Boston University
Study Problems for Macroeconomics
Section 1: Friday, September 19

1. *A simple dynamic programming model of capital accumulation* Consider the following economy. Individuals have preferences

$$U = \sum_{t=0}^{\infty} \beta^t \log(c_t)$$

and a constraint of the form

$$k_{t+1} = ak_t^{\alpha} - c_t$$

(a) Write the Bellman equation for this economy.

(b) Find the FOC(s) that must be satisfied for an optimal consumption and capital plan;

(c) Show the following consumption policy

$$c_t = \phi a k_t^{\alpha}$$

is consistent with the condition(s) that you produced in (b).

(d) Derive the value function by substituting the optimal consumption policy into the objective. Show that it takes the form

$$v = \gamma + \theta \log(k_t)$$

and determine the values of γ and θ.

(e) Show that the optimal policy maximizes

$$\log(c_t) + \beta v(k_{t+1})$$

where $v(k_{t+1})$ is the value function derived in part (d).

(f) Now suppose that there is an arbitrary value function of the form

$$v_{n-1}(k_{t+1}) = \gamma_n + \theta_{n-1} \log(k_t)$$

which appears on the right hand side of the Bellman equation. Show that

$$v_n(k_t) = \max\{\log(c_t) + \beta v_{n-1}(k_{t+1})\}$$

takes the form

$$v_n(k_{t+1}) = \gamma_n + \theta_n \log(k_t)$$

and determine the coefficients γ_n and θ_n.

(g) Show that the sequence of coefficients $\{\gamma_n\}$ and $\{\theta_n\}$ generated in this manner converges to the γ and θ values that you determined in part (b) from any initial γ_0 and θ_0.

(h) Explain how to interpret the solution for a fixed number of iterations N as providing the capital accumulation decision rule for a finite horizon version of the problem above, in which

$$U = \sum_{t=0}^{T} \beta^t \log(c_t)$$

and in which $k_{T+1} \geq 0$. What is the relationship between N and T?