Boston University
Study Problems for Macroeconomics
Section 1: Friday, September 19

1. *Labor supply and choice over time*

Consider a household that is maximizing its lifetime utility,

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t, l_t^*, l_t)$$

where c_t is consumption, l_t is leisure and

$$l_{t+1}^* = \rho l_t^* + \theta l_t$$

depends on current and past leisure ($l_t = 1 - n_t$, where n_t is market work). Suppose further that the household is optimizing within a regime of sequential markets, with its wealth evolving according to

$$a_{t+1} = (1 + r_{t,t+1})[a_t + w_t n_t - c_t + \pi_t]$$

(a) What economic considerations might lead l_t^* to enter in the utility function?

Answer: This specification says that a particular function (distributed lag) of past leisure enters the utility function. Economic reasons might include exhaustion, health and various kinds of household capital.

(b) Which variables are controlled state variables from the point of household? What choices affect the motion of these controlled states

Answer: the controlled state variables are financial assets (a) and the variable (l^*) The motion of these state variables is influenced by leisure (work) and consumption.

(c) Supposing that $1 + r_{t,t+1}, w_t$, and π_t are all functions of exogenous state variables ς_t, write a Bellman equation suitable for this problem. Append each of the dynamic equations to form a Lagrangian, with a different multiplier attached to each.

Answer: For convenience, let's write R=1+r. Then, the Bellman equation is

$$\begin{aligned} V(a_t, l_t^*, \varsigma_t) &= \max\{u(c_t, l_t^*, l_t) + \beta E_t V(a_{t+1}, l_{t+1}^*, \varsigma_{t+1})\} \\ s.t. a_{t+1} &= (1 + r(\varsigma_t)[a_t + w(\varsigma_t)(1 - l_t) - c_t + \pi(\varsigma_t) \\ l_{t+1}^* &= \rho l_t + \theta l_t \end{aligned}$$

The associated Lagrangian is

$$\begin{aligned} L_t = & \{u(c_t, l_t^*, l_t) + \beta E_t V(a_{t+1}, l_{t+1}^*, \varsigma_{t+1})\} \\ & + \lambda_t[(1 + r(\varsigma_t)[a_t + w(\varsigma_t)n_t - c_t + \pi(\varsigma_t) - a_{t+1}] \\ & + \phi_t[\rho l_t^* + \theta l_t - l_{t+1}^*] \end{aligned}$$

(d) Find the first-order conditions that describe the efficient choices.

Answer: The FOCs are

$$\begin{aligned}
c_t &: \frac{\partial u((c_t, l_t^*, l_t)}{\partial c_t} - \lambda_t = 0 \\
l_t &: \frac{\partial u((c_t, l_t^*, l_t)}{\partial l_t} - \lambda_t w(\varsigma_t) + \phi_t \theta = 0 \\
l_{t+1}^* &: -\phi_t + E_t \frac{\partial v(a_{t+1}, l_{t+1}^*, \varsigma_{t+1})}{\partial l_{t+1}^*} = 0 \\
a_{t+1} &: -\lambda_t + E_t \frac{\partial v(a_{t+1}, l_{t+1}^*, \varsigma_{t+1})}{\partial a_{t+1}} = 0
\end{aligned}$$

(e) Use the envelope theorem to determine how the first derivatives of the value function depend on economic variables, including the multipliers from part (c).

Answer:

$$\begin{aligned}
\frac{\partial v(a_t, l_t^*, \varsigma_t)}{\partial a_t} &= R(\varsigma_t)\lambda_t \\
\frac{\partial v(a_t, l_t^*, \varsigma_t)}{\partial l_t^*} &= \frac{\partial u((c_t, l_t^*, l_t)}{\partial l_t^*} + \phi_t \rho
\end{aligned}$$