Boston University
Study Problems for Macroeconomics
Section 1

1. *A simple dynamic programming model of capital accumulation* Consider the following economy. Individuals have preferences

$$U = \sum_{t=0}^{\infty} \beta^t \log(c_t)$$

and a constraint of the form

$$k_{t+1} = ak_t^{\alpha} - c_t$$

(a) Write the Bellman equation for this economy.

Answer: The Bellman equation is

$$v(k_t) = \max\{\log(c_t) + \beta v(k_{t+1})\}$$

where the maximization takes place subject to $k_{t+1} = ak_t^{\alpha} - c_t$.

(b) Find the FOC(s) that must be satisfied for an optimal consumption and capital plan;

Answer: The FOCs are variously written depending on the details of the maximization, but always imply that

$$\frac{1}{c_t} = \beta(\frac{1}{c_{t+1}} \alpha a k_{t+1}^{\alpha-1})$$

(c) Show the following consumption policy

$$c_t = \phi a k_t^{\alpha}$$

is consistent with the condition(s) that you produced in (b).

Answer: The efficiency condition implies that

$$\frac{y_t}{c_t} = \beta\alpha \frac{y_{t+1}}{c_{t+1}} \frac{y_t}{k_{t+1}})$$

so

$$\frac{1}{\phi} = \beta\alpha \frac{1}{\phi} \frac{1}{1-\phi}$$

which implies that

$$\phi = (1 - \beta\alpha)$$

(d) Derive the value function by substituting the optimal consumption policy into the objective. Show that it takes the form

$$v = \gamma + \theta \log(k_t)$$

and determine the values of γ and θ.

Answer: We know from above that

$$\log(c_t) = \log(\phi a) + \alpha \log(k_t)$$

and that

$$\log(k_{t+1}) = \log((1-\phi)a) + \alpha \log(k_t)$$

Let $\log(k) = \frac{1}{1-\alpha}\log((1-\phi)a)$ so that

$$\log(k_{t+1}/k) = \alpha \log(k_t/k)$$

and let $\log(c) = \log(\phi a) + \frac{\alpha}{1-\alpha}\log((1-\phi)a)$, so that

$$\log(c_t/c) = \alpha \log(k_t/k)$$

Then, we can write the objective as

$$\begin{aligned} U &= \sum_{t=0}^{\infty} \beta^t [\log(c_t/c) + \log(c)] \\ &= \frac{1}{1-\beta}\log(c) + \sum_{t=0}^{\infty} \beta^t [\alpha \log(k_t/k)] \\ &= \frac{1}{1-\beta}\log(c) + \sum_{t=0}^{\infty} (\beta\alpha)^t [\alpha \log(k_0/k)] \\ &= \frac{1}{1-\beta}\log(c) + \frac{\alpha}{1-\beta\alpha}[\log(k_0/k)] \end{aligned}$$

Thus, $\theta = \frac{\alpha}{1-\beta\alpha}$ and $\gamma = \frac{1}{1-\beta}\log(c) - \frac{\alpha}{1-\beta\alpha}[\log(k)]$

(e) Show that the optimal policy maximizes

$$\log(c_t) + \beta v(k_{t+1})$$

where $v(k_{t+1})$ is the value function derived in part (d).

Answer: Maximizing

$$\log(c_t) + \beta\gamma + \beta\theta \log(k_{t+1})$$

subject to the constraint is the same as maximizing

$$\log(c_t) + \beta\gamma + \beta\theta \log(a k_t^{\alpha} - c_t)$$

with respect to c_t. The FOC leads to

$$\frac{1}{c_t} = \beta\theta \frac{1}{a k_t^{\alpha} - c_t}$$

or

$$c_t = \frac{1}{1+\beta\theta} a k_t^{\alpha} = \frac{1}{1+\beta\frac{\alpha}{1-\beta\alpha}} a k_t^{\alpha} = (1-\beta\alpha) a k_t^{\alpha} = \phi a k_t^{\alpha}$$

(f) Now suppose that there is an arbitrary value function of the form

$$v_{n-1}(k_{t+1}) = \gamma_n + \theta_{n-1}\log(k_t)$$

which appears on the right hand side of the Bellman equation. Show that

$$v_n(k_t) = \max\{\log(c_t) + \beta v_{n-1}(k_{t+1})\}$$

takes the form

$$v_n(k_{t+1}) = \gamma_n + \theta_n \log(k_t)$$

and determine the coefficients γ_n and θ_n.

Answer: We can proceed as in the analysis above, except that we leave the decision rules as

$$\begin{aligned} c_t &= \frac{1}{1+\beta\theta_{n-1}} a k_t^{\alpha} \\ k_{t+1} &= \frac{\beta\theta_{n-1}}{1+\beta\theta_{n-1}} a k_t^{\alpha} \end{aligned}$$

so that

$$\begin{aligned} & \log(c_t) + \beta[\gamma_{n-1} + \theta_{n-1}\log(k_{t+1})] \\ = \ & (1+\beta\theta_{n-1})(\log(a) + \alpha\log(k_t)) \\ & + \beta\gamma_{n-1} + \log(\frac{1}{1+\beta\theta_{n-1}}) + \beta\log(\frac{\beta\theta_{n-1}}{1+\beta\theta_{n-1}}) \end{aligned}$$

Hence, we can see that

$$\theta_n = (1+\beta\theta_{n-1})\alpha$$

and that there is a more complicated expression for γ_n:

$$\gamma_n = \beta\gamma_{n-1} + \log(\frac{1}{1+\beta\theta_{n-1}}) + \beta\theta_{n-1}\log(\frac{\beta\theta_{n-1}}{1+\beta\theta_{n-1}})$$

(g) Show that the sequence of coefficients $\{\gamma_n\}$ and $\{\theta_n\}$ generated in this manner converges to the γ and θ values that you determined in part (b) from any initial γ_0 and θ_0.

Answer: The expression above is a difference equation in θ_n, which can be written as

$$\theta_n - \theta = \beta\alpha(\theta_{n-1} - \theta)$$

with $\theta = 1/(1-\beta\alpha)$. From any positive θ_0 this difference equation will converge to θ so long as $\beta\alpha < 1$.
The value of γ_n can similarly be shown to converge to γ.

(h) Explain how to interpret the solution for a fixed number of iterations N as providing the capital accumulation decision rule for a finite horizon version of the problem above, in which

$$U = \sum_{t=0}^{T} \beta^t \log(c_t)$$

and in which $k_{T+1} \geq 0$. What is the relationship between N and T? **Answer:** N= T+1. In the finite horizon problem one can simply iterate backwards starting at the final period T. The value function in period T, call it V_1 is:

$$\begin{aligned} V_1(k_T) &= \log(ak_T^{\alpha}) \\ V_2(k_{T-1}) &= \underset{c_{T-1}}{max} \ (\log(c_{T-1}) + \beta V_1(k_T)) \ s.t. \ k_T = ak_{T-1}^{\alpha} - c_{T-1} \\ V_2(k_{T-1}) &= \underset{c_{T-1}}{max} \ (\log(c_{T-1}) + \beta log(ak_T^{\alpha})) \ s.t. \ k_T = ak_{T-1}^{\alpha} - c_{T-1} \\ &\vdots \\ V_{T+1}(k_0) &= \underset{c_0}{max} \ (\log(c_0) + \beta V_T(k_1)) \ s.t. \ k_1 = ak_0^{\alpha} - c_0 \end{aligned}$$