Boston University
Study Problems for Macroeconomics

Section 3: Friday, October 3

1 Neoclassical Economy with Finite Horizon

Consider a finite horizon version of the neoclassical model of capital accumulation otherwise identical to that studied in class.

$$\max_{\{c_t\}} \sum_{t=0}^{T} \beta^t u(c_t)$$
$$\text{s.t.} \quad k_{t+1} + c_t \leq \Phi(k_t) \equiv f(k_t) + (1-\delta)k_t$$

Suppose further that there is a terminal constraint

$$k_{T+1} = \tilde{k}$$

where $\tilde{k}$ is a terminal capital stock.

Suppose further that k_0 and $\tilde{k}$ are sufficiently close to the modified golden rule k^* so that it is appropriate to linearize the first-order conditions around k^* and to then study the resulting dynamics. As a consequence, we can write the solution for the capital stock as

$$k_t - k^* = \theta_{ku}\mu_u^t + \theta_{ks}\mu_s^t$$

where μ_u (μ_s) is the unstable (stable) eigenvalue of the problem and θ_{ku}, θ_{ks} are coefficients yet to be determined. This is a direct implication of the fact that the neoclassical model has the following reduced form

$$\begin{bmatrix} \lambda_{t+1} - \lambda^* \\ k_{t+1} - k^* \end{bmatrix} = \begin{bmatrix} w_{\lambda\lambda} & w_{\lambda k} \\ w_{k\lambda} & w_{kk} \end{bmatrix} \begin{bmatrix} \lambda_t - \lambda^* \\ k_t - k^* \end{bmatrix}$$

and that the matrix W has two eigenvalues, one stable and one unstable. The expression $k_t - k^* = \theta_{ku}\mu_u^t + \theta_{ks}\mu_s^t$ then is an immediate implication of the last row of

$$\begin{bmatrix} \lambda_t - \lambda^* \\ k_t - k^* \end{bmatrix} = W^t \begin{bmatrix} \lambda_0 - \lambda^* \\ k_0 - k^* \end{bmatrix}.$$

(a) Derive the (non-linear) first-order conditions. Denote the multiplier on the capital constraint λ_t.

(b) The conditions

$$k_{T+1} = \tilde{k}$$
$$k_0 \quad \text{given}$$

are terminal and initial conditions on the path of capital accumulation. Use these constraints to determine the values of the θ coefficients in the linearized decision rules (for given eigenvalues μ_u and μ_s).

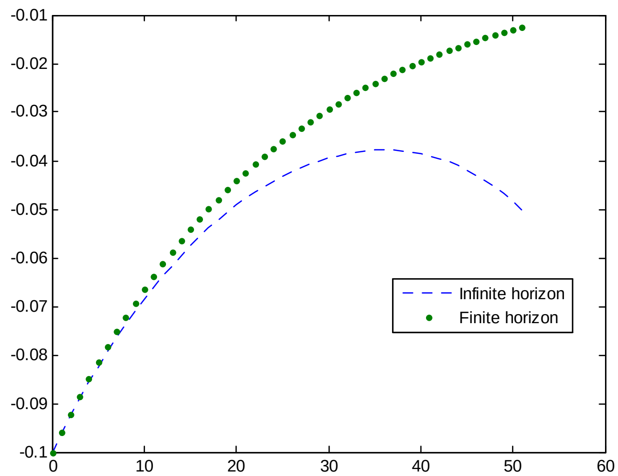
(c) Determine the initial shadow price and consumption consistent with this path of capital.

(d) Show that the limiting path of capital for $T \to \infty$ is the one derived for the infinite horizon model in class.

(e) Suppose that $k_0 < \tilde{k} < k^{*}$., Explain – in terms of economics and mathematics – why the path of capital would take the form of the dashed line below if T is large. In particular explain why

1. the finite horizon path (dashed path) would always lie below the infinite horizon path (dotted line) and
2. there is some interval of time for which $k_t > \tilde{k}$

Note: $k_0 - k^* = -.10$, $\widetilde{k} - k^* = -.05$, $\mu_s = .95$, $\mu_u = 1.02/\mu_s$, $T = 50$.



2 Capital Formation in the Neoclassical Model

The lecture discussion of the neoclassical model of capital formation introduced the idea of a "sustainable level" of consumption associated with every capital stock. Consider a capital stock $k_0 < k^*$ (the latter being the optimal, stationary capital stock). Consider the model written as

$$\max_{\{c_t\}} U = \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$\text{s.t.} \quad c_t + (k_{t+1} - k_t) \leq f(k_t) - \delta k_t = \phi(k_t)$$

(a) Define sustainable consumption as

$$W(k_0) = \frac{1}{1-\beta} u(\phi(k_0))$$

is an increasing, strictly concave function of capital – if u and ϕ are increasing and strictly concave functions (assume that these are differentiable functions if necessary).

(b) Given a situation where consumption equals the sustainable level, consider making a one-time shift from consumption to capital by a positive amount Δ. Accordingly, utility will be

$$u(\phi(k_0) - \Delta) + \beta W(k_0 + \Delta)$$

Show that a small, positive deviation is desirable starting from an initial position of $\Delta = 0$.

(c) Consider a situation in which it is feasible to move from k_0 to k^* in a single period. That is

$$c = \phi(k_0) + k_0 - k^* > 0$$

Relative to this "full adjustment in one step", consider a reduced capital accumulation for one step – reduced by by a small, positive amount Δ – and then keeping consumption at its sustainable level with respect to the new capital stock. Utility is then

$$u(\phi(k_0) + k_0 - k^* + \Delta) + \beta W(k^* - \Delta)$$

Show that some small deviation of this kind is beneficial compared to the "full adjustment" in one step.

(d) Consider the Bellman equation

$$\begin{aligned} V(k) &= \max c, k' \{u(c) + \beta V(k')\} \\ \text{s.t.} \quad k' - k &= \phi(k) - c \end{aligned}$$

and contemplate a small increase in capital accumulation similar to that in part (b) of this problem. Determine the effect of this on $u(c) + \beta V(k')$ and compare it your results in (b). Are there any differences? Why?