Boston University
Economics 702
Fall 2008

Introductory Study Problem #2: A simple model of labor supply

Suppose that an individual has a utility function of the form

$$u(c,l) = \log(c) - \frac{\chi}{1+\gamma}(1-l)^{1+\gamma}$$

where c is consumption and l is leisure. The utility function parameters χ and γ are both positive. The household also faces the constraint that the amount of its market work

$$n = 1 - l$$

cannot exceed one (the endowment of time).

The individual's budget constraint is

$$c = wn + \pi$$

where π is nonwage income.

(2-a) The Lagrangian suitable for an analysis of the optimal interior consumption ($c > 0$) and leisure ($0 < l < 1$) decisions is given by

$$L = u(c,l) + \lambda[w(1-l) + \pi - c]$$

Discuss how this is similar to the standard "two good" problem for a consumer with endowments. What is the numeraire? What are the endowments?

Answer: this is like a two good demand problem (c and l) with consumption as numeraire (its price is one). The endowment of the leisure good is one (its value is w) and the endowmen of the consumption good is π.

(2-b) Find the first order conditions to the problem in (2-a). Solve one for consumption given λ and the other for work given λ and w. Given these functions, how would λ be determined? Would a higher value of λ raise or lower consumption? work?

Answer: The FOCs are

$$\begin{aligned} u_c(c,l) &= \lambda \\ u_l(c,l) &= \lambda w \end{aligned}$$

With the specific functional forms, these are

$$\begin{aligned} \frac{1}{c} &= \lambda \\ \chi(1-l)^{\gamma} &= \lambda w \end{aligned}$$

and these can thus be solved as

$$\begin{aligned} c &= \frac{1}{\lambda} \\ (1-l) &= [\frac{\lambda w}{\chi}]^{\frac{1}{\gamma}} \end{aligned}$$

The budget constraint would determine the value of λ. However, at the level of these functions, a higher level of λ would lower consumption (c) and leisure (l).

(2-c) Given λ, what is the elasticity of labor supply with respect to the wage rate?

Answer: The last condition can be written as

$$n = [\frac{\lambda w}{\chi}]^{\frac{1}{\gamma}}$$

so that the elasticity of labor supply to w, holding fixed λ, is $1/\gamma$.

(2-d) Suppose now that there is only labor income ($\pi = 0$). What is the elasticity of labor supply with respect to the wage rate if λ is endogenously determined?

Answer: It is zero, because this utility function has exactly offsetting income and substitution effects of the wage change (there is a positive income effect because the individual has a positive endowment of time to be split between work and leisure). To see this, the budget constraint is

$$c = wn$$

The easiest way of showing the result is to use $c = \frac{1}{\lambda}$ and to substitute $\lambda = \frac{1}{wn}$ into substituting in the above expressions, we get that

$$n = [\frac{\lambda w}{\chi}]^{\frac{1}{\gamma}} = [\frac{w/(wn)}{\chi}]^{\frac{1}{\gamma}} = [\frac{1/n}{\chi}]^{\frac{1}{\gamma}}$$

which can be solved for

$$n = [\frac{1}{\chi}]^{\frac{1}{\gamma+1}}$$

The fact that w is absent from this expression means that the labor supply elasticity is zero. Consumptiuon is then

$$c = w[\frac{1}{\chi}]^{\frac{1}{\gamma+1}}$$

so that it is unit elastic in the wage.

(2-e) If the individual in part (d) were given a small amount of nonwage income, would he work harder or less hard? Show your answer in a figure and using "comparative statics".

Answer: We have seen above that lower lambda raises both consumption and leisure (lowers work). The budget constraint implies that

$$c = wn + \pi$$

To compute the response to a small change in π, we can differentiate this expression:

$$dc = wdn + d\pi$$

Similarly, differentiating the FOCs above, we get

$$-dc\frac{1}{c^2} = d\lambda$$

and

$$\gamma\chi n^{\gamma 1}dn = wd\lambda$$

Solving these conditions, we deduce that

$$\begin{aligned}
\gamma\chi n^{\gamma-1}dn &= wd\lambda \\
&= -dc\frac{w}{c^2} \\
&= -(wdn + d\pi)\frac{w}{c^2} \\
[\gamma\chi n^{\gamma-1} + \frac{w^2}{c^2}]dn &= -d\pi\frac{w}{c^2}
\end{aligned}$$

That is: labor falls with nonwage income. On the other hand, consumption rises with nonwage income

$$\begin{aligned}
dc &= wdn + d\pi \\
&= \frac{-\frac{w^2}{c^2}}{[\gamma\chi n^{\gamma-1} + \frac{w^2}{c^2}]}d\pi + d\pi \\
&= \frac{\gamma\chi n^{\gamma-1}}{[\gamma\chi n^{\gamma-1} + \frac{w^2}{c^2}]}d\pi > 0
\end{aligned}$$

In terms of a graph, there is a simple one. Place leisure on the horizontal axis and consumption on the vertical one. Draw the budget line $c = wn = w(1-l)$ and an indifference curve tagent to it. Now, shift up the budget line to $c = w(1-l) + \pi$. With consumption and leisure as "normal goods", the demand for each rises, so that consumption rises and work falls with nonwage income.