Boston University
Economics 702
Fall 2008

Introductory Study Problem #3: A Wealth Difference Equation

Consider an agent who can accumulate wealth (a_t) according to

$$a_{t+1} = (1 + r_t)[a_t + z_t]$$

where z_t is his saving and r_t is the rate of return.

(a) Assuming that the rate of return is constant over time, solve this difference equation from an initial condition a_0 given an exogenous sequence of savings, $\{z_t\}_{t=0}^{T}$, to determine wealth at date $T + 1$.

Answer: This equation can be solved recursively, as follows

$$\begin{aligned}
a_1 &= (1 + r)(a_0 + z_0) \\
a_2 &= (1 + r)(a_1 + z_1) = (1 + r)^2(a_0 + z_0) + (1 + r)z_1 \\
a_3 &= (1 + r)(a_2 + z_2) = (1 + r)^3(a_0 + z_0) + (1 + r)^2 z_1 + (1 + r)z_2 \\
&\quad \ldots \\
a_t &= (1 + r)^t a_0 + \sum_{s=0}^{t-1}(1 + r)^{t-s} z_s
\end{aligned}$$

(b) Supposing that wealth must be positive at date $T + 1$, show that the condition

$$a_{T+1} \geq 0$$

is equivalent to an economic requirement on the present value of saving

$$a_0 + \sum_{t=0}^{T}(\frac{1}{1 + r})^t z_t \geq 0$$

Answer: Use the solution in part (a) to write the condition as

$$a_{T+1} = (1 + r)^{T+1} a_0 + \sum_{s=0}^{T}(1 + r)^{T+1-s} z_s \geq 0$$

Divide by $(1+r)^{T+1}$ so that the condition is

$$a_0 + \sum_{s=0}^{T} (1+r)^{-s} z_s \geq 0$$

and then change the index to t and write $(1+r)^{-s} = (\frac{1}{1+r})^t$.

(c) Now suppose that the rate of return is not constant through time. Solve this difference equation from an initial condition a_0 given an exogenous sequence of savings, $\{z_t\}_{t=0}^{T}$, to determine wealth at date $T+1$.

Answer: This equation can be solved recursively, as follows

$$\begin{aligned}
a_1 &= (1+r_0)(a_0+z_0) \\
a_2 &= (1+r_1)(a_1+z_1) = (1+r_1)(1+r_0)(a_0+z_0) + (1+r_1)z_1 \\
a_3 &= (1+r_2)(a_2+z_2) = (1+r_2)(1+r_1)(1+r_0)(a_0+z_0) \\
&\quad +(1+r_2)(1+r_1)(1+r)z_1 + (1+r_2)z_2 \\
&\quad ... \\
a_t &= R_{t,0}a_0 + \sum_{s=0}^{t-1} R_{t,s} z_s
\end{aligned}$$

with

$$R_{t,s} = (1+r_s)(1+r_{s+1})...(1+r_{t-1})$$

for all $0 \leq s \leq t-1$

(d) Show that the requirement that wealth must be positive at date T+1, $a_{T+1} \geq 0$, is equivalent to an economic requirement on the present value of saving

$$a_0 + \sum_{t=0}^{T} p_t z_t \geq 0$$

and determine how the p_t relate to the r_t.

Answer: Use the solution in part (c) to write the condition as

$$a_{T+1} = R_{T+1,0}a_0 + \sum_{s=0}^{T} R_{T+1,s} z_s \geq 0$$

Divide by $R_{T+1,0}$ so that the condition is

$$a_0 + \sum_{s=0}^{T} \frac{R_{T+1,s}}{R_{T+1,0}} z_s \geq 0$$

Then, use the definition above to get

$$\begin{aligned} p_s &= \frac{(1+r_s)(1+r_{s+1})...(1+r_T)}{(1+r_0)(1+r_1)...(1+r_T)} \\ &= \frac{1}{(1+r_0)(1+r_1)...(1+r_{s-1})} \end{aligned}$$

as the interest rate based "present value price" of wealth at date s.