0$ imply that $\bar{c}$ solves: max $u(c)$ s.t. $pc \leqq p\bar{c}, c \geqq 0$. They then use $p\bar{c} >$ to construct a real positive t such that $u(\bar{c}) - tp\bar{c} \geqq u(c) - tpc$ for all $c \geqq 0$. Set $\bar{p} = tp$. Since u is increasing, $\bar{p} \gg 0$. Since the "income" $p\bar{c} > 0$ it is relatively harmless to assume that $\bar{c} \gg 0$ because $\bar{c}$ solves max $u(c)$ s.t· $pc \leqq p\bar{c}, c \geqq 0$. Thus if u is differentiable then $\bar{c}$ maximizes $u(c) - \bar{p}c$ implies that $\bar{p} = \partial u/\partial c(\bar{c})$.

Now we would like to investigate conditions so that $\bar{k} \gg 0$. Furthermore, when can the analysis of pp. 542–546 above be carried out starting from a Peleg-Ryder solution? Suppose some $\bar{k}_j = 0$. Then $\bar{k}_{ji} = 0$ for all $i = 1, 2, \ldots, n$. It is reasonable to assume that j is essential in at least one industry. I.e., there is i such that $F^i = 0$ when $\bar{k}_{ji} = 0$. We state this formally:

Assumption 2.2(2). Every capital good is essential in at least one industry. I.e., for each j there is i such that $F^i = 0$ if $k_{ji} = 0$.

Assumption 2.2(2) implies that $\bar{k} \gg 0$. If not then $\bar{k}_j = 0$ so that some $\bar{y}_i = 0$ which contradicts $\bar{c} \gg 0$. Let $\bar{k}_{ji}, \bar{k}_i$ be a solution to problem (19). By the Kuhn-Tucker theorem there is $\bar{W}_0 \geqq 0$ such that $\bar{k}_{ji}, \bar{k}_i$ solves

$$\text{(20)} \qquad \text{maximize} \sum_{i=1}^{n} \left[\bar{p}_i F^i(k_{0i}, \ldots, k_{ni}) - \sum_{s=1}^{n} \bar{W}_s k_{si}\right] + \bar{W}_0 \left[1 - \sum_{i=1}^{n} k_{0i}\right]$$

over *all* nonnegative $k_{ji}, k_i, j = 0, 1, \ldots, n$; $i = 1, 2, \ldots n$. This may be rewritten by rearrangment as

$$\text{(21)} \qquad \text{maximize} \sum_{i=1}^{n} \left[\bar{p}_i F^i - \sum_{s=0}^{n} \bar{W}_s k_{si}\right] + \bar{W}_0 .$$

Let $C_i(\bar{W}_0, \bar{W}_1, \ldots, \bar{W}_n) \equiv \text{infimum} \{\sum_{j=0}^{n} a_{ji} \bar{W}_j \,|\, F^i(a_{0i}, \ldots, a_{ni}) \geqq 1\}$. Since $\bar{W}_0$ is a constant solving (21) amounts to maximizing the first term. Constant returns may be used in the usual way to write the term of (21) as

$$\text{(22)} \qquad \max \sum_{i=1}^{n} [\bar{p}_i - C_i] F^i(k_{0i}, \ldots, k_{ni})$$

over all $k_{ji} \geqq 0$.

Because the maximum of (19) is finite we have $\bar{p}_i \leqq C_i$ provided that F^i can be made arbitrarily large for arbitrary choice of $(k_{0i}, \ldots, k_{ni})$—which we assume. Furthermore $\bar{p}_i \not< C_i$ because $\bar{y}_i > 0$. Thus $\bar{p}_i > C_i, i = 1, 2, \ldots, n$. The infimum in the definition of C_i would be attained provided that $\bar{W}_0 > 0$, $\bar{W}_i > 0, i = 1, 2, \ldots, n$. Now $\bar{W} = \eta_i \bar{p}_i > 0, i = 1, 2, \ldots, n$. Also $\bar{W}_0$ is the increase in the maximum of (19) when additional labor is added to the one unit available. Thus it is reasonable to assume that this is positive.

Assumption 2.2(3). $\bar{W}_0 > 0$.

Let a_{ji} be a set of input output coefficients that attain the infimum, C_i, for each i. Then $\bar{p}_i = C_i$ implies

$$\bar{p}_i = \bar{a}_{0i}\bar{W}_0 + \sum_{j=1}^{n} \eta_j \bar{p}_j \bar{a}_{ji}\,, \qquad i = 1, 2, \ldots, n. \tag{23}$$

In matrix form this becomes upon setting $\bar{q} = \bar{p}/\bar{W}_0$

$$\bar{q} = \bar{a}_0 + \bar{q}\eta\bar{a} \quad \text{or} \quad \bar{q}[I - \eta\bar{a}] = \bar{a}_0\,.$$

Now Assumption 2.2(1) implies $\bar{a}_0 \gg 0$, and $\bar{q} \gg 0$ by construction. Thus Gale [7, (296)]: $[I - \eta\bar{a}]^{-1}$ exists and is nonnegative. Since $g + \delta_i < \rho + g + \delta_i$, $i = 1, 2, \ldots, n$, therefore, $[I - (g + \delta)\bar{a}]^{-1}$ exists and is nonnegative. Thus the CPF: $\bar{a}_0[I - (g + \delta)\bar{a}]^{-1}c = 1$ is well defined. Also Assumption 2.2(1) implies that $\bar{b} \equiv \bar{a}_0[I - (g + \delta)\bar{a}]^{-1} \gg 0$, $\bar{s} \equiv \bar{a}_0[I - \eta\bar{a}]^{-1} \gg 0$. It is now easy to see that $\bar{c}$ is a point that solves max $u(c)$ s.t. $\bar{s}c \leqq M$ for some M where $\bar{b}\bar{c} = 1$.

To summarize: Under Assumption 2.2(1)–(3) we may use Peleg and Ryder's existence method to prove existence of a steady state $\bar{c}$, $\bar{k}$, $\bar{p}$, $\bar{y}$ such that $\bar{p}_i = C_i = \sum_{j=0}^{n} \bar{a}_{ji}\bar{W}_i$, $\bar{W}_i = \eta_i\bar{p}_i$, $\bar{b}\bar{c} = 1$, $\bar{c}$ solves max $u(c)$ s.t. $\bar{s}c = M$ for some M and $\bar{c} \gg 0$, $\bar{k} \gg 0$, $\bar{p} \gg 0$, $\bar{y} \gg 0$.

2.3. STEADY STATE EQUILIBRIA ARE GENERAL EQUILIBRIA WITH DISTORTIONS

In this section we point out a relation between steady state analysis in multisector models and general equilibrium theory when there are distortions present. We will argue that this relation is useful in the sequel.

To gain understanding look at the simple one sector problem

$$\begin{aligned} &\text{Max} \int_0^\infty e^{-\rho t}u(c)dt \\ &\text{s.t.} \quad c + \dot{k} = f(k) - \delta_1 k \\ &\qquad k(0) = k_0\,. \end{aligned} \tag{1}$$

The steady state equations are

$$\begin{aligned} \bar{c} + \delta_1\bar{k} &= y \equiv f(\bar{k}) \\ f'(k) - \rho &= \delta_1\,. \end{aligned} \tag{2}$$

Look at the following model of a competitive economy

$$\begin{aligned} &\text{consumers:} && \text{maximize } u(c) \\ &&& \text{s.t. } pc \leqq \pi + (w - \delta_1 p)k + T \\ &\text{firms:} && \text{maximize } pf(k) - wk - \rho pk\,. \end{aligned} \tag{3}$$

Let c^d, k^s solve the consumers' problem and k^d solve the firms' problem. The consumer forms expectations on p, w, π, T and then chooses c^d, k^s i.e., consumption demand and capital supply to maximize his welfare. T represents exogenous lump sum income to the consumer and π is lump sum income from profits. The consumer obtains instantaneous rent w for each unit of k but on the other hand maintenance cost $\delta_1 p$ must be paid out. Obviously $k^s = \infty$ if $w/p > \delta_1$, $k^s = 0$

if $w/p < \delta_1$ and k^s is any nonnegative number if $w/p = \delta_1$. Thus capital supply is *perfectly* elastic. The firm demands k^d to maximize profits. But an *ad valorem* tax ρ is levied on the factor k. The tax ρpk is then given to the consumer. This tax causes a distortion and thus a resultant deadweight loss. We may now define equilibrium. Normalize prices by setting $p = 1$.

DEFINITION. An equilibrium distorted by ρ (a ρ-equilibrium for short) is a set of values $k^s, k^d, c^s, c^d, \bar{T}, \bar{\pi}, \bar{w}$ such that k^s, c^d solves the consumers' problem and k^d solves the firms' problem and

$$(3) \qquad k^s = k^d \equiv \bar{k}, \bar{y} \equiv f(\bar{k}) = c^d + \delta_1 k^s , \quad \frac{\bar{\pi}}{\bar{p}} = f(\bar{k}) - \bar{w}\bar{k} - \rho\bar{k}$$
$$\bar{T} = \bar{p}\rho\dot{k} .$$

It is trivial to see that $\bar{c}, \bar{k}$ is an equilibrium distorted by ρ if and only if $\bar{c}, \bar{k}$ solves (2). To see this note that if $\bar{c}, \bar{k}$ is an equilibrium distorted by ρ then $f'(\bar{k}) - \rho = \bar{w}/\bar{p} = \delta_1$. The latter equation $\bar{w}/\bar{p} = \delta_1$ follows from $k^s = k^d = \bar{k} < \infty$. Now in equilibrium $f(\bar{k}) = \bar{c} + \delta_1\bar{k}$ from the budget constraint. Thus $\bar{c}, \bar{k}$ solves (2). The other implication is equally trivial. Now let us apply this equivalence to the model with one primary factor in Section 2.2.

DEFINITION. A ρ-equilibrium is a solution to the following general equilibrium problem.

(a) consumer: maximize $u(c_1, c_2, \ldots, c_n)$

$$\text{s. t. } \sum p_i c_i = \pi + w_0 1 + \sum w_i k_i - \sum (\delta_i + g) p_i k_i + T$$

(b) firm: maximize $\sum p_i F_i(k_{0i}, k_{1i}, \ldots, k_{ni}) - \sum_{j=0}^{n} w_j k_j - \rho \sum_{j=1}^{n} p_j k_j$

$$\text{s. t. } \sum_i k_{ji} \leqq k_j . \qquad j = 0, 1, 2, \ldots, n$$

(c) $T = \rho \sum_{j=1}^{n} p_j k_j .$

Thus a ρ-equilibrium is a vector $(k^s, k^d, y^s, c^d, \bar{T}, \bar{\pi}, \bar{p}, \bar{w})$ such that c^d, k^s solves (a), k^d solves (b),

$$y^s \equiv (F^1, \ldots, F^n), k^s = k^d, c^d + (\delta + g)k^s = y^s, \bar{T} = \rho \sum_{j=1}^{n} \bar{p}_j k_j{}^d .$$

Here the consumer has one unit of labor that receives wage $\bar{w}_0$. The other symbols are self explanatory.

It is trivial to point out and prove the following theorem.

THEOREM (Equivalence Theorem). *A ρ-equilibrium is a steady state for the optimal growth problem of Section* 2.2 *when future utilities are discounted by ρ. I.e., a ρ equilibrium solves equations* (6) *Section* 2.1 *in steady state. Furthermore a steady state solution to* (6) *is a ρ-equilibrium.*

Let us point out why we think that this equivalence is important. First, this equivalence theorem will generalize to cases of joint production and many or no primary factors. Thus it may be applied to general growth models such as McKenzie [**18**] for example. Second, there is already a small but growing literature on existence of general equilibrium with distortions. See Shoven and Whalley [**27**], [**28**], Shoven [**26**], and Sontheimer [**29**], [**30**]. Thus the equivalence theorem allows us to apply this literature to existence of steady states in optimal growth under discounting of future utilities. Third, we predict that a literature on uniqueness of distorted-equilibria will develop along the lines of the standard analysis of uniqueness of competitive equilibria exposited in Arrow and Hahn [**1**, (chapter 9)]. Once this has been done (if it hasn't already been done) we may read off theorems on uniqueness of steady states in very general models of optimum growth under discounting. The equivalence theorem shows that the non uniqueness problem under discounting is not much more mysterious than uniqueness of competitive equilibrium, and we think that it is fair to say that we have a good understanding of uniqueness of competitive equilibrium. Fourth we may be able to use some of the standard stability of competitive equilibrium results in Arrow and Hahn [**1**, (Chapter 11)] to study local stability of steady states but we are uncertain about the potential of use of the equivalence theorem in this direction.

3.1. CONDITIONS FOR UNIQUENESS OF STEADY STATE INVOLVING THE NON SINGULARITY OF A CERTAIN JACOBIAN MATRIX

We may develop an approach to uniqueness of steady states in analogy with the approach to uniqueness of general equilibrium involving the nonsingularity of a certain Jacobian matrix. See Arrow and Hahn [**1**, (236)] for example. The analogy between our method and theirs is a little forced but is worth mentioning nonetheless.

Look at problem (2) of Section 2.2. Any optimum program must satisfy

$$
\begin{aligned}
&y_1 \equiv \text{maximize } F^1(l_1, k_{11}, \ldots, k_{n1}) \\
&\text{s. t. } y_i \leqq F^i(l_i, k_{1i}, \ldots, k_{ni}), && i = 2, 3, \ldots, n \\
&\sum_{i=1}^{n} k_{ji} \leqq k_j, && j = 1, 2, \ldots, n \\
&\sum_{s=1}^{n} l_s \leqq 1, y_j \equiv c_j + \dot{k}_j + (\delta_j + g)k_j, && j = 1, 2, \ldots, n.
\end{aligned}
\tag{1}
$$

Thus we may write $y_1 = T[y_2, \ldots, y_n; k_1, \ldots, k_n]$ in the usual way of describing problem (1) by a transformation function T(see [**4**, (285)]). In this way problem (2) of Section 2.2 may be rewritten as

$$
\begin{aligned}
\text{Maximize} \int_0^\infty e^{-\rho t} u[T(c_2 + (g + \delta_2)k_2 + \dot{k}_2, \ldots, c_n + (g + \delta_n)k_n \\
+ \dot{k}_n; k_1, \ldots, k_n) - (g + \delta_1)k_1 - \dot{k}_1, c_2, \ldots, c_n]dt\,.
\end{aligned}
\tag{2}
$$

Let $e^{-\rho t}F(\dot{k}, k)$ be the integrand of (2). Then the Euler equations for the calculus of variations problem: maximize $\int_0^\infty e^{-\rho t}F(\dot{k}, k)dt$ are

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{k}_i}\right) = \frac{\partial F}{\partial k_i} + \rho \frac{\partial F}{\partial \dot{k}_i}, \qquad i = 1, 2, \ldots, n.$$

But

$$\frac{\partial F}{\partial \dot{k}_1} = -\frac{\partial u}{\partial c_1};$$

$$\frac{\partial F}{\partial k_1} = \frac{\partial u}{\partial c_1}\left[\frac{\partial T}{\partial k_1} - (g + \delta_1)\right]; \quad \frac{\partial F}{\partial \dot{k}_i} = \frac{\partial u}{\partial c_1}\frac{\partial T}{\partial y_i}, \qquad i = 2, 3, \ldots, n;$$

$$\frac{\partial F}{\partial k_i} = \frac{\partial u}{\partial c_1}\left[\frac{\partial T}{\partial k_1} + \frac{\partial T}{\partial y_i}(g + \delta_i)\right], \qquad i = 2, 3, \ldots, n.$$ [9]

The steady state equations boil down to

$$(3) \qquad \frac{\partial T}{\partial k_1} = \rho + g + \delta_1, \quad \frac{\partial T}{\partial y_i}[\rho + g + \delta_i] + \frac{\partial T}{\partial k_i} = 0$$

where given $k_1, \ldots, k_n$, the vector $c_2, \ldots, c_n$ is determined by

$$(4) \qquad \underset{(c_2,\ldots,c_n)}{\text{Maximize}}\, u[T(c_2 + (g + \delta_2)k_2, \ldots, c_n + (g + \delta_n)k_n; k_1, \ldots, k_n) - (g + \delta_1)k_1, c_2, \ldots, c_n].$$

Let ${}_2c = (c_2, \ldots, c_n)$. Let $H({}_2c, k) = T(c_2 + (g + \delta_2)k_2, \ldots, c_n + (g + \delta_n)k_n; k_1, \ldots, k_n) - (g + \delta_1)k_1$. Let $h(k)$ be the optimum ${}_2c$ for (4). Consider the problem maximize $u[H({}_2c, k), {}_2c]$ and note that it generates equation (3) for the no ${}_2c, k$ discounting case $\rho = 0$. It can be shown that $u[H({}_2c, k), {}_2c]$ is concave in its arguments. This follows directly from the definition of T, the concavity of $F^i, i = 1, 2, \ldots, n$, and the concavity of u. Thus if $u[H({}_2c, k), {}_2c]$ is strictly concave there will be only one maximum. Furthermore, setting all partial derivatives equal to zero yields it. The steady state equations (3), (4) can be written in the compact form

$$(5) \qquad G_1(k, \rho) = 0, \ldots, G_n(k, \rho) = 0$$

where

$$G_1(k, \rho) = -[\rho + g + \delta_1] + \frac{\partial T}{\partial k_1}, \quad G_i(k, \rho) = \frac{\partial T}{\partial y_i}[\rho + g + \delta_i] + \frac{\partial T}{\partial k_i}, \qquad i = 2, 3, \ldots, n.$$

The reader will note that the case $\rho = 0$ is the borderline Ramsey case studied by McKenzie [**18**], Gale [**8**] and Brock [3] in the context of discrete time models

[9] This follows from $0 = \partial F/\partial k_i + \rho\, \partial F/\partial \dot{k}_i = \partial u/\partial c_1[\partial T/\partial k_i + \partial T/\partial y_i(g + \delta_i)] + \rho\, \partial u/\partial c_1\, \partial T/\partial y_i$ $i = 2, 3, \ldots, n$. Note that the term $\partial u/\partial c_1$ cancels. We are left with $\partial T/\partial k_i + \partial T/\partial y_i[\rho + g + \delta_i] = 0, i = 2, 3, \ldots, n$. The expression $\partial T/\partial k_1 = \rho + g + \delta_1$ is obtained in a similar manner.

In [**18**], [**8**], [3], it was shown that optimum paths converge to a unique steady state for the nondiscounted case. Rockafellar [**24**] and Samuelson [**25**] have studied continuous time models for the case $\rho = 0$. In this section we shall treat the uniqueness question for $\rho > 0$.

The reason that this is much more difficult than the uniqueness question for $\rho = 0$ is because equations (5) are generated by the concave programming problem maximize $k\, u[H({}_2c, k), {}_2c]$ over ${}_2c, k$ for the case $\rho = 0$ whereas equations (5) for $\rho > 0$ are not generated by setting the partial derivatives of some function $E(k)$ equal to zero. For if such E exists and E is continuously differentiable then $\partial^2 E/\partial k_i \partial k_j = \partial^2 E/\partial k_j \partial k_i$. Thus $\partial G_i/\partial k_j = \partial G_j/\partial k_i$ must hold. The latter holds for $\rho = 0$ but not for $\rho > 0$. This is so because $\partial E/\partial k_i = G_i$ for the case, $\rho = 0$. It may well be that one could find an "integrating" factor $\mu(k)$ such that $\partial/\partial k_i(\mu G_i) = \partial/\partial k_j(\mu G_j)$ so that an $E(k)$ could be found so that $\partial E/\partial k_j = \mu G_i$. But such a method cannot yield an E such that there is only one k such that $\partial E/\partial k = 0$ for all concave utility functions u because of the multiple steady state example of Section 2.2.

Before we move on let us point out that the formulation (2) is very general and can handle many cases of joint production. Furthermore the class of problems yielding steady state equations of the form (5) is so general that we cannot think of any problem that does not yield steady state equations of the form (5).

Look at the Jacobian matrix $\partial G/\partial k$ of (5). We shall assume that this is nonsingular on $\{k/G(k, \rho) = 0.\ \rho \geqq 0\}$.[10] For the case $\rho = 0$ because G is the gradient of some $E(k)$, $\partial G/\partial k$ will be singular for a "measure zero" set of problems.

[10] Lionel McKenzie asked us if there might not be something in the economics of the problem that would make $\partial G/\partial k$ "naturally" singular. This question is best dealt with by considering explicitly the maximization problem that generates the steady state when $\rho = 0$.

This problem is

(a) $$\text{maximize } u(F^1(k_{01}, \ldots, k_{n1}) - (g + \delta_1)k_1, \ldots, F^n(k_{0n}, \ldots, k_{nn}) - (g + \delta_n)k_n)$$
$$\text{s. t. } \sum_{i=1}^{n} k_{ji} \leqq k_j, \qquad j = 0,1,2,\ldots,n.$$

Here $k^0 = 1, k_{0i} = l_i$.

If u is strictly concave and F^i are strictly concave (except on rays—recall that they are homogeneous of degree 1) then one would expect that the maximum value $V(k_1, \ldots, k_n)$ of the objective in problem (a) would be strictly concave in $k_1, \ldots, k_n$. Thus one would think that there is a good chance that the Hessian of V would be nonsingular. Since $\partial V/\partial k_i = \partial G/\partial k_i$ one would think that there is hope for the truth of the conjecture that "most" strictly concave problems of the form (a) generate nonsingular Hessians for $\rho = 0$.

It is easy to generate examples of problems (a) with a nonsingular Hessian at the optimum. The simplest is

(b) $$\text{maximize} \log [k_{01}^{1/3} k_{11}^{1/3} k_{21}^{1/3} - \delta k_1] + \log [k_{02}^{1/3} k_{12}^{1/3} k_{22}^{1/3} - \delta k_2]$$
$$\text{s. t. } k_{01} + k_{02} = 1, k_{11} + k_{12} = k_1, k_{21} + k_{22} = k_2.$$

Since everything is symmetric

$$k_{01} = k_{02} = \frac{1}{2},\ k_{11} = k_{12} = \frac{1}{2}k_1,\ k_{21} = k_{22} = \frac{1}{2}k_2,\ k_1 = k_2 \equiv k\,.$$

(*Continued on next page*)

Thus for $\rho = 0$, nonsingularity of $\partial G/\partial k$ seems relatively harmless. For $\rho > 0$ it is a purely mathematical condition. We can only hope that most economically interpretable hypotheses will imply that $\partial G/\partial k$ is nonsingular on the set of equilibria. This turned out to be the case for uniqueness of general equilibrium (see Arrow and Hahn [**1**, (Chapter 9)], Pearce and Wise [**21**], Mas-Collel [**15**]). Because ρ steady states are general equilibria distorted by ρ in the sense of Section 2.3 it is quite likely that economically interpretable hypotheses will imply $\partial G/\partial k$ is nonsingular on the set of steady states. Enough for the defense—let us get on with the theorem.

THEOREM 1. *Let the G_1 of equations (5) be defined on $Q \equiv \{(k, \rho) \mid k \geqq 0, \rho \geqq 0\}$, be continuously differentiable on $\{(k, \rho) \mid (k, \rho)$ in $Q, k \gg 0\}$. Assume that for all $\rho \geqq 0$ there is at least one $k \geqq 0$ such that $G(k, \rho) = 0$ and for all $\rho \geqq 0$ all solutions of $G(k, \rho) = 0$ are strictly positive. Furthermore assume for all $\rho \geqq 0$ that $\partial G/\partial k$ is nonsingular on $\{k \mid G(k, \rho) = 0, (k, \rho)$ in $Q\}$ and for $\rho = 0$ assume that there is just one $(k, 0)$ in Q such that $G(k, 0) = 0$. Then there is just one steady state for each $\rho \geqq 0$.*

Before we prove this let us outline the strategy. Look at the set $M_\rho = \{k \mid G(k, \rho) = 0\}$. When $\rho = 0$, the number of elements in M_ρ is one. Since G is continuously differentiable $|\partial G/\partial k|$ is a continuous function of k, ρ. Thus the determinant is nonzero in a neighborhood of $(k_0, 0)$ where $M_0 = \{k_0\}$. Therefore there is a neighborhood of 0 such that there is only one solution of $G(k, \rho) = 0$ for all ρ in this neighborhood. Let $\bar{\rho} \equiv \sup \{\rho \mid G(k, \rho) = 0$ has only one solution on $[0, \rho)\}$. We show that $\bar{\rho} = \infty$.

PROOF OF THE THEOREM. In order to rid ourselves of pathological cases we shall assume that $\bigcup_{\rho \in B} M_\rho$ is bounded if the set B is closed and bounded, i.e., compact. This insures that steady state capital labor ratios cannot go off to infinity at finite rates of discount.[11] This can be obtained from the diminishing returns properties of the production functions. Now let us get on with the proof.

(*Continued*)
The objective reduces to

$$2 \log \left(\left(\frac{1}{2} \right) k^{2/3} - \delta k \right). \tag{c}$$

Differentiating and setting the derivative equal to zero we get.

$$\frac{1}{3} k^{-(1/3)} = \delta. \tag{d}$$

Thus $k = (3\delta)^{-3}$. It is easy to check that this is, indeed, the solution to (b). We leave to the reader the straightforward, but tedious, job of calculating the Hessian and evaluating it at $k = (3\delta)^{-3}$. It is nonsingular. Thus our conjecture is not empty.

It is beyond the scope of this paper to attempt a formal proof of our conjecture. However it is plausible, at least for the case $\rho = 0$. The strict concave utility function, and the labor constraint, $\sum_{i=1}^{n} k_{0i} = 1$ tend to remove the "natural tendency for singularity" due to constant returns F^i.

[11] See [**18**, (357)] for a standard type of assumption that rids us of this problem.

We claim that there is a neighborhood $[0, \varepsilon_0)$ of $\rho = 0$ such that $\rho \in [0, \varepsilon_0)$ implies $G(k, \rho) = 0$ has only one solution. By the implicit function theorem there is a neighborhood N of $(k_0, 0)$ in $n + 1$ dimensional space such that there is a function $k(\rho)$ having the property that $G(k(\rho), \rho) = 0$ on N. Suppose there is no neighborhood $[0, \varepsilon_0)$ of $\varepsilon = 0$ such that $\rho \in [0, \varepsilon_0)$ implies there is only one k_ρ such that $G(k_\rho, \rho) = 0$. Then there is a sequence $\varepsilon_n \rightarrow 0$ such that there are two solutions k_n, k^n of $G(k, \varepsilon_n) = 0$. Since $\bigcup_{n=1}^{\infty} M_{\varepsilon_n}$ is bounded the sequences $\{k_n\}$, $\{k^n\}$ lie in a bounded set. Hence a subsequence of one of these may be picked that converges to a limit point k that lies outside of the interior of N. The reader will recall that the equation $G(k, \rho) = 0$ has only one solution for $(k, \rho) \in N$. Thus we have constructed a pair of solutions to $G(k, 0) = 0$. This is a contradiction to uniqueness of the solution for $\rho = 0$. Thus a neighborhood $[0, \varepsilon_0)$ exists such that M_ρ has only one element for $\rho \in [0, \varepsilon_0)$. Let $\bar{\rho} = \sup \{\tilde{\rho} \mid M_\rho$ has only one element for $\rho \in [0, \tilde{\rho})\}$. If $\bar{\rho} = \infty$ we are finished. Suppose $\bar{\rho} < \infty$. Let $\varepsilon > 0$ then for $\rho < \bar{\rho}$, M_ρ has only one element and for $\rho \in (\bar{\rho}, \bar{\rho} + \varepsilon)$ there is ρ_ε such that M_{ρ_σ} has more than one element. Since $\partial G(k, \bar{\rho})/\partial k$ is non singular for $\bar{k} \in M_\rho$ there is a neighborhood N of $(\bar{k}, \bar{\rho})$ such that on N there is a function $k(\rho)$ such that $G(k(\rho), \rho) = 0$. By definition of $\bar{\rho}$ we can choose ε so small that $(\bar{\rho} - \varepsilon, \bar{\rho} + \varepsilon)$ is in the projection of N on the ρ axis. There is $\rho_\varepsilon \in (\bar{\rho}, \bar{\rho} + \varepsilon)$ such that M_{ρ_ε} has more than one element, and one of these element $\tilde{k}_\varepsilon$ is outside N by definition of N. Choose a subsequence $\varepsilon_n \rightarrow 0$, $\tilde{k}_{\varepsilon_n} \rightarrow \tilde{k}$. Since $\tilde{k}_{\varepsilon_n}$ is outside of N for each n the limit point $\tilde{k}$ is outside the interior of N. Now $G(\tilde{k}, \bar{\rho}) = 0$. Using the nonsingularity of $\partial G/\partial k$ at $(k, \bar{\rho})$ and the implicit function theorem we may choose a neighborhood M such that there is a function $\tilde{k}(\rho)$ such that $G(\tilde{k}(\rho), \rho) = 0$ on M, and $M \cap N = \phi$. Now look at the projection of M, N on the ρ axis. We may select an open set 0, containing $\bar{\rho}$ such that for $\rho \in 0$ both $G(k(\rho), \rho) = 0$, $G(\tilde{k}(\rho), \rho) = 0$ and $k(\rho) \neq \tilde{k}(\rho)$. Thus we have created two distinct solutions for $\rho < \bar{\rho}$. This contradicts the definition of $\bar{\rho}$ as the "largest" ρ such that on $[0, \rho)$ there is only one solution of $G(k, \rho) = 0$.[12] This ends the proof.

The nonsingularity of $\partial G/\partial k$ across the solution set of $G(k, \rho) = 0$ is an obscure assumption. It would be worthwhile to relate it to the assumptions of Section 2.2. Nonetheless it is worthwhile to know its implications at least. It may well turn out to be a weak assumption that may turn out to be implied by the assumption of Section 2.2 for example.

[12] At the risk of being pedantic it may be worthwhile to remark why this theorem "works." Basically the solution set and domain of definition must be such that the implicit function theorem may be applied. I.e., for each (k, ρ) such that $G(k, \rho) = 0$ there must be an open neighborhood N of (k, ρ) such that N is contained in the domain of definition of the mapping. The strict positivity of solutions k for each $\rho \geqq 0$ ensures this. By use of the open neighborhood property of the domain and solution set we may use the implicit function theorem to assert that $\bar{\rho} = \infty$. There may be a question about the open neighborhood property at $\rho = 0$. But this may be taken care of by extending $G(k, \rho)$ to $\rho < 0, \rho$ small. Of course in this case the steady states may not be optimal for $\rho < 0$ but that doesn't harm us.

Note too that the boundedness of $\bigcup_{\rho \in B} M_\rho$ is essential.

4. SOME REMARKS ON THE MEANING OF ALL THIS

We see the study of uniqueness and stability in models of the type analyzed in this paper as being interesting not only for optimal growth but also for laying the foundations for a serious general equilibrium theory over time with capital accumulation entering in a nontrivial way. One may look at the model of Section 2.1, for example, as the solution to an equilibrium problem where there is one consumer who spends his income from profits and rentals obtained from capital lent out to firms on consumption and capital accumulation. Let the firms be static profit maximizers. Let the consumer be a price taker who is in charge of capital accumulation. Equilibrium is a set of prices on consumption and rental rates on equipment such that markets clear at all moments of time. I.e., this is the perfect foresight case. See Brock [**2**] and Lucas and Prescott [**14**] for expositions on perfect foresight as an equilibrium concept. In the one consumer model the equilibrium sequence of prices, rents, consumptions and capitals is unique. But we may ask if it converges to a steady state as time passes. It certainly cannot converge to the same steady state independent of initial conditions if the steady state is not unique. Thus our problem arises as part of general equilibrium theory.

If the steady state is not unique then "comparative statics" across parameter changes may be meaningless. One certainly is interested in the impact on the long run equilibrium consumption, capital constellation due to a parameter change in a world of more than one or two goods. One could say that the uniqueness of steady state problems is just as basic for long run problems as uniqueness of competitive equilibrium is in the usual static models. It is reassuring that in the model studied in this paper, at least, that "normality" type assumptions will give a unique steady state.

More elaborate equilibrium models may be constructed. Our analysis, we hope, should be helpful in studying uniqueness of the long run equilibrium capital stock configuration in more sophisticated models. See Walter Heller's Ph.D. thesis [**9**] for work viewing growth theory as a branch of equilibrium theory and its possibilities of use in constructing a general equilibrium theory where capital accumulation over time is brought in an essential way.

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REFERENCES

[**1**] ARROW, K. AND F. HAHN, *General Competitive Analysis* (San Francisco: Holden-Day, Inc. 1971).

[**2**] BROCK, W., "On Models of Expectations Generated by Maximizing Behavior of Economic Agents Over Time," *Journal of Economic Theory*, V (June, 1972), 479–513.

[**3**] ———, "On Existence of Weakly Maximal Programmes in Multisector Economy," *Review of Economic Studies*, XXXVII (April, 1970), 275–280.

[**4**] BURMEISTER, E. AND R. DOBELL, *Mathematical Theories of Economic Growth* (New York: The Macmillan Co., 1970).

[**5**] ——— AND K. KUGA, "The Factor Price Frontier in a Neoclassical Multisector Model," *International Economic Review*, XI (February, 1970), 162–174.

[6] Burmeister, E. and S. Turnovsky, "Capital Deepening Response in an Economy with Heterogeneous Capital Goods," (mimeographed, Department of Economics, University of Pennsylvania, Philadelphia, Pennsylvania and School of General Studies, Australian National University, Canberra, Australia, (December, 1971), forthcoming in *American Economic Review*.

[7] Gale, D., *The Theory of Linear Economic Models* (New York: McGraw-Hill, 1960).

[8] ———, "On Optimal Development in a Multi-Sector Economy," *Review of Economic Studies*, XXXIV (January, 1967), 1-18.

[9] Heller, W. P., "Disequilibrium Dynamics of Competitive Growth Paths," forthcoming in *Econometrica*.

[10] Inada, K., "On a Two-Sector Model of Economic Growth: Comment and a Generalization," *Review of Economic Studies*, XXX (June, 1963), 119-127.

[11] Kurz, M., "Optimal Economic Growth ond Wealth Effects," *International Economic Review*, IX (October, 1968), 348-357.

[12] Levhari, D. and N. Liviatan, "On Stability in the Saddle Point Sense," *Journal of Economic Theory*, IV (February, 1972), 88-93.

[13] Liviatan, N. and P. A. Samuelson, "Notes on Turnpikes: Stable and Unstable," *Journal of Economic Theory*, I (December, 1969), 454-475,

[14] Lucas, R. and E. Prescott, "Investment Under Uncertainty," *Econometrica*, XXXIX (September, 1971), 659-681.

[15] Mas-Colell, A., "On the Uniqueness of Competitive Equilibrium and the Jacobian Condition," (mimeographed, Department of Economics, University of Minnesota, Minneapolis, Minnesota, May, 1972).

[16] ———, "Local Uniqueness and Global Properties of Equilibria," (mimeographed, Department of Economics, University of Minnesota, Minneapolis, Minnesota, May, 1972).

[17] McKenzie, L., "Matrices with Dominant Diagonals and Economic Theory, in K. Arrow, S. Karlin, and P. Suppes eds., *Mathematical Methods in the Social Sciences, 1959* (Stanford: Stanford University Press, 1960), 47-62.

[18] ———, "Accumulation Programs of Maximum Utility and the Von Neumann Facet," in *Value, Capital, and Growth: Essays in Honor of Sir John Hicks* (Chicago: Aldine, 1968), 353-383.

[19] Mirrlees, J., "The Dynamic Nonsubstitution Theorem," *Review of Economic Studies*, XXVI (January, 1969), 67-76.

[20] Morishima, M., *Equilibrium, Stability, and Growth* (Oxford: Clarendon Press, 1964).

[21] Pearce, I. and J. Wise, "On the Uniqueness of Competitive Equilibrium," forthcoming

[22] Peleg, B. and H. Ryder, "The Modified Golden Rule of a Multisector Economy, Research Memorandum No. 79, Department of Mathematics, The Hebrew University of Jerusalem, Israel.

[23] Pontryagin, L. S. *et al.*, *The Mathematical Theory of Optimal* Processes (New York: Interscience Publishers, Inc.), 1962.

[24] Rockafellar, R. T., "Saddle Points of Hamiltonian Systems in Convex Problems of La Grange," (mimeographed, Department of Mathematics, University of Washington, Seattle, Washington, September, 1971).

[25] Samuelson, P. A., "The General Saddlepoint Property of Optimal Control Motions," *Journal of Economic Theory*, V (August, 1972), 102-120.

[26] Shoven, J. B., "A Proof of the Existence of A General Equilibrium with Ad Valorem Commodity Taxes," (mimeographed, Yale University, Department of Economics, 1972).

[27] ——— and J. Whalley, "A General Equilibrium Calculation of the Effects of Differential Taxation of Income from Capital in the U.S.," *Journal of Public Economics*, I (November, 1972), 281-322.

[28] ———, "General Equilibrium with Taxes: A Computational Procedure and an Existence Proof, (mimeographed, Yale University, Department of Economics, 1972), forthcoming in *Review of Economic Studies*.

[29] Sontheimer, K. C., "The Existence of International Trade Equilibrium with Trade Tax-Subsidy Distortions," *Econometrica*, XXXIX (November, 1971), 1015-1035.

[30] ———, "An Existence Theorem for the Second Best," *Journal of Economic Theory*, III (March, 1971), 1-21.

[31] STIGLITZ, J. E., "Non-Substitution Theorems with Durable Goods," *Review of Economic Studies*, XXVII (October, 1970), 543-554.

[32] SUTHERLAND R. S., "On Optimal Development in a Multisectoral Economy: The Discounted Case," *Review of Economic Studies*, XXXVII (October, 1970), 585-589.