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Using the generalized Schur form to solve a multivariate linear rational expectations model☆

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Abstract

In this paper, I show how to use the generalized Schur form to solve a system of linear expectational difference equations (a multivariate linear rational expectations model). The method is simple to understand and to use, and is applicable to a large class of rational expectations models. The only hard part is taken care of by just two standard algorithms, both of which are available as freeware on the Internet as part of LAPACK. Like other matrix decomposition based methods, it is also very fast to execute.

JEL classification: C32; C63

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1. Introduction

In this paper, I show how to use the complex generalized Schur form to solve a system of linear expectational difference equations (a multivariate linear

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rational expectations model). The approach is simple to understand and to use, and is robust in the sense that it works well for almost any multivariate linear rational expectations model that has a solution at all. It also has the advantage of exploiting readily available state-of-the-art numerical algorithms to take care of the main calculations.

The method advocated in this paper is a small but nevertheless significant improvement upon the available alternatives. The approach is closely related especially to those of Sims (1996) and King and Watson (1995a,b). Like Sims' approach (but unlike King and Watson's), the approach presented here is based on the generalized Schur form or QZ decomposition of a matrix pencil. An alternative, presented in King and Watson (1995a) is to use the canonical form of a matrix pencil (see Gantmacher, 1959). Conceptually, the Schur form has the advantage of treating infinite and finite unstable eigenvalues in a unified way. Moreover, it is universally agreed among numerical analysis specialists that the Schur form is preferable from a computational point of view.[1] This fact has led to the development of readily available state-of-the-art algorithms for the calculation and manipulation of the generalized Schur form. The methods recommended in this paper are therefore very easy to implement and achieve a high degree of computational efficiency.[2]

By generalizing the assumption that certain variables are predetermined, the approach here is slightly more general than that of King and Watson (1995a,b), and can easily be extended to achieve the same level of generality as Sections 1–3 of Sims (1996).[3] The differences in terms of generality are the following. Like Blanchard and Kahn (1980), King and Watson (1995a,b) assume that certain variables have an exogenously given initial value and zero one-step-ahead prediction error. (These variables are called "predetermined".) Sims' (1996) setup implies the assumption that a given linear combination of variables has an exogenously given one-step-ahead prediction error, whereas it emerges endogenously what linear combination of initial values has to be given exogenously to pin down the solution. Meanwhile, in the present paper, I assume that certain

[1] The problem with the canonical form is that, unlike the Schur form, it is not continuous in the entries of the matrices to be factorized. It is therefore numerically unstable. For that reason, King and Watson (1995a) do not in fact recommend using the canonical form for calculations. An alternative approach, based on the singular value decomposition, is presented in King and Watson (1995b).

[2] The required LAPACK routines are xGEGS (which calculates the generalized Schur form) and xTGSEN (which reorders it). The x should be replaced by a *Z* if the double precision complex version is desired, a C for the single precision complex version, a D for the double precision real version or an S for the single precision real version. LAPACK can be freely downloaded from http://www.netlib.org/lapack. For documentation of these routines, see Kågström and Poromaa (1996a,b).

[3] Section 4 in Sims allows different variables to have different upper limits on their long-run growth rates. Section 5 in Sims generalizes to continuous time.

variables have an exogenously given initial value and an exogenously given (but not necessarily zero) one-step-ahead prediction error.

Moreover, this paper attempts to improve upon the approach of Sims (1996) in several ways. Firstly, it aspires to be conceptually more straightforward. Generally speaking, the treatment follows that of Blanchard and Kahn (1980) much more closely than that of Sims. Those familiar with their work should therefore find it easier to follow the discussion of the present paper than that of Sims.

In detail, the main difference lies in the method employed to pin down the one-step-ahead prediction error of the endogenous variables. In Sims (1996), the approach is as follows. First, the equation is written without expectational terms but with an endogenous prediction error process. The system is then transformed into a triangular one using the generalized Schur form, and the unstable block of equations is isolated. This block is then solved forward, and the endogenous prediction error process is solved for by imposing the informational restriction that the solution must be adapted to the given filtration. At this stage, no extraneous assumptions (e.g. about what variables are predetermined) are invoked; all information about the solution is given in the coefficient matrices of the difference equation itself.

By contrast, the approach in this paper follows very closely the one used in Blanchard and Kahn (1980). No endogenous prediction error is introduced, and the unstable block of the triangular system is solved forward without having to solve for the prediction error separately. Instead, the endogenous prediction error process is solved for when solving the *stable* block of equations and use is then made of extraneous assumptions which generalize Blanchard and Kahn's assumption of certain variables being predetermined.

Secondly, whereas Sims' solution is presented only in terms of an infinite sum of future values of a rather general driving process, I focus mainly on the case where the driving process is (representable as) a VAR(1). In this case, the infinite sums can be eliminated by calculating the relevant geometric sum of matrices, and I discuss efficient ways of doing this.

Finally, I point out the usefulness of the real generalized Schur form, which is about 4–5 times faster to calculate than the complex generalized Schur form.[4]

The methods I present are tailored towards circumstances in which there is a *unique stable* solution. However, like Sims (1996) I also characterize the set of stable solutions when this set has more than one element.

Other related papers include Binder and Pesaran (1994, 1996, 1997), Gilli and Pauletto (1997), and Uhlig (1995). Binder and Pesaran (1997) and Gilli and Pauletto (1997) transform the expectational difference equation into a large

[4] A caveat here is that the reordering algorithm for the real generalized Schur form is about half as fast as that for the complex one. Nevertheless, this leaves us with roughly a doubling of the speed.

sparse system of linear equations and then proceed to solve this sparse linear system. This approach is conceptually straightforward, involves only elementary matrix operations, and is probably the best one in a finite-horizon setting. However, some simple computational experiments suggest that the Binder and Pesaran (1997) algorithm is about six times slower than the method advocated in the present paper for a linearized version of the infinite-horizon stochastic growth model with standard parameter values (for details, see Section 6.1).

Meanwhile, Binder and Pesaran (1994, 1996) and Uhlig (1995) use an undetermined coefficients approach and reduce the solution of the expectational difference equation to the solution of a matrix quadratic equation. Apparently, however, matrix quadratic equations seem ultimately to boil down to matrix factorization problems. Uhlig (1995) demonstrates that a matrix quadratic equation can be solved by solving a (generalized) eigenvalue/eigenvector problem. Presumably, the generalized Schur form could also be used; this would probably be slightly preferable from a computational point of view. In any case, the very fact that matrix quadratic equations boil down to matrix factorization problems indicates that they constitute a detour and that we may as well use methods based directly on matrix factorizations.

The paper is organized as follows. In Section 2, I define the problem. Section 3 presents the generalized Schur form. Section 4 states the assumptions I make. These include sufficient conditions for there to be a unique stable solution. Section 5 uses the generalized Schur form to solve the problem. Section 6 discusses some applications. Section 7 concludes.

2. The problem

Throughout the paper, let $(\Omega, \mathscr{F}, P)$ be an a priori given probability space, and let $\underline{\mathscr{F}} = \{\mathscr{F}_t;\ t = 0, 1, \dots\}$ be a filtration of $\mathscr{F}$. Let $\mathbf{z} = \{z_t;\ t = 0, 1, \dots\}$ be an exogenously given n_z-dimensional $\underline{\mathscr{F}}$-adapted stochastic process. Let A, B be $n \times n$ matrices and let C be an $n \times n_z$ matrix.[5] Our goal is to solve, for an n-dimensional process $\mathbf{x}$, equations of the following form[6]

$$A\,\mathrm{E}[x_{t+1}|\mathscr{F}_t] = Bx_t + Cz_t, \quad t = 0, 1, \dots\ . \tag{2.1}$$

[5] Setting $C = I$ involves no loss of generality, of course. But in many applications it is convenient to leave the C matrix in.

[6] The fact that the equation is of first-order involves no loss of generality. Any kth order system can be reduced to a first-order system. Because of the possible presence of expectations with respect to information available at different times, this reduction is not entirely trivial, but the problem has been solved by Binder and Pesaran (1994). Their canonical form is of second order; see Appendix A of the present paper for a discussion of how to reduce this to a first-order system.

These systems of course arise in many contexts. One rich set of examples comes from the linearization of the individual optimization conditions and market clearing conditions in a (possibly distorted) dynamic equilibrium model. Notice that, unlike Blanchard and Kahn (1980), but like Sims (1996) and King and Watson (1995a,b) I allow the matrix A to be singular. Roughly speaking, this generalization allows static (intratemporal) equilibrium conditions to be included among the dynamic relationships. Technically, these singularities show up as zeroth-order equations in the triangularization of our system, reflecting that some equations in the original system state relationships among the variables in x_t with no reference to $\mathrm{E}[x_{t+1}|\mathscr{F}_t]$.

The idea in this paper is to use the complex generalized Schur form in order to reduce (2.1) into blocks of equations, separating the system into an unstable and a stable block of equations. The stable solution is then found by solving the unstable block forward and the stable block backward.

Throughout, our focus will be on the case when there is a unique stable solution to our system of difference equations, although we shall also characterize the set of stable solutions when this set has more than one element. Also, we shall mainly focus on the case where the exogenous driving process is a VAR(1), but will state the solution in a more general case as well.

2.1. An existence condition

As pointed out in King and Watson (1995a), if there is no $z \in \mathbb{C}$ such that $|Az - B| \neq 0$, then (2.1) has no solution for generic exogenous processes $\mathbf{z}$. To understand this, note that if the matrix polynomial $Az - B$ is singular in this sense, then its rows are linearly dependent, and hence there exists a row vector polynomial $\alpha(z)$ such that $\alpha(z)^{\mathrm{T}}(Az - B) = 0$ identically in z.[7] To apply this result to our case, define the forward shift operator as follows:

$$Fx_t := \mathrm{E}[x_{t+1}|\mathscr{F}_t]. \tag{2.2}$$

Our expectational difference equation can then be written as

$$(AF - B)x_t = Cz_t. \tag{2.3}$$

Now, suppose $Az - B$ is singular. Then there exists a row vector forward shift polynomial $\alpha(F)$ such that our equation implies that

$$\alpha(F)^{\mathrm{T}}Cz_t = 0, \quad t = 0, 1, \ldots, \tag{2.4}$$

which is false for generic matrices C and processes $\mathbf{z}$.

We will see below that if $Az - B$ is not singular (regular), then there does exist a solution to (2.1), although not necessarily a stable one.

[7] This is true in spite of the fact that the set of polynomials is a ring rather than a field.

3. The generalized Schur form

As indicated above, the idea in this paper is to try to reduce ("uncouple") the system (2.1) into a (block) triangular system of equations, and then to solve the system recursively in the sense that we first solve the second block, and then the first using the solution for the second. Theorem (3.3) below implies that such a reduction is always possible. The notation h_{ij} will denote the element in the ith row and jth column of any matrix H.

Definition 3.1. Let $P:\mathbb{C}\to\mathbb{C}^{n\times n}$ be a matrix-valued function of a complex variable (a matrix pencil). Then the set of its generalized eigenvalues $\lambda(P)$ is defined via $\lambda(P)=\{z\in\mathbb{C}\colon |P(z)|=0\}$.

When $P(z)=Az-B$, we sometimes write $\lambda(A,B)$ for the set of generalized eigenvalues. In this case, $\lambda\in\mathbb{C}$ is a generalized eigenvalue of P just in case there is a nonzero vector $x\in\mathbb{C}^n$ such that $Bx=\lambda Ax$.

Definition 3.2. Let $P(z)$ be a matrix pencil. Then P is said to be regular if there is a $z\in\mathbb{C}$ such that $|P(z)|\neq 0$, i.e. if $\lambda(P)\neq\mathbb{C}$.

Theorem 3.3 (the complex generalized Schur form). *Let A and B be $n\times n$ matrices of complex numbers such that $P(z)=Az-B$ is a regular matrix pencil. Then there exist unitary $n\times n$ matrices of complex numbers Q and Z such that*

1. *$QAZ=S$ is upper triangular,*
2. *$QBZ=T$ is upper triangular,*
3. *For each i, s_{ii} and t_{ii} are not both zero,*
4. $\lambda(A,B)=\left\{\dfrac{t_{ii}}{s_{ii}}\colon s_{ii}\neq 0\right\}$,
5. *The pairs (s_{ii},t_{ii}), $i=1,\ldots,n$ can be arranged in any order.*

Proof. See Golub and van Loan (1996). □

Note that the set $\lambda(A,B)$ may have fewer than n elements, since if A is singular, we may have $s_{ii}=0$ for some i. Acknowledging the abuse of language, the missing generalized eigenvalues will be called "infinite". Meanwhile, the finite generalized eigenvalues λ_i such that $|\lambda_i|>1$ (and sometimes also those with $|\lambda_i|=1$) will be called finite and unstable. The infinite and finite unstable generalized eigenvalues will be called unstable. The remaining generalized eigenvalues will be called stable.[8] In order to avoid dealing with infinite

[8] Eigenvalues with $|\lambda_i|=1$ are problematic and will be ignored in this paper.

numbers, we will sometimes represent an eigenvalue by the pair (s_{ii}, t_{ii}) rather than by their ratio.

The possibility that s_{ii} and t_{ii} are both zero for some i is ruled out by assuming $Az - B$ to be regular. We will see below that the case $s_{ii} = t_{ii} = 0$ corresponds to a row in a transformed version of (2.1) having the form $w_t = 0$ where w_t is exogenously given. See Section 5.2.

Notice that I have presented the *complex* generalized Schur form. If A and B are real, one can alternatively calculate the *real* Schur form, for which S and T are *block* upper triangular. This is always good enough when there is a unique solution to the system. Especially for very large systems, the real Schur form is faster to compute. However, the sorting of the eigenvalues in the real generalized Schur form presents some (surmountable) difficulties. In the first place, the complex generalized eigenvalues do not show up directly as pairs of diagonal elements, but as generalized eigenvalues of pairs of 2×2 blocks in S and T. Secondly, the sorting algorithm for the real generalized Schur form seems to be only about half as fast as that for the complex generalized Schur form.

4. Assumptions

In this section we make some assumptions, and we will see that these constitute sufficient conditions for there to be a unique stable solution. These assumptions follow closely the assumptions of Blanchard and Kahn (1980) and King and Watson (1995a,b). Conceptually, the relationship between the conditions stated here and those stated in the survey by Binder and Pesaran (1994) is also a very close one. Throughout this section, the matrices A, B, the process $\mathbf{x}$, etc. all refer to Eq. (2.1).

We begin by stating a few definitions. First, we need to define stability.

Definition 4.1. Let $\mathbf{x}$ be a stochastic process with values in $\mathbb{R}^n$. We will call $\mathbf{x}$ stable if there is an M such that

$$||x_t||_{\max} \leq M \tag{4.1}$$

for all $t = 0, 1, \ldots$, where $\|\cdot\|_{\max}$ is defined via

$$||x||_{\max} = \max_i \sqrt{\mathrm{E}[|x_i|]}. \tag{4.2}$$

The intuitive meaning of stability is, of course, that the unconditionally expected values of the moduli of the elements of x_t do not blow up as t increases beyond all bounds.

Next, we define what is meant by a martingale difference and a white noise process.

Definition 4.1. Let $(\Omega, \mathscr{F}, P, \underline{\mathscr{F}})$ be a filtered probability space. A vector process $\boldsymbol{\xi}$ is called a $(P, \underline{\mathscr{F}})$-martingale difference process if

1. $\boldsymbol{\xi}$ is adapted to $\underline{\mathscr{F}}$,
2. $\mathrm{E}[\xi_{t+1} \mid \mathscr{F}_t] = 0$ for each $t = 0, 1, \ldots$.

Definition 4.2. A $(P, \underline{\mathscr{F}})$-martingale difference process $\boldsymbol{\varepsilon}$ is called a $(P, \underline{\mathscr{F}})$-white noise process if there is a matrix Σ such that

$$\mathrm{E}[\varepsilon_{t+1}\varepsilon_{t+1}^{\mathrm{T}} \mid \mathscr{F}_t] = \Sigma \quad \text{for each } t = 0, 1, \ldots . \tag{4.3}$$

Note that a white noise process is stable, and that a martingale difference process is stable if, for instance, it is uniformly bounded. Finally, we define what we mean by 'predetermined' or 'backward-looking'. The definition is a generalization of that given in Blanchard and Kahn (1980).

Definition 4.3. Let $(\Omega, \mathscr{F}, P, \underline{\mathscr{F}})$ be a filtered probability space. A process $\mathbf{k}$ is called *backward-looking* if

1. The prediction error $\boldsymbol{\xi}$ defined via $\xi_{t+1} := k_{t+1} - \mathrm{E}[k_{t+1} \mid \mathscr{F}_t]$ is an exogenous martingale difference process, and
2. $k_0 \in \mathscr{F}_0$ is exogenously given.

Notice that backward-lookingness is not strictly speaking a property of the process $\mathbf{x}$, but a property of the model to which $\mathbf{x}$ is a solution. Specifying what variables in a system are backward-looking amounts to specifying what is exogenously given in the model. Note also that, if $\xi_{t+1} \equiv 0$, the definition here reduces to that given in Blanchard and Kahn (1980). In this sense the definition here is a generalization of theirs.

We now turn to the assumptions. The first one states that the exogenous driving sequence is stable; this improves the prospects for the existence of a stable solution for the endogenous process.

Assumption 4.1. The exogenous n_z-dimensional process $\mathbf{z}$ is stable and adapted to the given filtration $\underline{\mathscr{F}}$.

Most of the time we will be specializing even further, and assume that $\mathbf{z}$ is represented as a VAR(1) with mean zero, autocorrelation matrix Φ with eigenvalues strictly inside the unit circle, an exogenously given $(P, \mathscr{F})$-white noise prediction error process $\boldsymbol{\varepsilon}$ and exogenously given initial value z_0.

Now, suppose some of the variables in x_t, say the n_k first ones, have an exogenously given expectations error and initial value. To write this down

formally, we partition x_t according to

$$x_t = \begin{bmatrix} k_t \\ {\scriptstyle n_k \times 1} \\ d_t \end{bmatrix}. \tag{4.4}$$

We now assume that k_t is 'backward-looking' in the sense defined above, i.e. that k_t has an exogenous initial value and prediction error.

Assumption 4.2. k_0 is an exogenously given $\mathscr{F}_0$-measurable random variable and

$$k_{t+1} - \mathrm{E}[k_{t+1} \mid \mathscr{F}_t] = \xi_{t+1} \tag{4.5}$$

where $\boldsymbol{\xi}$ is an exogenous $(P, \mathscr{F})$-martingale difference process.

We now assume that the requirement of Theorem (3.3) is satisfied so that the problem is well formulated.

Assumption 4.3.

$$\text{There exists a } z \in \mathbb{C} \text{ such that } |Az - B| \neq 0. \tag{4.6}$$

For a discussion of the meaning of this condition, see Section 2.1.

Moreover, to avoid the case where any solution is unstable we assume that there are no unit roots, i.e. we make the following assumption.

Assumption 4.4.

$$\text{There is no } z \in \mathbb{C} \text{ with } |z| = 1 \quad \text{and} \quad |Az - B| = 0. \tag{4.7}$$

Now, let Q, Z, S, T be the matrices whose existence is guaranteed by Theorem (3.3). Let S and T be arranged in such a way that the n_s stable generalized eigenvalues come first. By a stable generalized eigenvalue, we mean a pair (s_{ii}, t_{ii}) with $|s_{ii}| > |t_{ii}|$. The remaining generalized eigenvalues are unstable. Note that our no-unit-root assumption rules out the possibility $|s_{ii}| = |t_{ii}|$. Let $n_u = n - n_s$. Partition the rows of Z conformably with the classification of eigenvalues and the columns conformably with the partition of x_t in (4.4) so that, for example, the upper left-hand block of Z is $n_s \times n_k$. We write

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}. \tag{4.8}$$

Assumption 4.5.

$$Z_{11} \text{ is square and invertible.} \tag{4.9}$$

Note that Assumption (4.5) implies that $n_s = n_k$. This means that there are as many 'predetermined' variables (variables with exogenously given initial value and prediction error) as there are stable eigenvalues.[9] For a more detailed discussion of this condition, see Section 5.3.1.

The question may arise how likely it is that $n_s = n_k$ in a typical application. In fact, it is very likely. If our system of equations is derived from a linear-quadratic dynamic optimization problem of the form presented in Section 6.2, we are guaranteed, unless some eigenvalues have unit modulus, that $n_s = n_k$. The problem can then be transformed into an equivalent one for which the optimality conditions are such that the relevant matrix pencil is *symplectic* so that the generalized eigenvalues appear in reciprocal pairs.[10] See Anderson et al. (1997), pp. 187–189.

Theorem 4.1. Under the above assumptions, there is, for each given k_0 *and* n_k*-dimensional* $(P, \underline{\mathscr{F}})$*-white noise process* $\boldsymbol{\varepsilon}$*, an almost surely* (P) *unique stable process* $\mathbf{x}$ *satisfying* (2.1).

Proof. See the construction of the solution below. □

5. Using the generalized Schur form

5.1. Triangularizing the system

Our first step is to find an upper triangular system of expectational difference equations in the auxiliary variables y_t defined via

$$y_t := Z^H x_t. \tag{5.1}$$

In order to expose the unstable and stable blocks, we partition y_t conformably with that of Z^H so that

$$y_t = \begin{bmatrix} s_t \\ u_t \end{bmatrix}, \tag{5.2}$$

[9] If there should be more stable eigenvalues than there are backward-looking variables, one can always convert forward-looking variables into backward-looking ones by fixing their initial value and prediction error exogenously. More generally, an excess of stable eigenvalues gives us degrees of freedom when pinning down a stable solution. For an ingenious use of these degrees of freedom, see Farmer and Guo (1994).

[10] A subtlety here is if the $n_u \times n_u$ matrix R (see Section 6.2) is not invertible. Then the representation described in Anderson et al. (1997) is not feasible. But this is in fact not important, since the representation presented in Section 6 has the same generalized eigenvalues as theirs plus an additional n_u infinite eigenvalues corresponding to the n_u non-predetermined control variables u_t.

where s_t is $n_s \times 1$ and u_t is $n_u \times 1$. Now, premultiply our system by Q. (Since Q is invertible, this leads to an equivalent system.) We get

$$S\mathrm{E}[y_{t+1}|\mathscr{F}_t] = Ty_t + QCz_t, \tag{5.3}$$

where S and T are upper triangular. The partitioned version is

$$\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \mathrm{E}\left[\begin{bmatrix} s_{t+1} \\ u_{t+1} \end{bmatrix} \middle| \mathscr{F}_t\right] = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} s_t \\ u_t \end{bmatrix} + \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} Cz_t. \tag{5.4}$$

Note that S_{11} and T_{22} are invertible by construction. Note also that for the above representation, we do not really need that S and T are upper triangular; what we need is upper block triangularity. This is why the real generalized Schur form is sufficient; all the 2×2 blocks on the diagonal of S and T correspond to complex conjugate pairs of eigenvalues. Since these eigenvalues have the same absolute value, they never appear on opposite sides of the partition.

The treatment below operates under the assumption that $\mathbf{z}$ is non-trivial. Of course, when $\mathbf{z}$ is a VAR(1), endogenous and exogenous variables can be consolidated and a new system be constructed with $z_t \equiv 0$. This simplifies the analytical expressions considerably, but typically slows down the numerical calculations by requiring the Schur factorization to be performed on bigger matrices.

5.2. Solving the triangular system

5.2.1. Solving for u_t

Since the generalized eigenvalues of the matrix pencil $S_{22}z - T_{22}$ are all unstable, the unique stable solution for u_t is found by solving 'forward,' and we find that

$$u_t = -T_{22}^{-1} \sum_{k=0}^{\infty} [T_{22}^{-1} S_{22}]^k Q_2 C\mathrm{E}[z_{t+k}|\mathscr{F}_t]. \tag{5.5}$$

Notice that this result is independent of whether there are any singularities in S_{22}, corresponding to singularities in the A matrix, noting the mathematical convention that $F^0 = I$ for any square matrix F, including the zero matrix.

When $\mathbf{z}$ is a stationary VAR process with autocorrelation matrix Φ, Eq. (5.5) implies that

$$u_t = Mz_t, \tag{5.6}$$

where

$$\mathrm{vec}\, M = [(\Phi^{\mathrm{T}} \otimes S_{22}) - I_{n_z} \otimes T_{22}]^{-1} \mathrm{vec}[Q_2 C]. \tag{5.7}$$

For a derivation of this result, see Appendix B. When n_u is very large, however, this is not the best formula to use for calculations. It is more efficient then to

proceed recursively, following King and Watson (1995b). The method works as follows. Consider the second block of (5.4). We have the equation

$$S_{22}\mathrm{E}[u_{t+1}|\mathscr{F}_t] = T_{22}u_t + Q_2Cz_t, \tag{5.8}$$

where the important thing for our purposes is that S_{22} and T_{22} are upper triangular. To solve for u_t in terms of z_t, we proceed line by line, where each line i has the structure

$$\sum_{j=i+1}^{n_u} s_{ij}\mathrm{E}[u_{j,t+1}|\mathscr{F}_t] + s_{ii}\mathrm{E}[u_{i,t+1}|\mathscr{F}_t] = t_{ii}u_{i,t} + \sum_{j=i+1}^{n_u} t_{ij}u_{j,t} + g_i^{\mathrm{T}}z_t, \tag{5.9}$$

where s_{ij} is the row i, column j element of S_{22}, t_{ij} is the row i, column j element of T_{22}, and g_i^{T} is the ith row of Q_2C. Now, suppose we have solved for elements $n_u, n_u-1, \ldots, i+1$ of u_t and want to solve for the ith element. That is, we have solved for the last $n_u - i$ lines m_j^{T} of M and seek the ith row m_i^{T}. Consolidating the known terms, we get

$$s_{ii}\mathrm{E}[u_{i,t+1}|\mathscr{F}_t] = t_{ii}u_{i,t} + r_i^{\mathrm{T}}z_t, \tag{5.10}$$

where

$$r_i^{\mathrm{T}} = \sum_{j=i+1}^{n_u} (t_{ij}m_j^{\mathrm{T}} - s_{ij}m_j^{\mathrm{T}}\Phi) + g_i^{\mathrm{T}}. \tag{5.11}$$

Notice that if it were to happen that $s_{ii} = t_{ii} = 0$, we would be in deep trouble, since it implies that $r_i^{\mathrm{T}}z_t = 0$. However, this possibility is ruled out by Assumption 4.3. Indeed, by construction $|s_{ii}/t_{ii}| < 1$ for the values of i pertaining to the unstable (lower) block of equations that we are considering in this section, so the unique stable solution is

$$u_{i,t} = -\frac{1}{t_{ii}}\sum_{k=0}^{\infty}\left(\frac{s_{ii}}{t_{ii}}\right)^k r_i^{\mathrm{T}}\mathrm{E}[z_{t+k}|\mathscr{F}_t] = -\frac{1}{t_{ii}}\sum_{k=0}^{\infty}\left(\frac{s_{ii}}{t_{ii}}\right)^k r_i^{\mathrm{T}}\Phi^k z_t. \tag{5.12}$$

It follows that

$$u_{i,t} = r_i^{\mathrm{T}}[s_{ii}\Phi - t_{ii}I_{n_z}]^{-1}z_t \tag{5.13}$$

so that $m_i^{\mathrm{T}} = r_i^{\mathrm{T}}[s_{ii}\Phi - t_{ii}I_{n_z}]^{-1}$. Notice that, using this procedure, the calculation of M requires inverting an $n_z \times n_z$ matrix n_u times whereas the formula (5.7) involves inverting an $n_z n_u \times n_z n_u$ matrix once. Hence when n_u is large, it takes much less time to do the former than the latter.

5.2.2. *Solving for s_t*

In the preceding section we solved for u_t in terms of expectations of z_t. This means that we can treat u_t and its expectations as given from now on. By the first

block of (5.4),

$$S_{11}\mathrm{E}[s_{t+1}|\mathscr{F}_t] + S_{12}\mathrm{E}[u_{t+1}|\mathscr{F}_t] = T_{11}s_t + T_{12}u_t + Q_1Cz_t. \tag{5.14}$$

Reshuffling a bit, this becomes

$$\mathrm{E}[s_{t+1}|\mathscr{F}_t] = S_{11}^{-1}T_{11}s_t + S_{11}^{-1}T_{12}u_t - S_{11}^{-1}S_{12}\mathrm{E}[u_{t+1}|\mathscr{F}_t] + S_{11}^{-1}Q_1Cz_t. \tag{5.15}$$

Since $S_{11}^{-1}T_{11}$ is a stable matrix by construction, we cannot use the stability requirement to fix the undetermined initial value and expectations error. Instead it is time to use our assumption of an exogenously given expectations error. Recalling the definition of y_t, we have

$$k_{t+1} = [Z_{11}\ Z_{12}]\begin{bmatrix} s_{t+1} \\ u_{t+1} \end{bmatrix}. \tag{5.16}$$

Hence

$$Z_{11}(s_{t+1} - \mathrm{E}[[s_{t+1}]|\mathscr{F}_t]) + Z_{12}(u_{t+1} - \mathrm{E}[u_{t+1}|\mathscr{F}_t]) = \xi_{t+1}. \tag{5.17}$$

We now invoke Assumption 4.5 which says that Z_{11} is invertible, and proceed to find the unique solution. (See Section (5.3.1) for a discussion of what this Assumption means.) Since Z_{11} is invertible, Eqs. (5.6), (5.15), and (5.17) taken together recursively define the unique solution for s_t given s_0 and the exogenous process $\boldsymbol{\varepsilon}$. In particular, we have

$$\begin{aligned} s_{t+1} = {} & S_{11}^{-1}T_{11}s_t + S_{11}^{-1}[T_{12}M - S_{12}M\Phi + Q_1C]z_t \\ & - Z_{11}^{-1}Z_{12}M\varepsilon_{t+1} + Z_{11}^{-1}\xi_{t+1}. \end{aligned} \tag{5.18}$$

All that remains, then, is to find s_0 in terms of k_0 and z_0 (which are given). It is not hard to confirm that

$$s_0 = Z_{11}^{-1}[k_0 - Z_{12}Mz_0]. \tag{5.19}$$

Having solved for the full vector y_t, it is easy to invert the transformation to find x_t.

5.3. *Eliminating the auxiliary process* y_t

If we want to avoid having to deal with the auxiliary process y_t, we can use the relationship $x_t = Zy_t$ to find a recursive representation for x_t with no reference to y_t. The result is the following.

Theorem 5.1. Under the assumptions made in Section 4, *the unique solution to* (2.1) *can be written as*

$$d_t = Z_{21}Z_{11}^{-1}k_t + Nz_t, \tag{5.20}$$

and

$$k_{t+1} = Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1}k_t + Lz_t + \xi_{t+1}, \tag{5.21}$$

where

$$N = (Z_{22} - Z_{21}Z_{11}^{-1}Z_{12})M \tag{5.22}$$

and

$$L = -Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1}Z_{12}M + Z_{11}S_{11}^{-1}[T_{12}M - S_{12}M\Phi + Q_1C] + Z_{12}M\Phi. \tag{5.23}$$

Proof. Partition the Hermitian transpose of Z, Z^{H}, via

$$Z^{\mathrm{H}} = \begin{bmatrix} Z_{11}^{\mathrm{H}} & Z_{12}^{\mathrm{H}} \\ Z_{21}^{\mathrm{H}} & Z_{22}^{\mathrm{H}} \end{bmatrix}.$$

By the definition of the auxiliary variables, we have

$$Z_{21}^{\mathrm{H}}k_t + Z_{22}^{\mathrm{H}}d_t = u_t. \tag{5.24}$$

But since $u_t = Mz_t$, this is equivalent to (5.20). To see this, we substitute (5.20) into (5.24). We get

$$[Z_{21}^{\mathrm{H}} + Z_{22}^{\mathrm{H}}Z_{21}Z_{11}^{-1}]k_t + Z_{22}^{\mathrm{H}}(Z_{22} - Z_{21}Z_{11}^{-1}Z_{12})Mz_t = Mz_t,$$

and of course we want to show that this is an identity, i.e. that

$$Z_{21}^{\mathrm{H}} + Z_{22}^{\mathrm{H}}Z_{21}Z_{11}^{-1} = 0, \tag{5.25}$$

and that

$$Z_{22}^{\mathrm{H}}(Z_{22} - Z_{21}Z_{11}^{-1}Z_{12}) = I_{n_u}. \tag{5.26}$$

But since Z_{11} is invertible, (5.25) is equivalent to

$$Z_{21}^{\mathrm{H}}Z_{11} + Z_{22}^{\mathrm{H}}Z_{21} = 0, \tag{5.27}$$

and this follows from the fact that $Z^{\mathrm{H}}Z = I$. For the same reason, $Z_{22}^{\mathrm{H}}Z_{22} = I_{n_u} - Z_{21}^{\mathrm{H}}Z_{12}$, so the left-hand side of (5.26) becomes

$$I_{n_u} - Z_{21}^{\mathrm{H}}Z_{12} + Z_{21}^{\mathrm{H}}Z_{11}Z_{11}^{-1}Z_{12} = I_{n_u}. \tag{5.28}$$

The expression for k_{t+1}, (5.21), is proved in a similar fashion. □

5.3.1. Non-existence and indeterminacy

We begin this section with a discussion of the meaning of Assumption 4.5. We have already noted that it implies (but is of course not implied by) $n_k = n_s$ which means that there are just as many 'backward-looking' variables as there are

stable eigenvalues. From an intuitive point of view, this makes sense. In the deterministic case, there is an n-dimensional set of solutions. A unique solution should be pinned down by imposing n linear restrictions. The requirement of stability imposes as many linear restrictions as there are unstable eigenvalues — n_u. The given initial values of the 'backward-looking' variables impose another n_k linear restrictions. Thus, it would seem that if $n_u + n_k = n$ (which is equivalent to $n_k = n_s$), there should be a unique solution. The question therefore arises if there are any interesting cases where $n_k = n_s$ but nevertheless Z_{11} is singular so that there is no stable solution. A deterministic example (taken from King and Watson (1995b) illustrates the point using the ordinary (non-generalized) Schur form.

$$\begin{bmatrix} x_{1,t+1} \\ x_{2,t+1} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix}, \tag{5.29}$$

where $x_{1,0}$ is given exogenously. Here the sorted Schur form (with the eigenvalues in order of descending modulus) of the coefficient matrix is

$$\begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{5.30}$$

so that although $n_s = n_k$, what corresponds to Z_{11} is zero. In this case, there is no stable solution. To see why, consider the transformed system

$$\begin{bmatrix} s_{t+1} \\ u_{t+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} s_t \\ u_t \end{bmatrix}. \tag{5.31}$$

To solve this uniquely, we need a value for s_0. But since Z_{11} is singular, we cannot translate the given value $x_{1,0}$ into the required value s_0. An intuitive description of what is going on is that the unstable eigenvalue cannot be transferred from the predetermined variable to the non-predetermined variable.

A subtlety in this context is that the generalized Schur form is not unique even if a particular ordering of the eigenvalues is imposed. It is therefore an open question whether there might be two generalized Schur forms of the same matrix pencil, one with Z_{11} invertible and the other with Z_{11} singular. A reasonable conjecture is that this cannot happen, but apparently there is no known proof of this.

In the case we just considered, there was no stable solution because Z_{11} had short rank; its column space was not rich enough to house a value of s_0 that would satisfy

$$Z_{12}Mz_0 + Z_{11}s_0 = k_0. \tag{5.32}$$

Alternatively, we might have rank $(Z_{11}) \geq n_k$, in which case there will be a non-empty set of stable solutions, and the set of permissible initial values s_0 and prediction error $v_{t+1} = s_{t+1} - \mathrm{E}[s_{t+1}|\mathscr{F}_t]$ are fully characterized by

(5.32) and (5.17) together with the requirement that $\boldsymbol{v}$ is a stable $(P, \underline{\mathscr{F}})$-martingale difference process.

6. Examples

6.1. A simple RBC model

An example of this approach in action is the following. Suppose a representative agent solves

$$\max_{\mathbf{c}\in\underline{\mathscr{F}}} \mathrm{E}\left[\sum_{t=0}^{\infty} \beta^t [c_t^{\alpha}(1-h_t)^{1-\alpha}]^{1-\sigma}\right]$$

$$\text{s.t.} \quad \begin{cases} k_{t+1} = z_t k_t^{\theta} h_t^{1-\theta} + (1-\delta)k_t, \\ k_0 \text{ given}, \\ \mathbf{k} \text{ stable}, \end{cases} \tag{6.1}$$

where $\ln z_t$ is an exogenous AR(1) process with mean 0, autocorrelation φ and expectations error $\boldsymbol{\varepsilon}$. Note that the no-Ponzi-game condition has been replaced by the stronger (but in practice equivalent) condition that $\mathbf{k}$ be stable. $\underline{\mathscr{F}}$ is the filtration generated by the exogenous random variables k_0, z_0, and the process $\boldsymbol{\varepsilon}$. Note that when the equilibrium conditions of this model are (log-)linearized, we get a singular A matrix since h_t is, loosely speaking, an 'intratemporal' choice variable.

Having found the steady state (how long this takes depends strongly on the initial guess) and linearized the equilibrium conditions (which takes no more than two hundredths of a second using numerical derivatives), finding the linear decision rule for this model typically takes about 0.025 s using Matlab on a PC with a Pentium 166 MHz processor.[11]

6.2. Linear-quadratic problems

Since it is such a popular class of examples, it is interesting to see how the method presented in this paper applies to the case of a (single-agent) linear-quadratic dynamic optimization problem.

[11] To replicate this result, set $\alpha = 0.33$, $\beta = 1.03^{1/4}$, $\sigma = 2$, $\delta = 0.025$, $\theta = 0.36$, $\varphi = 0.95$. The results are not particularly sensitive to parameter values, except that if $\sigma = 1$, the decision rule is computed much faster; it then takes only about 0.008 s. Using the same parameter values as above, the Bowden method in Binder and Pesaran (1997) takes about 0.16 s. If σ is increased further, the difference between the two methods increases as the algorithm of Binder and Pesaran (1997) becomes slower due to the increase in the required forecasting horizon and the method advocated here actually becomes faster.

Apparently van Dooren (1981) was the first to note the usefulness of the generalized Schur form in solving linear-quadratic control problems, and the approach here is similar to his. Meanwhile, Anderson et al. (1997), pp. 194–200 also discuss various algorithms for solving linear-quadratic optimization problems. By dropping the requirement that the R matrix (see below) be invertible, the approach presented here is slightly more general than any of these methods.[1,2]

With the usual filtered probability space $(\Omega, \mathscr{F}, P, \underline{\mathscr{F}})$ in the background, our problem is

$$\max_{\mathbf{u}\in\underline{\mathscr{F}}} \left\{ \frac{1}{2}\mathrm{E}\sum_{t=0}^{\infty} \beta^t [x_t^{\mathrm{T}}\ u_t^{\mathrm{T}}] \begin{bmatrix} Q & W \\ W^{\mathrm{T}} & R \end{bmatrix} \begin{bmatrix} x_t \\ u_t \end{bmatrix} \right\}$$

$$\text{s.t.} \quad \begin{cases} x_{t+1} = Ax_t + Bu_t + C\xi_{t+1}, \\ x_0 \text{ given}, \\ \mathbf{x} \text{ stable}, \end{cases} \tag{6.2}$$

where $\boldsymbol{\xi} = \{\xi_t;\ t = 0, 1, \ldots\}$ is an exogenous $(P, \underline{\mathscr{F}})$-white noise process. Suppose the matrix $\left[\begin{smallmatrix} Q & W \\ W^{\mathrm{T}} & R \end{smallmatrix}\right]$ is symmetric and negative semidefinite so that the first-order conditions (together with the transversality condition) are sufficient for an optimum. Note that if $\mathbf{x}$ and $\mathbf{u}$ are $\underline{\mathscr{F}}$-adapted (which of course we require them to be), then $\mathbf{x}$ has the exogenously given expectations error $C\boldsymbol{\xi}$. Introducing λ_t as the current shadow value of the constraint $x_t = Ax_{t-1} + Bu_{t-1} + C\xi_t$ and taking the first-order conditions for a maximum, we get

$$\begin{bmatrix} 0 & -\beta B^{\mathrm{T}} & 0 \\ 0 & \beta A^{\mathrm{T}} & 0 \\ 0 & 0 & I_{n_x} \end{bmatrix} \left[\mathrm{E}\left[\begin{bmatrix} u_{t+1} \\ \lambda_{t+1} \\ x_{t+1} \end{bmatrix} \middle| \mathscr{F}_t \right] \right] = \begin{bmatrix} R & 0 & W^{\mathrm{T}} \\ -W & I_{n_x} & -Q \\ B & 0 & A \end{bmatrix} \begin{bmatrix} u_t \\ \lambda_t \\ x_t \end{bmatrix}, \tag{6.3}$$

and this is of course exactly the format we want, recalling that we also need to know x_0 and that $x_{t+1} - \mathrm{E}[x_{t+1}|\mathscr{F}_t] = C\xi_{t+1}$. With the notation above, the 'forward-looking' variables are $d_t = \left[\begin{smallmatrix} u_t \\ \lambda_t \end{smallmatrix}\right]$ and the 'backward-looking' variables are $k_t = x_t$.

[1,2] It is not entirely clear whether the approach of Amman and Neudecker (1997) requires R to be non-singular. What the authors say is that "Normally, ... [R is] assumed to be ... positive definite". However, the authors omit to state whether they themselves need this assumption. In any case, it should be conceded that the level of generality achieved here is more a matter of conceptual neatness than of economic relevance. The point is that the invertibility of R is irrelevant to the existence of a solution; therefore a good solution algorithm should not require it to be invertible.

7. Conclusion

The approach presented in this paper aspires to be somewhat easier to grasp and apply than Sims (1996). By exploiting standard algorithms, it has the advantage over King and Watson (1995a,b) in that it achieves state-of-the-art computational efficiency. Moreover, it avoids the detour of solving the matrix quadratic equations derived in Uhlig (1995) and Binder and Pesaran (1994, 1996). Also, it seems to be somewhat faster for infinite-horizon problems than the approach of Binder and Pesaran (1997). All in all, it might be useful to many applied macroeconomists (and others) who want to learn a general method of solving a multivariate linear rational expectations model quickly.

Appendix A. The Binder–Pesaran canonical form

Binder and Pesaran (1994) show how to reduce a linear system of expectational difference equations with arbitrary leads and lags (and expectations taken with respect to information available at different times) to the following canonical form.

$$x_t = Ax_{t-1} + B\mathrm{E}[x_{t+1}|\mathscr{F}_t] + z_t, \tag{A.1}$$

where x_{-1} (but not x_0) is given and x_t is assumed to be adapted to $\langle\mathscr{F}_t\rangle$. Apart from this adaptedness requirement, no prediction errors are given exogenously. This Binder–Pesaran canonical form can easily be reduced to the canonical form (2.1) by defining

$$\tilde{x}_t = \begin{bmatrix} x_t \\ x_{t-1} \end{bmatrix}. \tag{A.2}$$

Let n_x be the dimension of x_t. Then the last n_x rows of $\tilde{x}_t$ have an exogenously given initial value (equal to x_{-1}) and exogenously given prediction error (equal to zero) because of the adaptedness requirement.

Having defined $\tilde{x}_t$, the reduced system becomes

$$\begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}\mathrm{E}[\tilde{x}_{t+1}|\mathscr{F}_t] = \begin{bmatrix} I & -A \\ I & 0 \end{bmatrix}\tilde{x}_t + \begin{bmatrix} -z_t \\ 0 \end{bmatrix} \tag{A.3}$$

with the last n_x elements of $\tilde{x}_t$ being 'backward-looking'.

Appendix B. Geometric sums of matrices

In this Section I derive a formula for the geometric sums of matrices.

Define

$$S = \sum_{k=0}^{\infty} \Phi^k A \Psi^k, \tag{B.1}$$

where we assume that $\lim_{k\to\infty} \Phi^k A \Psi^k = 0$. It follows from Eq. (B.1) that

$$S - \Phi S \Psi = A. \tag{B.2}$$

To solve for S, we invoke the matrix identity $\mathrm{vec}(ABC) = [C^{\mathrm{T}} \otimes A]\,\mathrm{vec}(B)$. We get

$$\mathrm{vec}(S) - (\Psi^{\mathrm{T}} \otimes \Phi)\mathrm{vec}(S) = \mathrm{vec}\ A. \tag{B.3}$$

If $[I - \Psi^{\mathrm{T}} \otimes \Phi]$ is invertible, we may conclude that

$$\mathrm{vec}(S) = [I - \Psi^{\mathrm{T}} \otimes \Phi]^{-1}\mathrm{vec}\ A. \tag{B.4}$$

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