Boston University
Economics 702
Fall 2008

Introductory Study Problem #1:
A two period model of optimal consumption over time

Suppose that a household has a utility function depending on consumption (c) in two periods,

$$U = \frac{1}{1-\sigma} c_0^{1-\sigma} + \beta \frac{1}{1-\sigma} c_1^{1-\sigma}$$

with $\sigma > 0$ and $0 < \beta < 1$.[1] The parameter β controls the extent to which the household "discounts" the future, so that β can be viewed as $\beta = \frac{1}{1+b}$ with b being a rate at which the individual discounts the future. As explored further below, the parameter σ controls the extent to which the individual substitutes across time.

Suppose also that the two-period budget constraint takes the form

$$c_0 + \frac{1}{1+r} c_1 = y_0 + \frac{1}{1+r} y_1$$

where y is income and r is the real interest rate.

(1-a) Discuss how this is similar to the standard "two good" problem for a consumer with endowments. What is the numeraire? What are the endowments? What is the relative price and how does it depend on the interest rate?

Answer: the two good problem for a consumer with endowments y_0 and y_1 is as follows. The consumer maximizes $U(c_0, c_1)$ subject to

$$p_0 c_0 + p_1 c_1 \leq [p_0 y_0 + p_1 y_1]$$

where p_i is the price of good $i = 0$ or $i = 1$. This formulation makes clear that date 0 consumption is the numeraire in the problem above, $p_0 = 1$. The relative price of consumption in period 1 in terms of period 0 is

$$\frac{1}{1+r}$$

so that it is higher when the interest rate is lower.

[1] The utility function is not well-defined at $\sigma = 1$, but if it is written as

$$U = \frac{1}{1-\sigma}(c_0^{1-\sigma} - 1) + \beta \frac{1}{1-\sigma}(c_1^{1-\sigma} - 1)$$

then we can use L'Hopital's rule to argue that the $\lim_{\sigma \to 1}\{\frac{1}{1-\sigma}(c_0^{1-\sigma} - 1)\} = \log(c)$. Thus, we are considering $u(c) = \log(c)$ when $\sigma = 1$.

(1-b-1) Draw the budget constraint for an individual with c_0 on the horizontal axis and c_1 on the vertical axis. What is the slope of the budget constraint?

Answer: Solve the BC for c_1: $c_1 = (1+r)y_0 + y_1 + (1+r)c_0$. Therefore the slope is $1+r$. The BC intersect the vertical axis ($c_0 = 0$) at $c_1 = (1+r)y_0 = y_1$ and the horizontal axis ($c_1 = 0$) at $c_0 = y_0 + \frac{1}{1+r}y_1$.

(1-b-2) Draw a set of indifference curves for this individual, with c_0 on the horizontal axis and c_1 on the vertical axis. What is the slope of the indifference curve at $\mathrm{c}_0 = c_1$ in terms of β? in terms of b? What is the relationship between the parameter σ and a measure of the individual's willingness to substitute over time?

Answer: Use the Implicit function theorem:

$$\frac{\partial c_1}{\partial c_0} = -\frac{\frac{\partial F}{\partial c_1}}{\frac{\partial F}{\partial c_0}} = -\frac{c_0^{-\sigma}}{\beta c_1^{-\sigma}}$$

Where $F = \frac{1}{1-\sigma}c_0^{1-\sigma} + \beta\frac{1}{1-\sigma}c_1^{1-\sigma} - U$. Evaluate this at $c_0 = c_1$ to get that the slope of the indifference curve is $-1/\beta = -(1+b)$. The parameter σ controls the elasticity of substitution: a higher σ corresponds to a lower elasticity of substitution. The elasticity of substitution (σ^{-1}) is defined by the reciprocal of the proportionate change in the magnitude of the slope of the indifference curve in response to a proportionate change in the ratio $\frac{c_0}{c_1}$,i.e.:

$$\frac{1}{\sigma} = -\left[\frac{c_0/c_1}{U_{c_0}/U_{c_1}}\frac{\partial(U_{c_0}/U_{c_1})}{\partial(c_0/c_1)}\right]^{-1}$$

Where U_{c_i} indicates the first derivative of the utility function with respect to c_i. From the first order condition of the optimization problem you can see that this can be rewritten as:

$$\frac{1}{\sigma} = -\left[\frac{c_0/c_1}{p_0/p_1}\frac{\partial(p_0/p_1)}{\partial(c_0/c_1)}\right]^{-1}$$

For more details see Romer: Advanced Macroeconomics, ch. 2.

(1-c) Formulate a Lagrangian suitable for deriving the demand for consumption at each date, using Λ as the multiplier on the budget constraint.

Answer:

$$L = U + [y_0 + \frac{1}{1+r}y_1 - c_0 - \frac{1}{1+r}c_1]$$

(1-d) For an interior optimum, $c_i > 0$, what are the first order conditions? Why do these differ in form?

Answer:

$$
\begin{aligned}
\frac{\partial U}{\partial c_0} - \Lambda &= 0 \\
\frac{\partial U}{\partial c_2} - \Lambda \frac{1}{1+r} &= 0
\end{aligned}
$$

They are different because consumption at date 0 is the numeraire. More generally,

$$
\frac{\partial U}{\partial c_i} - \Lambda p_i = 0
$$

in the demand problem discussion in (1-c) above.

(1-d) Use each first order condition to derive an expression that relates efficient consumption at date t=0,1 to Λ and other aspects of the consumer's problem.

Answer:

$$
\begin{aligned}
c_0 &= [\Lambda]^{-1/\sigma} \\
c_1 &= [\Lambda \frac{1}{1+r}]^{-1/\sigma}
\end{aligned}
$$

(1-e) Irving Fisher argued in his *Theory of Interest* that the growth rate of consumption would depend on a gap between "market and personal interest rates" and on a measure of the willingness of an individual to substitute over time. Using the ratio of the first order conditions, work out "Fisher's rule" for the growth rate of consumption, identifying each of his channels with an aspect of the two period model.

Answer: The FOCs from part (1-c) above imply that

$$
(1+r) = \frac{\frac{\partial U}{\partial c_0}}{\frac{\partial U}{\partial c1}} = \frac{c_0^{-\sigma}}{\beta c_1^{-\sigma}}
$$

so that

$$
\frac{c_1}{c_0} = [\beta(1+r)]^{1/\sigma} = [\frac{(1+r)}{1+b}]^{1/\sigma}
$$

Consumption grows if $r > b$, with a degree of response that depends on σ.

A particularly convient form comes about if logs are taken,

$$
\log(c_1/c_0) = \frac{1}{\sigma}[\log(1+r) - \log(1+b)] \simeq \frac{1}{\sigma}[r-b]
$$

where the last line comes from $\log(1+x) \simeq x$ for small x.

(1-f) Using the results of part (1-d) and the budget constraint, work out a value of Λ that is consistent with the consumer spending all of his life time income. Why does the form of the utility function assure that the consumer will do this?

Answer:

$$\begin{aligned} c_0 + \frac{1}{1+r} c_1 &= y_0 + \frac{1}{1+r} y_1 \Rightarrow \\ [\Lambda]^{-1/\sigma} + \frac{1}{1+r}[\Lambda \frac{1}{1+r}]^{-1/\sigma} &= y_0 + \frac{1}{1+r} y_1 \end{aligned}$$

$$\Lambda = \left[\frac{1 + (\frac{1}{1+r})^{\frac{\sigma-1}{\sigma}}}{y_0 + \frac{1}{1+r} y_1} \right]^{\sigma}$$

The consumer has positive marginal utility of consumption at any finite level of consumption and wealth.