Boston University
Study Problems for Macroeconomics
Section 1

2. *Optimal search for a new job*

Consider an unemployed individual who is well suited to work in some firms, but poorly suited to work in others. In each period, if the individual searches ($h = 1$), then he will be matched with probability q with a new job at wage known w. If he does not search ($h = 0$), then he will not get a new job. There is no borrowing or lending. All employed individuals work $n = 1$ units of time at their jobs.

Suppose further that employed individuals lose their jobs with probability δ, becoming unemployed at the start of any period. Finally, assume that all unemployed individuals receive unemployment benefits of amount b.

Individuals have a momentary utility function that depends positively on consumption and negatively on time spent working or searching (they find each of these equally distasteful). That utility function is written as $u(w) - \phi(1)$ for employed individuals, $u(b) - \phi(1)$ for unemployed individuals that search; and as $u(b) - \phi(0)$ for unemployed individuals that do not search.

(a) Explain why the utility value of being an employed worker, $\acute{V}_w$, obeys the Bellman equation

$$V_w = \{u(w) - \phi(1)\} + \beta\delta V_b' + \beta(1-\delta)V_w' \tag{1}$$

with $u(w) - \phi(1) > 0$ being the utility of consumption less the utility costs of working and with V_b' being the future utility value of being unemployed.

Answer: This Bellman equation reflects the idea that the "net rewards to work" are $\{u(w) - \phi(1)\}$, which is the momentary objective in the Bellman equation. It is assumed positive, so that it is a good thing to work. There are two possible outcomes in the future: job loss, which occurs with probability δ and results in future value of $V_b^{'}$, and job continuation, which occurs with probability $(1 - \delta)$ and results in a future value of V_w'.

(b) Explain why the utility value of being an unemployed individual can be written as

$$V_b = \max_h \left\{ \begin{array}{lr} u(b) - \phi(1) + \beta q V_w^{'} + \beta(1-q)V_b^{'} & \text{if } h = 1 \\ u(b) - \phi(0) + \beta V_b^{'} & \text{if } h = 0 \end{array} \right\} \tag{2}$$

Answer: This Bellman equation reflects the idea that the "net momentary rewards to job search" are $\{u(b) - \phi(1)\}$. If one searches, there are are two possible outcomes in the future: job finding, which occurs with probability q and results in future value of $V_w^{'}$, and unemployment continuation, which occurs with probability $(1 - q)$ and results in a future value of V_b'. The Bellman equation also describes the outcomes if one does not search, which is that one gets $u(b) - \phi(0)$ within the period, but does not get a job in the future, so that the future value is simply $V_b^{'}$.

(c) Assume that it is optimal for the individual to search. What stochastic process will describe the employment status of individuals? What role will the parameters q and δ play in this process? In what fraction of periods will the individual be unemployed?

Answser: If individual search, then we can describe how the fraction of the individuals that employed and unemployed evolves through time. Let

$$\phi_t = \begin{bmatrix} \phi_{wt} \\ \phi_{bt} \end{bmatrix}$$

be a vector containing thes fractions (which must sum to one). These fractions will then evolve through time according to

$$\begin{aligned} \phi_{w,t+1} &= (1-\delta)\phi_{wt} + q\phi_{bt} \\ \phi_{b,t+1} &= \delta\phi_{wt} + (1-q)\phi_{bt} \end{aligned}$$

Let Υ be a matrix of transition probabilities

$$\Upsilon = \begin{bmatrix} (1-\delta) & q \\ \delta & (1-q) \end{bmatrix}$$

Then, we can write the dynamics of fractions as

$$\phi_{t+1} = \Upsilon \phi_t$$

and the stationary distribution is a vector

$$\phi = \Upsilon \phi$$

which is recognized as indicating that ϕ is the eigenvector corresponding to a unit eigenvalue. Using MAPLE, the eigenvectors and eigenvalues of Υ are

$$\begin{aligned} \left\{ \begin{bmatrix} \frac{1}{d} q \\ 1 \end{bmatrix} \right\} &\quad \leftrightarrow \quad 1 \\ \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} &\quad \leftrightarrow \quad 1-q-d \end{aligned}$$

Now, for the elements of ϕ to sum to one (as is required), we must renormalize the computed unit eigenvector to

$$\begin{bmatrix} \frac{q}{q+d} \\ \frac{d}{q+d} \end{bmatrix} = \begin{bmatrix} \phi_{wt} \\ \phi_{bt} \end{bmatrix} = \phi$$

This expression makes intuitive sense. For one example, if no one ever loses a job, then $\phi_b = 0$ in the long run. For another, if the rate at which jobs are terminated, δ, increases, then the fraction of individuals in unemployment will rise.

(d) Continuing to assume that search is optimal, determine the values of V_w and V_b. (Note that nothing ever changes in this economy, except the individual's employment status, so that these are numbers rather than functions).

Answer: For this purpose, it is convenient to write the two Bellman equations as

$$V = m + \beta \Upsilon^T V$$

where V is a column vector of values m is a column vector of momentary rewards and Υ^T is the transpose of the transition matrix. Hence,

$$V = [I - \beta \Upsilon^T]^{-1} m$$

That is, with .

$$I - \beta \Upsilon^T = \begin{bmatrix} 1-\beta(1-\delta) & -\beta\delta \\ -\beta q & 1-\beta(1-q) \end{bmatrix}$$

and

$$[I - \beta \Upsilon^T]^{-1} = \frac{1}{D} \begin{bmatrix} 1-\beta(1-q) & \beta\delta \\ \beta q & 1-\beta(1-\delta) \end{bmatrix}$$

with D=$[1-\beta(1-\delta)][1-\beta(1-\delta)] - \beta^2 q\delta$

$$\begin{aligned} V_w &= \frac{1}{D}\{[1-\beta(1-q)](u(w)-\phi(1)) + \beta\delta(u(b)-\phi(1))\} \\ V_b &= \frac{1}{D}\{\beta q(u(w)-\phi(1) + [1-\beta(1-\delta)](u(b)-\phi(1)\} \end{aligned}$$

(e) Assume that the individual now chooses whether to search or not. It is useful to specify the Bellman equation in a slightly different manner to highlight this decision,

$$V_b = \max_h\{-[h\phi(1) - (1-h)\phi(0)] + \beta(hq)V_w^{'} + \beta[h(1-q) + (1-h)]V_b^{'}\}$$

Search is desirable if

$$-[\phi(0) - \phi(1)] + \beta q[V_w' - V_b'] > 0$$

In words, search is desirable if the cost in terms of current search time is less than the gain to finding a job. To determine whether this is fulfilled, we must determine the relevant future values.

Under what condition on the level of b will be optimal for the individual not to search?

Answer: An individual will choose not to search if the current cost of search

$$\phi(1) - \phi(0)$$

is greater than the expected future gain from search

$$\beta q(V_w - V_b)$$

Now, it is important that these future values reflect future optimization. To think about the appropriate future values, we use U to denote the welfare of an unemployed individual who does not search now or in the future. We use V to denote the welfare of an employed individual that knows that, if unemployed, he will not search in the future.

If the unemployed individual does not search now or in the future, then his welfare is

$$U = (u(b) - \phi(0)) + \beta U = \frac{1}{1-\beta}(u(b) - \phi(0))$$

If it is optimal not to search when unemployed, then the value of a job is

$$\begin{aligned} V &= u(w) - \phi(1) + \beta(1-\delta)V + \beta\delta U \\ &= \frac{1}{1-\beta(1-\delta)}[u(w) - \phi(1) + \beta\delta U] \end{aligned}$$

We are then interested in the gain

$$\begin{aligned} V - U &= \frac{1}{1-\beta(1-\delta)}[u(w) - \phi(1)] + \frac{1-\beta}{1-\beta(1-\delta)}U \\ &= \frac{1}{1-\beta(1-\delta)}\{[u(w) - \phi(1)] - [u(b) - \phi(0)]\} \\ &= \frac{1}{1-\beta(1-\delta)}\{[u(w) - u(b)] - [\phi(1) - \phi(0)]\} \end{aligned}$$

To find the critical level for benefits, we then look at

$$\begin{aligned} \phi(1) - \phi(0) &> \beta q[V - U] \\ &= \frac{\beta q}{1-\beta(1-\delta)}\{[u(w) - u(b)] - [\phi(1) - \phi(0)]\} \\ &= \frac{q}{r+\delta}\{[u(w) - u(b)] - [\phi(1) - \phi(0)]\} \end{aligned}$$

The economic interpretation of this condition is that the RHS is the expected present discounted value of rents from a future job. There are three terms. First, q is the probability of finding a job. Second, $\frac{1}{r+\delta}$ is interpretable as follows. Suppose there was a discount factor of β and that one had a stock that paid "d" j steps ahead with probability $(1-\delta)^{j-1}$. It would then be the case that this stock was worth

$$p = \sum_{j=1}^{\infty} \beta^j (1-\delta)^{j-1} d = \frac{\beta}{1-\beta(1-\delta)} d = \frac{1}{r+\delta} d$$

Hence, the factor $\frac{1}{r+\delta}$ expresses the net effect of discounting (at rate r) and the survival of a job (at rate $(1-\delta)$). Third, $[u(w)-u(b)]-[\phi(1)-\phi(0)]$ is the flow rents to a job. The expected present discounted value takes into all of these three factors. Hence, the economic condition is

$$u(b) < u(w) + \frac{q}{r+\delta+q}[\phi(1)-\phi(0)]$$

This takes into account the expected present discounted value of search costs $\frac{q}{r+\delta+q}[\phi(1)-\phi(0)]$