Boston University
Economics 702
Fall 2008

Introductory Study Problem #4: Optimal Consumption Over Time

Consider an agent who has preferences

$$U = \sum_{t=0}^{T} \beta^t u(c_t)$$

with

$$\begin{aligned} u(c_t) &= \frac{1}{1-\sigma}(c^{1-\sigma} - 1) \text{ for } \sigma \neq 1 \\ u(c_t) &= \log(c_t) \text{ for } \sigma = 1 \end{aligned}$$

where c_t is consumption at date t, u(c) is the momentary utility derived from this consumption, and β is a parameter that describes the rate at which future utility is discounted. Suppose further, as in problem 3, that this agent can accumulate wealth (a_t) according to

$$a_{t+1} = (1 + r_t)[a_t + y_t - c_t]$$

where saving (z_t) is the difference between income and consumption, $z_t = y_t - c_t$.

Suppose finally that this agent faces the constraint

$$a_{T+1} \geq 0 \tag{1}$$

That is: the agent can not leave life with negative assets.

(a) We know from problem 3 that this asset difference equation is equivalent to a present value constraint on consumption. Using the Lagrangian,

$$L = \sum_{t=0}^{T} \beta^t u(c_t) + \Lambda[a_0 + \sum_{t=0}^{T} p_t(y_t - c_t) \geq 0 \tag{2}$$

find the first order conditions (FOCs) that describe efficient consumption under the assumption (implied by the utility function above) that consumption is always positive.

(b) From these FOCS, derive an optimal level of consumption at each date. Discuss the nature of this consumption plan.

(c) Suppose $p_t = \beta^t$. What is the nature of the consumption plan? What aspects of consumption preferences does this plan reflect?

(d) Suppose that there are an even number of periods in the lifetime (T is odd). Suppose further that

$$y_t = \left\{ \begin{array}{ll} \underline{y} & \text{if t is even} \\ \overline{y} & \text{if t is odd} \end{array} \right\}$$

as well as that a_0 and that $p_t = \beta^t$. What will be the pattern of consumption, income and saving?

(e) Show that the FOCs imply

$$u_c(c_t) = \beta(1 + r_t)u_c(c_{t+1})$$

and use this to determine an expression for the growth rate of consumption. How does this expression compare to the one that you developed in problem 1? Show that a form of "Fisher's rule" holds for the multiperiod consumption problem.