Optimal Savings under Uncertainty[1]

I. INTRODUCTION

There has been extensive discussion in the literature of optimal savings behaviour under certainty in the context of infinite time horizon. To our knowledge the extension of the results under certainty to a situation of uncertainty has been attempted by Mirrlees [1] and Phelps [2]. Mirrlees considers a one-commodity neoclassical model with two factors of production, labour and capital; constant returns to scale in production; exponential labour force growth, and Harrod-neutral technical change. Uncertainty is introduced in the model form of a Wiener process over time for the logarithm of the Harrod-neutral technical change. The maximand is the expected value of the integral of discounted future *per capita* utilities. Mirrlees establishes a set of conditions characterizing an optimal consumption policy as a function of capital stock per unit of labour and the level of technology. These conditions correspond to the Euler equations and the transversality condition characterizing the optimal accumulation path under certainty. For the case of a constant (but negative) elasticity utility function and a Cobb-Douglas production function, Mirrlees shows that optimal savings can increase with increasing uncertainty, at least for some set of values of the capital-labour ratio and the level of technology. Phelps considers a pure capital model where an individual at any moment of time has the option of either consuming his wealth and current wage income or investing part of it. The return to investment is uncertain but the probability distribution of returns is assumed to be independently and identically distributed over time. The objective is again to maximize the expected value of the sum of discounted future utility over a finite horizon. Phelps takes the limit, as the horizon extends to infinity, of the finite-horizon optimal policy and discusses the behaviour of the limit policy as the " riskiness " of return to capital increases. The utility function is one for which the elasticity of marginal utility is constant. It is shown that the limit policy results in lower (higher) consumption for any given level of wealth as riskiness increases if the elasticity of marginal utility exceeds (falls short of) unity in absolute value. In the in-between case of unitary elasticity, consumption policy is invariant with respect to riskiness. Phelps somehow feels the last result to be " odd ".

Our purpose is first to re-examine a slightly simplified version of Phelps's model (we assume wage income to be zero) in the context of an infinite horizon, and to derive a set of sufficient conditions characterizing an optimal policy provided one exists. These conditions will be derived on the assumption of a strictly concave utility function, a class which includes, but is wider than, the class of constant elasticity utility functions. Second, we show that the limit policy obtained by Phelps is indeed the optimal policy for an infinite horizon for the class of constant elasticity utility functions. The set of sufficient conditions we derive is quite similar to the set derived by Mirrlees for his more general (and hence complicated) continuous time model. We believe that our derivation is of independent interest in that it yields all the Mirrlees-Phelps results in a considerably more simple and intuitively appealing way, though our model is not as rich as Mirrlees' because of its omission of diminishing returns to capital. In the last section we extend our model to include dynamic portfolio choice and discuss some possible implications about portfolio choice of change of the risk parameters of one of the assets.

[1] The authors are grateful to Professor K. J. Arrow for his valuable comments.

II. THE MODEL AND SOME GENERAL RESULTS

Consider the following model: At each period t, an individual (or a planner) has the option of either consuming all his wealth, k_t, or " investing " part of it. Denoting consumption by c_t, investment will then be $(k_t - c_t)$. This investment will result in his wealth in the next period, k_{t+1}, becoming $(k_t - c_t)r_t$. We shall assume that r_t is a non-negative independently and identically distributed random variable over time with the distribution function F. The planner has an instantaneous utility function, $u(c_t)$. This function is assumed to be strictly concave and non-decreasing. The objective is to maximize

$$E\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right], \qquad \ldots(1)$$

subject to a set of stochastic constraints

$$k_{t+1} = (k_t - c_t)r_t, \text{ with } k_0 \text{ given}, \qquad \ldots(2)$$

and the non-negativity constraints

$$k_t \geqq c_t \geqq 0. \qquad \ldots(3)$$

In (1), E denotes the expected value operator, the expectation being over the joint distribution of the random variables c_t, $t = 0, 1, 2, \ldots$. The parameter β is the discount factor. However, as (1) is not necessarily convergent let us introduce the following definitions:

Definition 1. A feasible policy $f(k, t)$ is said to *overtake* another feasible consumption policy $g(k, t)$, if, starting from the same k_0, the policies f and g yield (stochastic) consumption paths c_t and $\hat{c}_t$ that satisfy $E\left[\sum_{t=0}^{T} \beta^t\{u(c_t) - u(\hat{c})\}\right] > 0$ for all $T >$ some T_0.

Definition 2. A feasible consumption policy $f(k, t)$ is said to be optimal if it overtakes all other feasible consumption policies.

Proceeding somewhat informally, we shall first discuss a special case when

$$\max E\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right]$$

exists.

It is clear that in an optimal solution the consumption at any t will depend only on the wealth k_t at that t. First, given that r_t is independently and identically distributed over time with a *known* distribution function F, information on the past realizations of r_t has no bearing on its future behaviour. Second, given a constant discount factor β, calendar time is irrelevant for the determination of optimal consumption. Thus, we can restrict our search for the optimal solution to the class " consumption policies ". By a consumption policy we shall mean a function of wealth k, denoted by $f(k)$. Thus, consumption will equal $f(k)$ at any time t if the wealth at time t equals k.

Let us define $V(k_0)$ as the expected value of the sum of discounted utilities attainable from an initial wealth k_0, and following a policy $f(k)$. In other words,

$$V(k_0) \equiv E\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right], \qquad \ldots(4)$$

where

$$c_t = f(k_t), \quad t = 0, 1, 2, \ldots, \qquad \ldots(5)$$

and

$$k_{t+1} = [k_t - f(k_t)]r_t. \qquad \ldots(6)$$

It is clear that we can rewrite $V(k_0)$ as:

$$V(k_0) = u(c_0) + \beta EV[(k_0 - c_0)r_0]. \qquad \ldots(7)$$

If $f(k)$ is an optimal policy, then initial consumption $c_0 = f(k_0)$ must be such as to maximize $u(c)+\beta EV[(k_0-c)r]$ over $0 \leqq c \leqq k_0$. That is, for an optimal policy,

$$V(k_0) = \max_{0 \leqq c \leqq k_0} [u(c)+\beta EV[(k_0-c)r_0]]. \qquad \text{...(8)}$$

We shall assume that the maximum occurs at a c in the open interval $(0, k_0)$. A sufficient condition for this is that $u'(0) = \infty$. Proceeding informally, let us differentiate the right-hand side of (8) with respect to c and equate the derivative to zero and get [1]

$$u'(c)-\beta E[r_0 V'\{(k_0-c)r_0\}] = 0$$

or

$$u'(c) = \beta E[r_0 V'\{(k_0-c)r_0\}]. \qquad \text{...(9)}$$

Let $c = f(k_0)$ be the solution to (9). Then, substituting this in (7), we get

$$V[k_0] = u[f(k_0)]+\beta EV[\{k_0-f(k_0)\}r_0]. \qquad \text{...(10)}$$

Let us differentiate both sides of (10) with respect to k_0:

$$V'(k_0) = f'(k_0)u'\{f(k_0)\}+\beta E[r_0\{1-f'(k_0)\}V'[\{k_0-f(k_0)\}r_0]]. \qquad \text{...(11)}$$

Using (9), we can rewrite (11) as follows:

$$V'(k_0) = u'\{f(k_0)\} \qquad \text{...(12)}$$

or, alternatively,

$$u'[f(k_0)] = \beta E\{r_0 u'[f[\{k_0-f(k_0)\}r_0]]\}. \qquad \text{...(13)}$$

Equations (12) and (13) are easily interpreted. Equation (12) says that the marginal worth of initial wealth equals the marginal utility of initial consumption if the consumption policy is optimal. The reason for this equality is the following: Given an extra unit of initial wealth, one can always be assured of getting at least the marginal utility of initial consumption by allotting all of the extra wealth to consumption at period zero. By saving part of it, the wealth of the next period can be changed, but the extent of change is a random variable since r_0 is uncertain. But in an optimal policy, the expected gain in future utility by saving equals the loss in utility in period zero because of saving. Thus, the act of saving at the margin cannot increase expected welfare. Equation (13) restates (12) by eliminating the function $V(k_0)$. It will be easily recognized as the myopic rule in intertemporal utility maximization. It says that the marginal utility of initial consumption equals the expected value of the product of the discounted marginal utility of next period's consumption and the gross rate of return r_0 to saving. This is so because (i) $[k_0-f(k_0)]r_0$ is the wealth in period 1 if the rate of return happened to be r_0; (ii) $f[\{k_0-f(k_0)\}r_0]$ is the consumption in period 1 since the consumption policy is $f(k)$. A moment's reflection will suggest that (13) is in fact the functional equation which has the optimal consumption policy $f(k)$ as its solution. This is seen from the fact that, given our assumptions about the stochastic process relating to r_t and the constancy of the discount factor, the consumption policy is stationary. Thus, we can drop the subscript zero from k_0 and r_0 and rewrite (13) as the following fundamental equation:

$$u'[f(k)] = \beta E[ru'\{f\{\{k-f(k)\}r\}\}]. \qquad \text{...(14)}$$

Remark. Equation (12) implies that optimal policy, when it exists, is unique. This follows from the concavity of u (and hence the monotonicity of u').

The above informal discussion suggests that (14) is a necessary condition for a policy to be optimal. But is it also sufficient? This question is answered in the following theorem:

[1] Note that the implicit assumptions involved in this operation are (*a*) $V(k_0)$ is differentiable, and (*b*) the differential and expected value operators can be interchanged.

Theorem 1. *Let $f(k)$ be a feasible policy which satisfies the following conditions:*

(*a*) $u'[f(k)] = \beta \int ru'[f\{\{k-f(k)\}r\}]dF$;

(*b*) $E[\beta^t u'\{f(k_t)\}k_t] \to 0$ as $t \to \infty$.

Then $f(k)$ is the optimal policy.

Proof. Let c_t denote the consumption associated with the policy $f(k)$, starting from a wealth level k at time zero. Let $\hat{c}_t$ denote the consumption associated with an alternative policy, say $g(k, t)$, starting from the same wealth level. We shall prove that f is optimal, that is, f overtakes g:

$$E\left[\sum_{t=0}^{T-1} \beta^t\{u(c_t)-u(\hat{c}_t)\}\right] > 0 \text{ for all } T > \text{some } T_0. \qquad ...(15)$$

By concavity of U:

$$E\left[\sum_{t=0}^{T-1} \beta^t\{u(c_t)-u(\hat{c}_t)\}\right] \geqq E\left[\sum_{t=0}^{T-1} \beta^t u'(c_t)(c_t-\hat{c}_t)\right]. \qquad ...(16)$$

By definition, $k_t = [k_{t-1}-c_{t-1}]r_{t-1}$ and $\hat{k}_t = [\hat{k}_{t-1}-\hat{c}_{t-1}]r_{t-1}$. Note that the same r_{t-1} appears both in k_t and $\hat{k}_t$. The reason is that c_t and $\hat{c}_t$ are the consumption paths associated with the same realization $\{r_t\}$, $t = 0, 1, 2, ...$ of the random variable r. Hence, we can assert (ignoring sets of probability zero corresponding to values of r equal to zero),

$$\frac{1}{r_{t-1}}(k_t-\hat{k}_t) = (k_{t-1}-\hat{k}_{t-1})-(c_{t-1}-\hat{c}_{t-1})$$

or

$$c_{t-1}-\hat{c}_{t-1} = (k_{t-1}-\hat{k}_{t-1}) - \frac{1}{r_{t-1}}(k_t-\hat{k}_t). \qquad ...(17)$$

Hence,

$$\begin{aligned}\sum_{t=0}^{T-1} \beta^t u'(c_t)(c_t-\hat{c}_t) &= \sum_{t=0}^{T-1} \beta^t u'(c_t)\left\{(k_t-\hat{k}_t) - \frac{1}{r_t}(k_{t+1}-\hat{k}_{t+1})\right\} \\ &= \sum_{t=1}^{T-1} \beta^{t-1}\frac{(k_t-\hat{k}_t)}{r_{t-1}}[\beta r_{t-1}u'(c_t)-u'(c_{t-1})] - \beta^{T-1}\frac{u'(c_{T-1})}{r_{T-1}}(k_T-\hat{k}_T) \\ &\geqq \sum_{t=1}^{T-1} \beta^{t-1}\frac{(k_t-\hat{k}_t)}{r_{t-1}}[\beta r_{t-1}u'(c_t)-u'(c_{t-1})] - \frac{\beta^{T-1}u'(c_{T-1})}{r_{T-1}}k_T, \qquad ...(18)\end{aligned}$$

since $\hat{k}_t \geqq 0$, $u' \geqq 0$, $\beta > 0$, $r_{T-1} > 0$.

Now $\frac{k_T}{r_{T-1}} = k_{T-1}-c_{T-1} = k_{T-1}-f(k_{T-1})$. Thus, $\frac{k_T}{r_{T-1}}$ depends only on k_{T-1}. Also, $u'[c_{T-1}] = E[\{\beta r_{T-1}u'(c_T)\}|\ k_{T-1}]$, where the expectation operator relates to the distribution of r_{T-1}. The reason is $c_{T-1} = f(k_{T-1})$ and $c_T = f(k_T)$, and f by assumption satisfies the functional equation (12) above. Hence,

$$E\left\{\beta^{T-1}u'(c_{T-1})\frac{k_T}{r_{T-1}}\right\} = E\left\{\beta^{T-1}\frac{k_T}{r_{T-1}}E\{\beta r_{T-1}u'(c_T)|\ k_{T-1}\}\right\}, \qquad ...(19)$$

where the first expectation operator relates to the distribution of k_{T-1} over all possible realizations of $\{r_t\}$, $t = 0, ..., T-2$. From the fact that $\frac{k_T}{r_{T-1}}$ depends only on k_{T-1}, we can write [1]

$$E\left\{\beta^{T-1}u'(c_{T-1})\frac{k_T}{r_{T-1}}\right\} = E\left\{\beta^{T-1}\frac{k_T}{r_{T-1}}\beta r_{T-1}u'(c_T)\right\}$$
$$= E[\beta^T u'(c_T)k_T]. \quad \ldots(20)$$

But by hypothesis, $E\{\beta^T u'(c_T)k_T\}\to 0$ as $T\to\infty$. Similarly, $\frac{k_t-\hat{k}_t}{r_{t-1}}$ depends only on k_{t-1} and $\hat{k}_{t-1}$. Hence,

$$E\left[\frac{k_t-\hat{k}_t}{r_{t-1}}\{\beta r_{t-1}u'(c_t)-u'(c_{t-1})\}\right] = E\left[\frac{k_t-\hat{k}_t}{r_{t-1}}.E\{\beta r_{t-1}u'(c_t)-u'(c_{t+1})|\,k_{t-1}\}\right].$$

But

$$E\{\beta r_{t-1}u'(c_t)-u'(c_{t-1})|\,k_{t-1}\} = 0,$$

since

$$u'(c_{t-1}) = E\{\beta r_{t-1}u'(c_t)|\,k_{t-1}\}.$$

Hence,

$$E\left[\frac{k_t-\hat{k}_t}{r_{t-1}}\{\beta r_{t-1}u'(c_t)-u'(c_{t-1})\}\right] = 0. \quad \ldots(21)$$

Hence, for large T,

$$E\left[\sum_{t=0}^{T-1}\beta^t\{u(c_t)-u(\hat{c}_t)\}\right] \geqq 0.$$

Thus, f is the optimal policy.

Remark. The special case of $E\left\{\sum_{t=0}^{\infty}\beta^t u(c_t)\right\}$ finite—for all k will be satisfied if $\beta Er<1$, and either $u(c)$ is bounded below or there exists some policy for which

$$E\left[\sum_0^\infty \beta^t u(c_t)\right]$$

is finite.

Proof. Suppose $u(c)$ is bounded below, say $u(c) \geqq \underline{u}$ for all c. Then

$$E\left[\sum_0^\infty \beta^t u(c_t)\right] \geqq E\left[\sum_0^\infty \beta^t \underline{u}\right] = \frac{\underline{u}}{1-\beta}.$$

Further,

$$E\left[\sum_0^\infty \beta^t u(c_t)\right] \leqq U\left[E\sum_0^\infty \beta^t c_t\right]$$

by concavity of u. Feasibility requires that $0 \leqq c_t \leqq k_t$. Hence,

$$\sum_0^\infty \beta^t c_t \leqq \sum_0^\infty \beta^t k_t.$$

[1] Consider a bivariate density function $F(x, y)$. Let $G(y)$ be the marginal density of y and $H(x, y)$ be the conditional density of x, given y. In other words,

$$\int F(x, y)dx = G(y) \text{ and } H(x, y) = \frac{F(x, y)}{G(y)}.$$

Consider the expected value of $p(x, y)\,.\,q(y)$:

$$E\{p(x, y)q(y)\} = \iint p(x, y)q(y)F(x, y)dxdy$$
$$= \int q(y)G(y)dy \int p(x, y)H(x, y)dx$$
$$= \int q(y)E\{p(x, y)|\,y\}G(y)dy$$
$$= E[q(y)E\{p(x, y)|\,y\}].$$

In our example, k_{T-1} (or k_{t-1}, $\hat{k}_{t-1}$) corresponds to y and r_{T-1} to x. $q(y)$ can be identified with k_T/r_{T-1} (or $(k_t-\hat{k}_t)/r_{t-1}$) and $p(x, y)$ can be identified with $\beta u'(c_t)r_{t-1}$ or $\beta u'(c_t)r_{t-1}-u'(c_{t-1})$.

Now $k_t \leqq \bar{k}_t$ where $\bar{k}_t$ is the wealth at time t if $c_t = 0$ for all t. It is clear that

$$\bar{k}_t = k_0 \prod_{\tau=0}^{t-1} r_\tau.$$

This leads to

$$E\left[\sum_0^\infty \beta^t c_t\right] \leqq E\left[\sum_0^\infty \beta^t k_t\right]$$

$$\leqq E\left[\sum_0^\infty \beta^t \bar{k}_t\right] = E\left[\sum_0^\infty \beta^t k_0 \prod_{\tau=0}^{t-1} r_\tau\right].$$

But r_τ is independently and identically distributed over time. Hence,

$$E\left[\sum_0^\infty \beta^t c_t\right] \leqq \sum_0^\infty \beta^t (Er)^t.$$

If $0<\beta Er<1$, the sum on the right-hand side is finite. Thus, $E\left[\sum_0^\infty \beta^t u(c_t)\right]$ is bounded above, implying that it is finite for all k. If $u(c)$ is not bounded below, the lower bound for $E\left[\sum_0^\infty \beta^t u(c_t)\right]$ is provided by the policy for which this sum is assumed to be finite.

Theorem 2. *If an optimum policy exists, then the associated welfare function $V(k)$ is non-decreasing in k, and concave.*

Proof. The result $V(k)$ is non-decreasing in k follows from our earlier result (12) that $V'(k) = u'\{f(k)\} \geqq 0$.

We now show that $V[\lambda k_1+(1-\lambda)k_2] \geqq \lambda V(k_1)+(1-\lambda)V(k_2)$ for any k_1, $k_2>0$ and $0<\lambda<1$. Consider any realization $\{r_t\}$ of r. Let k_{it}, c_{it} denote the wealth and consumption paths resulting from an initial wealth k_i and following the optimal policy, given the realization $\{r_t\}$. From feasibility of the optimal policy we know $k_{it} \geqq c_{it} \geqq 0$. Now suppose the initial wealth is $\lambda k_1+(1-\lambda)k_2$. Then, for the same realization $\{r_t\}$,

$$\lambda c_{1t}+(1-\lambda)k_{2t}$$

is a feasible consumption path, and the associated wealth path will be $\lambda k_{1t}+(1-\lambda)k_{2t}$. Hence, the welfare $V[\lambda k_1+(1-\lambda)k_2]$ attainable by following the optimal policy f has to be greater than or equal to the welfare associated with the consumption path

$$\lambda c_{1t}+(1-\lambda)c_{2t}.$$

In other words,

$$V[\lambda k_1+(1-\lambda)k_2] \geqq E\left[\sum_{t=0}^\infty \beta^t u\{\lambda c_{1t}+(1-\lambda)c_{2t}\}\right]. \qquad \ldots(22)$$

But by concavity of u, the right-hand side of (22) is greater than or equal to

$$E\left[\sum_{t=0}^\infty \beta^t\{\lambda u(c_{1t})+(1-\lambda)u(c_{2t})\}\right].$$

Hence,

$$V[\lambda k_1+(1-\lambda)k_2] \geqq \lambda E\left\{\sum_{t=0}^\infty \beta^t u(c_{1t})\right\} +(1-\lambda)E\left\{\sum_{t=0}^\infty \beta^t u(c_{2t})\right\}$$

$$= \lambda V(k_1)+(1-\lambda)V(k_2). \qquad \ldots(23)$$

Corollary 1. The optimal policy $f(k)$ is an increasing function of k.

Proof. We know $V'(k) = u'[f(k)]$ by (12). Differentiating both sides with respect to k once again,

$$V''(k) = u''[f(k)]f'(k). \qquad \ldots(24)$$

By concavity of V and u, $V''<0$, $u''<0$. Hence, (24) implies $f'(k)>0$.

III. AN EXAMPLE

Let us now consider the case of a constant elasticity utility function. That is,

$$u(c) = \frac{1}{1-\alpha} c^{1-\alpha},$$

where $\alpha > 0$. Mirrlees considered the case where $\alpha > 1$. Following Phelps [1] we include the cases $\alpha = 1$, when $u(c) = \log c$, and $\alpha < 1$ as well.

Given $u(c) = \frac{1}{1-\alpha} c^{1-\alpha}$, $u'(c) = c^{-\alpha}$. Substituting this in our fundamental function equation (14), we get

$$[f(k)]^{-\alpha} \equiv \beta E[r[f\{\{k-f(k)\}r\}]^{-\alpha}]. \qquad \text{...(25)}$$

Let us consider a policy $f(k) = \lambda k$; i.e. consumption is a constant proportion of wealth. Such a policy will be feasible if and only if $0 < \lambda < 1$. Substituting for $f(k)$ in (25), we get:

$$\lambda^{-\alpha}k^{-\alpha} \equiv \beta E\{\lambda^{-\alpha}(1-\lambda)^{-\alpha}k^{-\alpha}r^{1-\alpha}\},$$

or

$$(1-\lambda)^{\alpha} \equiv \beta E(r^{1-\alpha}). \qquad \text{...(26)}$$

If $\beta E(r^{1-\alpha})$ is positive and less than unity, we obtain a λ which will yield a feasible policy. Our assumption that $r \geqq 0$ will ensure the positivity of $E(r^{1-\alpha})$. We shall *assume* now that $\beta E(r^{1-\alpha}) < 1$ so that we get a feasible policy. Let us apply our Theorem 1 to check the optimality of this solution. To do this, we have to verify condition (*B*), and show the finiteness of the supremum. First let us take condition (*B*). If the policy $f(k) = \lambda k$ is adopted,

$$k_t = [k_{t-1} - c_t]r_t = (1-\lambda)k_{t-1}r_{t-1}. \qquad \text{...(27)}$$

Iterating (27), we can easily write

$$k_t = (1-\lambda)^t k_0 \prod_{\tau=0}^{t-1} r_\tau. \qquad \text{...(28)}$$

Now,

$$u'[f(k_t)] = [f(k_t)]^{-\alpha} = \lambda^{-\alpha}k_t^{-\alpha}. \qquad \text{...(29)}$$

Thus,

$$\beta^t k_t u'[f(k_t)] = \beta^t \lambda^{-\alpha} k_t^{1-\alpha}$$
$$= \beta^t \lambda^{-\alpha}(1-\lambda)^{t(1-\alpha)} k_0^{1-\alpha} \prod_{\tau=0}^{t-1} r_\tau^{1-\alpha}.$$

Hence,

$$E[\beta^t k_t u'\{f(k_t)\}] = \beta^t \lambda^{-\alpha}(1-\lambda)^{t(1-\alpha)} k_0^{1-\alpha} E\left[\prod_{\tau=0}^{t-1} r_\tau^{1-\alpha}\right]. \qquad \text{...(30)}$$

But by the assumption that r_t is independently and identically distributed over time, (30) can be written as

$$E[\beta^t k_t u'\{f(k_t)\}] = \beta^t \lambda^{-\alpha}(1-\lambda)^{t(1-\alpha)} k_0^{1-\alpha}[E(r^{1-\alpha})]^t. \qquad \text{...(31)}$$

Using (26) we can simplify (31) and obtain

$$E[\beta^t k_t u'\{f(k_t)\}] = k_0^{1-\alpha}\lambda^{-\alpha}(1-\lambda)^t. \qquad \text{...(32)}$$

But λ is positive and less than unity. Hence, from (32) we can conclude that

$$\lim_{t\to\infty} E[\beta^t k_t u'\{f(k_t)\}] = 0.$$

[1] Phelps's utility function was of the form $u(c) = \beta + \frac{1}{1-\alpha} c^{1-\alpha}$.

This verifies condition (B). Applying Theorem 1, we get $f(k) = \lambda k$ as the optimal policy. It is easy to check $V(k)$ for this optimal policy.

$$V(k) = E\left\{\sum_{t=0}^{\infty} \beta^t u(c_t)\right\} = E\left[\sum_{t=0}^{\infty} \beta^t \left\{\frac{\lambda^{1-\alpha}}{1-\alpha} k_t^{1-\alpha}\right\}\right]$$

$$= \frac{\lambda^{1-\alpha}}{1-\alpha} E\left[\sum_{t=0}^{\infty} \beta^t k_t^{1-\alpha}\right]$$

$$= \frac{\lambda^{1-\alpha}}{1-\alpha} E\left[\sum_{t=0}^{\infty} \beta^t (1-\alpha)^{t(1-\alpha)} k^{1-\alpha} \prod_{\tau=0}^{t-1} R_\tau^{1-\alpha}\right]$$

$$= \frac{\lambda^{1-\alpha}}{1-\alpha}\left[k^{1-\alpha} \sum_{t=0}^{\infty} \beta^t (1-\alpha)^{t(1-\alpha)} E\left(\prod_{\tau=0}^{t-1} R_\tau^{1-\alpha}\right)\right]$$

$$= \frac{\lambda^{1-\alpha}}{1-\alpha} k^{1-\alpha}\left[\sum_{t=0}^{\infty} \beta^t (1-\alpha)^{t(1-\alpha)} \{E(R^{1-\alpha})\}^t\right]$$

$$= \frac{\lambda^{1-\alpha}}{1-\alpha} k^{1-\alpha}\left[\sum_{t=0}^{\infty} \beta^t (1-\alpha)^{t(1-\alpha)} \{E(R^{1-\alpha})\}^t\right]$$

$$= \frac{\lambda^{1-\alpha}}{1-\alpha} k^{1-\alpha}\left[\sum_{t=0}^{\infty} (1-\alpha)^t\right] = \frac{\lambda^{-\alpha} k^{1-\alpha}}{1-\alpha} < \infty.$$

Let us now examine the influence of uncertainty on the policy $f(k)$. From (26) it is obvious that if $\alpha = 1$, i.e., $u(c) = \log c$, then uncertainty has no influence on the optimal policy. The optimal policy in this case is $\lambda = 1-\beta$. This result can be intuitively seen as follows: We know that if $u(c) = \log c$ and $r = \bar{r} > 0$ with probability one, the optimal consumption policy is still $\lambda = 1-\beta$ and it does not depend on $\bar{r}$. If certain knowledge of $\bar{r}$ does not influence optimal policy, it stands to reason that uncertainty about r will have no impact on optimal policy either.

Suppose now $\alpha \neq 1$. In order to sharpen our discussion, let us make specific assumptions about the distribution of r. For instance, let $\log r$ be distributed normally with mean μ and variance σ^2. Then it is easy to verify that:

(i) $E(r) = e^{\mu+\sigma^2/2}$,

(ii) $\text{var } r = e^{(2\mu+\sigma^2)}[e^{\sigma^2}-1]$, and

(iii) $E(r^{1-\alpha}) = e^{\mu(1-\alpha)+(1-\alpha)^2\sigma^2/2}$.

If we denote by $\bar{r}$ the expected value of r, then

(iv) $\text{var } r = \bar{r}^2[e^{\sigma^2}-1]$, and

(v) $E(r^{1-\alpha}) = \bar{r}^{1-\alpha} e^{-\alpha(1-\alpha)\sigma^2/2}$.

Substituting the expression for $E(r^{1-\alpha})$ in (26), we get:

$$(1-\lambda)^\alpha = \beta \bar{r}^{(1-\alpha)} e^{-\alpha(1-\alpha)\sigma^2/2}. \qquad \ldots(33)$$

For $0<\lambda<1$, we must have $\beta(\bar{r})^{1-\alpha} e^{-\alpha\sigma^2(1-\alpha)/2} < 1$. Let us assume this.[1] We can visualize the situation of increasing uncertainty about r as one in which $\bar{r}$ remains constant, while

[1] Surprisingly, this condition is identical to the existence condition $n\alpha - v - n(n+1)\beta + r > 0$ of Mirrlees. This is seen if we recognize that our discount factor β can be identified with his $e^{-(r-v)}$, his n with our $(\alpha-1)$, his $(\alpha-\beta)$ with our μ, and his β with our $\sigma^2/2$. Then our feasibility condition is:

$$\beta E r^{1-\alpha} < 1 \text{ is equivalent to } e^{-(r-v)(\alpha-\beta)(-n)+\beta n^2} < 1;$$

that is, $r - v + (\alpha-\beta)n - \beta n^2 > 0$ or $r - v + n\beta - \beta n(n+1) > 0$.

var r increases. This means that σ increases in such a way that $\mu+\sigma^2/2$ is constant, thereby keeping $\bar{r}$ constant. Returning now to (33), we observe that as σ increases while $\bar{r}$ remains constant, λ will *increase* if $\alpha<1$, and *decrease* if $\alpha>1$. Thus, the proportion of wealth consumed (in the optimal policy) *increases* with increasing uncertainty if $\alpha<1$ and *decreases* with uncertainty if $\alpha>1$. Our result for $\alpha>1$ is thus consistent with those of Mirrlees.

Remark. Suppose $0<\alpha<1$. From (33) it follows that the condition for the existence of an optimal policy is the case of a *certain* return $\bar{y}$ is $\beta\bar{r}^{1-\alpha}<1$. If the return is uncertain but has mean $\bar{r}$, this condition is $\beta\bar{r}^{1-\alpha}e^{-\alpha(1-\alpha)\sigma^2/2}<1$. It may happen that if σ is large, the condition for existence may hold if there is uncertainty, while it is violated for the corresponding certainty case.

For the sake of variety let us consider the following distribution for r:

$$r = +\bar{r}+m\varepsilon \text{ with probability } 1/(2n+1),$$

where ε is a positive constant and $m = 0, \pm 1, \pm 2, \ldots, \pm n$. Assume also $\bar{r}-n\varepsilon>0$. Then $E(r) = \bar{r}$, $\text{var } r = \varepsilon^2 n(n+1)/6$. Substituting in (26), we get

$$(1-\lambda)^\alpha = \beta \sum_{m=0}^{n} [\bar{r}+m\varepsilon]^{1-\alpha}/2n+1.$$

Let us assume the feasibility condition is satisfied. Keep $\bar{r}$ constant and increase ε, i.e., increase uncertainty. It is easily seen that λ increases with ε if $\alpha<1$ and decreases as ε increases if $\alpha>1$.

IV. PORTFOLIO SELECTION

One of the more interesting areas for further research is the portfolio selection problem. The problem assumes simple form in the case of the constant elasticity utility function, in which case it is not hard to prove that the portfolio policy is independent of the amount of capital accumulated.

Assume the existence of two assets, one with a rate of return r_1, the other with rate of return r_2 . r_1 and r_2 are random variables with known joint distribution. Let δ be the proportion invested in the first asset, and $(1-\delta)$ in the second asset, $0 \leqq \delta \leqq 1$. Equation (26) for determination of the optimal consumption policy assumes the form:

$$(1-\lambda)^\alpha = \beta E[\delta r_1+(1-\delta)r_2]^{1-\alpha}, \qquad \ldots(34)$$

and

$$V(k) = \frac{\lambda^{-\alpha}k^{1-\alpha}}{1-\alpha}. \qquad \ldots(35)$$

Our aim is to maximize $V(k)$ by the proper choice of portfolio δ. Differentiating V with respect to δ,

$$\frac{\partial V}{\partial \delta} = \frac{\partial V}{\partial \lambda}\frac{\partial \lambda}{\partial \delta} = -\alpha\frac{V}{\lambda}\frac{\partial \lambda}{\partial \delta}. \qquad \ldots(36)$$

If $\alpha>1$, $V<0$, and hence $\frac{\partial V}{\partial \lambda}>0$. If $\alpha<1$, $V>0$, and hence $\frac{\partial V}{\partial \lambda}<0$.

In order to maximize V by suitable choice of δ, we should maximize λ if $\alpha>1$ and minimize λ if $\alpha<1$.

Now

$$\frac{\partial \lambda}{\partial \delta} = -\frac{1-\alpha}{\alpha(1-\lambda)^{\alpha-1}}\beta E(r_1-r_2)[\delta r_1+(1-\delta)r_2]^{-\alpha}, \qquad \ldots(37)$$

$$\frac{\partial^2\lambda}{\partial\delta^2} = \frac{1-\alpha}{(1-\lambda)^{\alpha-1}}\beta E(r_1-r_2)^2[\delta r_1+(1-\delta)r_2]^{-\alpha-1}$$

$$-\frac{(1-\alpha)^2}{\alpha(1-\lambda)^{\alpha}}\beta\frac{\partial\lambda}{\partial\delta}E(r_1-r_2)[\delta r_1+(1-\delta)r_2]^{-\alpha}$$

$$= \frac{1-\alpha}{(1-\alpha)^{\alpha-1}}\beta E(r_1-r_2)^2[\delta r_1+(1-\delta)r_2]^{-\alpha-1}$$

$$+\frac{(1-\alpha)^3}{\alpha^2(1-\lambda)^{2\alpha-1}}\beta^2\{E(r_1-r_2)[\delta r_1+(1-\delta)r_2]^{-\alpha}\}^2. \qquad ...(38)$$

Since $\delta r_1+(1-\delta)r_2$ is a non-negative random variable and $0<\lambda<1$ for all δ in $0 \leqq \delta \leqq 1$, $\beta>0$, it follows that $\frac{\partial^2\lambda}{\partial\delta^2}<0$ if $\alpha>1$, and $\frac{\partial^2\lambda}{\partial\delta^2}>0$ if $0<\alpha<1$. Let us take $\alpha>1$ first. If $\left.\frac{\partial\lambda}{\partial\delta}\right|_{\delta=1} = -\frac{1-\alpha}{\alpha(1-\lambda)^{\alpha-1}}E(r_1-r_2)r_1^{-\alpha} \geqq 0$, then λ is maximized at $\delta = 1$. If

$$\left.\frac{\partial\lambda}{\partial\delta}\right|_{\delta=0} = \frac{-(1-\alpha)}{\alpha(1-\lambda)^{\alpha-1}}E(r_1-r_2)r_2^{-\alpha} \leqq 0,$$

then λ is maximized at $\delta = 0$. If $\left.\frac{\partial\lambda}{\partial\delta}\right|_{\delta=0} = \frac{-(1-\alpha)}{\alpha(1-\lambda)^{\alpha-1}}E(r_1-r_2)r_2^{-\alpha} \leqq 0$, then λ is maximized at $\delta = 0$. If $\left.\frac{\partial\lambda}{\partial\delta}\right|_{\delta=0}>0$ and $\left.\frac{\partial\lambda}{\partial\delta}\right|_{\delta=1}<0$, the λ is maximized at $-0<\hat{\delta}<1$ when $\left.\frac{\partial\lambda}{\partial\delta}\right|_{\delta=\hat{\delta}}=0$.

For the case $\alpha<1$, the results are: If $\left.\frac{\partial\lambda}{\partial\delta}\right|_{\delta=1} \leqq 0$, then λ is minimized at $\delta = 1$. If $\left.\frac{\partial\lambda}{\partial\delta}\right|_{\delta=0} \geqq 0$, then λ is minimized at $\delta = 0$. If $\left.\frac{\partial\lambda}{\partial\delta}\right|_{\delta=1}>0$, and $\left.\frac{\partial\lambda}{\partial\delta}\right|_{\delta=0}<0$, then λ is minimized at $0<\hat{\delta}<1$ when $\left.\frac{\partial\lambda}{\partial\delta}\right|_{\delta=\hat{\delta}}=0$.

In order to examine the behaviour of the optimal value of the portfolio variable δ with respect to changes in the parameters of the distribution of r_1 and r_2, let us consider an example where $\log r_1$ and $\log r_2$ are independently and normally distributed with parameters (μ_1, σ_1^2) and (μ_2, σ_2^2). To simplify discussion further, let us consider the extreme case when the entire portfolio is invested in asset 1. We saw above that this can happen only if $E(r_1-r_2)r_2^{-\alpha} \geqq 0$. Given the assumptions on r_1 and r_2, this condition reduces to:

$$\bar{r}_1 e^{-\alpha\sigma_1^2} \geqq \bar{r}_2 \text{ where } \bar{r}_1 = E(r_1) \text{ and } \bar{r}_2 = E(r_2).$$

(A necessary condition for this is $\bar{r}_1>\bar{r}_2$).

Another way of looking at this condition is to interpret $e^{-\alpha\sigma_1^2}$ as a " risk " discount factor. Thus, if even after discounting for risk the mean return of the first asset with larger mean exceeds the mean return of the other asset, naturally the entire portfolio will be invested in this asset. This condition also shows that if σ_1 is increased sufficiently while keeping $\bar{r}_1$ constant (this is the same as increasing var (r_1) keeping the mean constant), the discounted mean return of this asset could be made to fall below the mean return of the other asset resulting in some part of the portfolio being invested in the second asset. It is our belief that this is in general true. That is, if the variance of the return on one of the assets is increased while keeping its mean constant, the optimal proportion invested in this asset will go down.

The Hebrew University, Jerusalem D. LEVHARI
Indian Statistical Institute T. N. SRINIVASAN

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