Boston University
Economics 702
Fall 2008

Introductory Study Problem #4: Optimal Consumption Over Time

Consider an agent who has preferences

$$U = \sum_{t=0}^{T} \beta^t u(c_t)$$

with

$$\begin{aligned} u(c_t) &= \frac{1}{1-\sigma}(c^{1-\sigma} - 1) \text{ for } \sigma \neq 1 \\ u(c_t) &= \log(c_t) \text{ for } \sigma = 1 \end{aligned}$$

where c_t is consumption at date t, u(c) is the momentary utility derived from this consumption, and β is a parameter that describes the rate at which future utility is discounted. Suppose further, as in problem 3, that this agent can accumulate wealth (a_t) according to

$$a_{t+1} = (1 + r_t)[a_t + y_t - c_t]$$

where saving (z_t) is the difference between income and consumption, $z_t = y_t - c_t$.

Suppose finally that this agent faces the constraint

$$a_{T+1} \geq 0 \tag{1}$$

That is: the agent can not leave life with negative assets.

(a) We know from problem 3 that this asset difference equation is equivalent to a present value constraint on consumption. Using the Lagrangian,

$$L = \sum_{t=0}^{T} \beta^t u(c_t) + \Lambda[a_0 + \sum_{t=0}^{T} p_t(y_t - c_t) \geq 0 \tag{2}$$

find the first order conditions (FOCs) that describe efficient consumption under the assumption (implied by the utility function above) that consumption is always positive.

Answer: The FOCs are

$$\begin{aligned}\frac{\partial L}{\partial c_t} &= \beta^t u_c(c_t) - \Lambda p_t = 0 \\ \frac{\partial L}{\partial c_t} &= a_0 + \sum_{t=0}^{T} p_t(y_t - c_t) = 0\end{aligned}$$

(b) From these FOCS, derive an optimal level of consumption at each date. Discuss the nature of this consumption plan.

Answer: The plan takes the form

$$c_t = [\frac{\Lambda p_t}{\beta^t}]^{-\frac{1}{\sigma}}$$

if it is written in terms of the multiplier. Under this specification, consumption is high in those periods during which p_t is low relative to β^t, with an elasticity of response that is dicated by σ. The level of the plan can also be written in terms of wealth, by solving out for Λ, as follows

$$\begin{aligned}a_0 + \sum_{t=0}^{T} p_t(y_t - [\frac{\Lambda p_t}{\beta^t}]^{-\frac{1}{\sigma}}) &= 0 \\ &\Rightarrow \Lambda^{-\frac{1}{\sigma}} = \frac{a_0 + \sum_{t=0}^{T} p_t y_t}{\sum_{t=0}^{T} p_t [\frac{p_t}{\beta^t}]^{-\frac{1}{\sigma}}} \\ &\Rightarrow c_t = [\frac{p_t}{\beta^t}]^{-\frac{1}{\sigma}} \frac{a_0 + \sum_{s=0}^{T} p_s y_s}{\sum_{s=0}^{T} p_s [\frac{p_s}{\beta^s}]^{-\frac{1}{\sigma}}}\end{aligned}$$

(c) Suppose $p_t = \beta^t$. What is the nature of the consumption plan? What aspects of consumption preferences does this plan reflect?

Answer: the consumption plan is flat, because the market discount factor p_t is exactly equal to the individual's personal discount factor.

(d) Suppose that there are an even number of periods in the lifetime (T is odd). Suppose further that

$$y_t = \left\{ \begin{array}{ll} \underline{y} & \text{if t is even} \\ \overline{y} & \text{if t is odd} \end{array} \right\}$$

as well as that a_0 and that $p_t = \beta^t$. What will be the pattern of consumption, income and saving?

Answer: the consumption plan is flat, as discussed above, so that

$$c_t = \frac{a_0 + \sum_{s=0}^{T} \beta^s y_s}{\sum_{s=0}^{T} \beta^s}$$

If income were zero, the individual would simply gradually run down assets toward the end of his life, so that there would be dissaving every period. If assets were zero and income fluctuated as above, saving would be negative in even periods if $\underline{y} < \overline{y}$ and positive in odd periods. The pattern would be reversed if the inequalitty was reversed. This must be the case because consumption is constant over time and the present value of saving will have to be zero, i.e., with $a_0 = 0$,

$$0 = \sum_{s=0}^{T} \beta^s (y_s - c)$$

so that positive save is offset with negative saving.

(e) Show that the FOCs imply

$$u_c(c_t) = \beta(1 + r_t)u_c(c_{t+1})$$

and use this to determine an expression for the growth rate of consumption. How does this expression compare to the one that you developed in problem 1? Show that a form of "Fisher's rule" holds for the multiperiod consumption problem.

Answer: let's look at the efficiency condition above,

$$\beta^t u_c(c_t) = \Lambda p_t$$

and then the price definition from problem 3,

$$p_t = \frac{1}{(1 + r_0)(1 + r_1)...(1 + r_{t-1})}$$

Taking the ratios of the t and t+1 FOCs, we have

$$\frac{\beta^{t+1} u_c(c_{t+1})}{\beta^t u_c(c_t)} = \frac{\Lambda p_{t+1}}{\Lambda p_t} = \frac{p_{t+1}}{p_t} = \frac{1}{(1 + r_t)}$$

so that

$$\beta \frac{u_c(c_{t+1})}{u_c(c_t)} = \frac{1}{(1 + r_t)}$$

as in problem 1. Hence, we can use the same analysis to conclude that

$$\begin{aligned} \log(c_{t+1}/c_t) &= \frac{1}{\sigma}[\log(1 + r_t) + \log(\beta)] \\ &\simeq \frac{1}{\sigma}[r_t - \nu] \end{aligned}$$

with $1 + \nu = 1/\beta$.