Boston University
Study Problems for Macroeconomics

Section 2: Friday, September 26

This collection of three problems concerns the "dynamic labor demand" problem, which Sargent has used in many contexts to exposit aspects of rational expectations models and dynamic optimization. In this problem, we use three approaches to determining labor demand: (i) a straightforward DP approach that determines the value and policy functions; (ii) a derivation of the labor demand Euler equation and a splitting of its dynamics into forward and backward components; (iii) the analysis of the Euler equation from the standpoint of rational expectations analysis to determine the solution form and the solution via undetermined coefficients.

In all three problems, consider a firm that seeks to maximize the expected present discounted value of its profits,

$$E_t \sum_{j=0}^{\infty} \beta^j [y_{t+j} - w_{t+j} n_{t+j} - \phi(h_{t+j})] \tag{1}$$

where $y_t = f_0 + (f_1 + a_t) n_t - \frac{1}{2} f_2 n_t^2$ is output for a firm subject to productivity shocks a_t with all the parameters $f_i > 0$; w_t is the wage paid to workers (each of whom works one unit of time); n_t is the number of workers; and h_t is the number of new hires in the period. The firm's employment evolves according to

$$n_t = (1 - \delta) n_{t-1} + h_t \tag{2}$$

and $\phi(h_t) = \frac{1}{2} d h_t^2$ is a cost of hiring new workers, which is increasing in the number of new workers hired ($d > 0$).

1. *Dynamic programming and labor demand with employment adjustment costs.*

For the purposes of the remainder of this problem, assume that the productivity and the wage rate are constant over time.

(a) Form Bellman equation. Identify: controls, endogenous states, exogenous states.

(b) Insert the transition equation for labor into the adjustment cost function ϕ to eliminate h. Find an efficiency condition for the labor stock. Use the ET to eliminate the derivative of the value function. Determine under what conditions on model parameters (including w and a) there is a positive stationary level of employment at the firm.

(c) Assume that the value function takes the form

$$v(n_{t-1}) = v_0 + v_1 n_{t-1} - \frac{1}{2} v_2 n_{t-1} \tag{3}$$

where $v_i \geq 0$. Determine the optimal decision rules for employment and hiring.

(d) Find restrictions on coefficients v_i which satisfy the Bellman equation. Show that there is only one choice of these coefficients for which the decision rule for employment is a stable difference equation.

Hint: proceed in the following matter. First, determine how the coefficient v_0 is determined if the coefficients v_1 and v_2 are known, using the fact that the Bellman equation must hold when employment is stationary. Second, assume that the employment decision rule takes the form $n_t = \theta + \mu(n_{t-1} - \theta)$ with the θ coefficient satisfying the restrictions that you worked out in the previous problem. Then, use the envelope theorem to find restrictions on v_1 and v_2. Solve for these coefficients in terms of θ and μ. Third, solve for θ in terms of μ. Fourth, show that the remaining restriction is a quadratic equation in μ that has only one stable root.

2. *Dynamic labor demand once again*

Reinstate time-varying wage rate and productivity, assuming that these variables are governed by a Markov process.

(a) Form Bellman equation. Identify: controls, endogenous states, exogenous states.

(b) Find efficiency conditions and use the envelope theorem to eliminate the derivative of the value function. Manipulate these equations to produce a "second order expectational difference equation."

$$\begin{aligned} & d(1-\delta)\beta E_t n_{t+1} - [f_2 + d(1 + \beta(1-\delta)^2)] n_t + d(1-\delta) n_{t-1} \\ = \; & w_t - a_t - f_1 \end{aligned}$$

What is the special form of this difference equation if $d = 0$? What is the form of the labor demand when $d = 0$? How is this related to economics of labor demand?

(c) Assuming $d > 0$ and that productivity and wage are known *sequences*, solve for a dynamic labor schedule of the form

$$n_t = \mu_1 n_{t-1} + \psi \sum_{j=0}^{\infty} \mu_2^{-j} (a_{t+j} - w_{t+j}) + \kappa \tag{4}$$

using Sargent's method of operators (chapter 9, pages 195-199). Determine the coefficients μ_1, μ_2, κ, and ψ

(d) Supposing that the decision rule under certainty takes the form

$$n_t = \mu_1 n_{t-1} + \psi E_t \sum_{j=0}^{\infty} \mu_2^{-j} (a_{t+j} - w_{t+j}) + \kappa \tag{5}$$

and that the exogenous variables are governed by

$$\begin{aligned} a_t &= \rho_a a_{t-1} + e_{at} \qquad (6) \\ w_t &= \rho_w w_{t-1} + e_{wt} \qquad (7) \end{aligned}$$

solve for the labor demand decision rule,

$$n_t = \mu_1 n_{t-1} + \psi_a a_t + \psi_w w_t + \gamma \tag{8}$$

3. *Dynamic Labor Demand and Linear Rational Expectations Models.*

We now consider the dynamic labor demand problem from the perspective of linear rational expectations models. The starting point for this problem is the stochastic Euler equation,

$$\begin{aligned} & d(1-\delta)\beta E_t n_{t+1} - [f_2 + d(1+\beta(1-\delta)^2)]n_t + d(1-\delta)n_{t-1} \\ = \; & w_t - a_t - f_1 \end{aligned}$$

(a) If Y_t is a two element vector containing n_t and one other variable, then this specification can be placed in the form

$$AE_tY_{t+1} = BY_t + Cx_t$$

where $x_t = w_t - a_t - f_1$ What is Y_t? What are the matrices A, B, C?

(b) If $d \neq 0$, is A singular? What are the roots of $|A$z-$B| = 0$?

(c) If $d = 0$, is A singular? What are the roots of $|Az - B| = 0$? What does this imply about the nature of the demand for labor.

(d) Supposing that d is not zero, show that one of the roots of $|Az-B| = 0$ satisfies the restriction $0 < \mu_1 < 1$ and the other satisfies $\mu_2 > 1/\beta$. What does this condition imply about about whether the dynamic labor demand model has a unique stable rational expectations solution?

(e) Describe the necessary steps to solving the dynamic labor demand model, if it is viewed as a linear rational expectations model.

(f) Suppose that $w_t = \rho_w w_{t-1} + e_{wt}$ and $a_t = \rho_a a_{t-1} + e_{at}$ as in problem 2 above. Suppose further that a researcher conjectures a solution of the form

$$n_t = \mu n_{t-1} + \gamma_w w_t + \gamma_a a_t + \kappa$$

Based on the theory of linear rational expectations models, do you think that this is a good guess? Why or why not?

(g) Solve the Euler equation using the method of undetermined coefficients to obtain the solution above. Indicate the restrictions on the μ, γ, and κ coefficients.