Lecture 8:
Solving Linear Rational Expectations Models

Five Components

- A. Core Ideas
- B. Nonsingular Systems Theory
  - (Blanchard-Kahn)
- C. Singular Systems Theory
  - (King-Watson)
- D. A Singular Systems Example
- E. Computation

A. Core Ideas

1. Recursive Forward Solution
2. Law of Iterated Expectations
3. Restrictions on Forcing Processes
4. Limiting Conditions
5. Fundamental v. nonfundamental solutions
6. Stable v. unstable roots
7. Predetermined v. nonpredetermined variables
8. Sargent’s procedure: unwind unstable roots forward

Basic Example

- Stock price as discounted sum of expected future dividends
- Let p_t be the (ex dividend) stock price and d_t be dividends per share.
- Basic approach to stock valuation

$$p_t = \sum_{j=1}^{\infty} \beta^j E_t d_{t+j} \ \ with \ \beta = \frac{1}{1+r}$$

- Intuitive reference model, although sometimes criticized for details and in applications

Origins of this model

- Investors (stockholders) must be indifferent between holding stock and earning an alternative rate of return "r" on some other asset.

$$\frac{E_t p_{t+1} + E_t d_{t+1} - p_t}{p_t} = r$$

- Left hand side is expected return on stock including dividends and capital gains.

Rewriting this as an expectational difference equation

- Takes first order form

$$\text{General:} \quad \text{a } \mathrm{E}_t y_{t+1} = b y_t + c_0 x_t + c_1 E_t x_{t+1}$$

$$\text{Specific:} \quad E_t p_{t+1} = (1+r) p_t - E_t d_{t+1}$$

- Alternatively

$$p_t = \frac{1}{1+r}[E_t p_{t+1} + E_t d_{t+1}]$$

1. Recursive Forward Solution

- The process is straightforward but tedious,

$$
\begin{aligned}
p_t &= \beta E_t[p_{t+1} + d_{t+1}] \\
&= \beta E_t[\beta(E_{t+1}(p_{t+2} + d_{t+2}) + d_{t+1}] \\
&= \dots \\
&= \sum_{j=1}^{J} \beta^j E_t d_{t+j} + \beta^J E_t p_{t+J}
\end{aligned}
$$

2. Law of Iterated Expectations

- General result on conditional expectations

$$E_t E_{t+1} E_{t+2} \ldots E_{t+j-1} x_{t+j} = E_t x_{t+j}$$

- Works in RE models where market expectations are treated as conditional expectations
- Lets us move to last line above.

3. Restrictions on Forcing Processes

- One issue in moving to infinite horizon: first part of price (the sum) above is well defined so long as dividends don't grow too fast, i.e.,

$$E_t d_{t+j} \leq h_t \gamma^j \;\; with \;\; \beta\gamma < 1$$

- Under this condition,

$$\lim_{J \to \infty} \sum_{j=1}^{J} \beta^j E_t d_{t+j} \leq \frac{\beta\gamma}{1-\beta\gamma} h_t < \infty$$

4. Limiting conditions

- For the stock price to match the basic prediction, the second term must be zero in limit.
- For a finite stock price, there must be some limit
- The conventional assumption is that

$$\lim_{J \to \infty} \beta^J E_t p_{t+J} = 0$$

- This is sometimes an implication (value of stock must be bounded at any point in time would do it, for example).

5. Fundamental v. nonfundamental solutions

- Nonfundamental solutions are sometimes called bubble solutions.
- In the current setting, let's consider adding an arbitrary sequence of random variables to the above: $p_t = f_t + b_t$
- These must be restricted by agents willingness to hold stock.

$$b_t = \frac{1}{1+r}[E_t b_{t+1}]$$

Form of nonfundamental solutions

- Stochastic difference equation

$$E_t b_{t+1} = (1+r) b_t \Rightarrow b_{t+1} = (1+r) b_t + \xi_{t+1}$$

- Bubble solution is expected to explode at just the right rate,

$$E_t \beta^j b_{t+j} = b_t$$

so that it is not possible to eliminate the last term in the above.

Comments on Bubbles

- A bursting bubble is one where there is a big decline due to a particular random event.
- Bubbles can’t be expected to burst (or they wouldn’t take place)
- Bursting bubbles are hard to distinguish empirically from anticipated increases in dividends that don’t materialize.

Ruling out bubbles of this form

- Formal arguments
  - Transversality condition
- Informal procedures
  - Unwind unstable roots forward (Sargent, see below)
  - Sometimes motivated by type of data that one seeks to explain (nonexplosive data)

6. Unstable and Stable Roots in RE models

- Stock price difference equation has unstable root
- Write as

$$E_t p_{t+1} = (1+r)[p_t + E_t d_{t+1}]$$

- Root is (1+r)>1 if r>0.

Stable root example

- Capital accumulation difference equation

$$k_{t+1} = (1-\delta)k_t + i_t$$

- Backward recursive solution

$$k_{t+1} = (1-\delta)^t k_0 + \sum_{j=o}^{t}(1-\delta)^j i_{t-j}$$

- Could well be part of RE model

7. Predetermined v. nonpredetermined variables

- Stock price: nonpredetermined
- Capital: predetermined
- General solution practice so far captures approach in literature

	Predetermined	Non Predetermined
Stable	Capital	
Unstable		Stock Price

8. Sargent's procedure

- In several contexts in the early 70s, Sargent made the suggestion that "unstable roots should be unwound forward."
- Examples:
  - Money and prices: similar to stocks
  - Labor demand: we will study this later.

B. Linear Difference Systems under Rational Expectations

- Blanchard-Kahn: key contribution in the literature on how to solve RE macroeconomic models with a mixture of predetermined variables and nonpredetermined ones.
- Variant of their framework that we will study

$$E_t Y_{t+1} = WY_t + \Psi_0 X_t + \Psi_1 E_t X_{t+1}$$

- Y is column vector of endogenous variables, X is column vector of exgenous variables
- Other elements are fixed matrices (I,W, Ψ) that are conformable with vectors (e.g. W is n(Y) by n(Y))

Types of Variables

- Predetermined (k): no response to x_t-$E_{t-1}x_t$
- Nonpredetermined (λ)
- Endogenous variable vector is partitioned as

$$Y_t = \begin{bmatrix} \lambda_t \\ k_t \end{bmatrix}$$

- Notation n(k) is number of k's etc. if we need to be specific about it.

Analytical Approach

- Transformation of system (of canonical variables form).
- In notation that we'll use later as well, let
  - Let T be an invertible matrix transforming equations
  - Let V be an invertible matrix transforming variables.
- New system in current context

$$E_t Y^*_{t+1} = W^* Y^*_t + \Psi^*_0 X_t + \Psi^*_1 E_t X_{t+1}$$

Transformed system of interest

- Can be based on eigenvectors: WP=Pμ
- T=inv(P) and V=inv(P)
- Then we have

$$E_t Y_{t+1} = W Y_t + \Psi_0 X_t + \Psi_1 E_t X_{t+1}$$

$$P^{-1} E_t Y_{t+1} = P^{-1} W P P^{-1} Y_t + P^{-1} \Psi_0 X_t + P^{-1} \Psi_1 E_t X_{t+1}$$

$$E_t Y^*_{t+1} = J Y^*_t + \Psi^*_0 X_t + \Psi^*_1 E_t X_{t+1}$$

with J block diagonal $J = \begin{bmatrix} J_u & 0 \\ 0 & J_s \end{bmatrix}$

Jordan form matrices

- Upper (or lower) diagonal
- Repeated root “blocks” may have ones as well as zeros above diagonal.
- Have inverses that are Jordan matrices also.
- Have zero limits if eigenvalues are all stable

Jordan form example

$$J = \begin{bmatrix} \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_1 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} \mu_2 & 1 \\ 0 & \mu_2 \end{bmatrix} \end{bmatrix}$$

Discussion

- Jordan blocks: matrices containing common eigenvalues.
- Form of Jordan block depends on structure of difference equation (or canonical variables)
- Two examples shown at right

$$B1: \; y_t = \mu_1 y_{t-1} + x_t$$
$$z_t = \mu_1 z_{t-1} + x_t$$

$$B2: \; y_t = \mu_1 y_{t-1} + z_t$$
$$z_t = \mu_1 z_{t-1} + x_t$$

Applying Sargent’s suggestion

- System is decoupled into equations describing variables with stable dynamics (s) and unstable dynamics (u)
- Taking the u part (with similarly partitioned matrices on x’s)

$$E_t u_{t+1} = J_u u_t + \Psi^*_{0u} X_t + \Psi^*_{1u} E_t X_{t+1}$$

- We can think about unwinding it forward.

Forward solution

- Comes from rewriting as

$$u_t = (J_u)^{-1} E_t u_{t+1} - (J_u)^{-1} \Psi_{0u}^* X_t - (J_u)^{-1} \Psi_{1u}^* E_t X_{t+1}$$

- Takes the form

$$u_t = -\sum_{h=0}^{\infty} [J_u^{-1}]^{h+1} E_t \{\Psi_{0u}^* X_{t+h} + \Psi_{1u}^* E_t X_{t+h+1}\}$$

- Suppresses unstable dynamics (any other initial condition for u implies explosive bubbles arising from these)

Stable block evolves according to

- The difference system

$$E_t s_{t+1} = J_s s_t + C_{0s}^* X_t + C_{1s}^* E_t X_{t+1}$$

Solving for the variables we really care about

- The u's and s's are related to the elements of Y according to

$$\begin{bmatrix} u \\ s \end{bmatrix} = \begin{bmatrix} V_{u\lambda} & V_{uk} \\ V_{s\lambda} & V_{sk} \end{bmatrix} \begin{bmatrix} \lambda \\ k \end{bmatrix}$$

$$\begin{bmatrix} \lambda \\ k \end{bmatrix} = \begin{bmatrix} R_{\lambda k} & R_{\lambda s} \\ R_{ku} & R_{ks} \end{bmatrix} \begin{bmatrix} u \\ s \end{bmatrix}$$

(R is inverse of V)

Solving for nonpredetermined variables

- Star Trek and Related Matters
- First "line" of matrix equation above

$$u_t = V_{u\lambda}\lambda_t + V_{uk}k_t.$$

- Solvable if we have two conditions
  - Same number of unstable and nonpredetermined variables
  - Invertible matrix $V_{u\lambda}$

Solving

- We get nonpredetermined variables as

$$\lambda_t = V_{u\lambda}^{-1}[u_t - V_{uk}k_t]$$

$$= -V_{u\lambda}^{-1}\sum_{h=0}^{\infty}[J_u^{-1}]^{h+1}E_t\{\Psi_{0u}^{*}X_{t+h} + \Psi_{1u}^{*}E_tX_{t+h+1}\}$$

$$-V_{u\lambda}^{-1}V_{uk}k_t$$

Solving

- We get the predetermined variables as

$$k_{t+1} = w_{kk}k_t + w_{k\lambda}\lambda_t + \Psi_{0k}x_t + \Psi_{1k}E_t x_{t+1}$$

- (Apparently, somewhat different solutions arise if you use other equations to get future k's but these are not really different)

BK provide

- A tight description of how to solve for RE in a rich multivariate setting, as discussed above.
- A counting rule for sensible models:

  number of predetermined=number of stable

  $$n(k)=n(s)$$

- Some additional discussion of “multiple equilibria” and “nonexistence”

Left Open

- What to do about unit roots? Generally (or at least in optimization settings) these are treated as stable, with idea that it is a notion of stability relative to a discount factor (β) that is relevant. Sometimes subtle.
- How cast models into form (1) or what to do about models which cannot be placed into form (1)?

C. Singular Linear RE Models

- Topic of active computational research in last 10 years.
- Now general form studied and used in computational work, although different computational approaches are taken.
- Theory provided in King-Watson, “Solution of Singular Linear Difference Systems under RE”, in a way which makes it a direct generalization of BK
- Not an accident that bulk of computational work undertaken by researchers studying “large” applied RE models with frictions like sticky prices and a monetary policy focus (Anderson-Moore, Sims, KW). These researchers got tired of working to put models in BK form.

Form of Singular System

- Direct generalization of BK difference system (but with A not necessarily invertible)

$$AE_t Y_{t+1} = BY_t + C_0 X_t + C_1 E_t X_{t+1}$$

Necessary condition for solvability

- There must be a “z” (scalar number) such that |Az-B| is not zero.
- Weaker than |A| not zero (required for inverse); can have |A|=0 or |B|=0 or both.
- If there is such a z, then one can construct matrices for transforming system in a useful way
  - T transforms equations
  - V transforms variables.

Transformed System

- General form

$$A^* E_t Y_{t+1}^* = B^* Y_t^* + C_0^* X_t + C_1^* E_t X_{t+1}$$

$$with\, A^* = TAV^{-1}; B^* = TBV^{-1}; C_i^* = TC_i$$

Form of Transformed System

- Key matrices are block diagonal
- Jordan matrices with stable and unstable eigenvalues just as in BK
- New matrix "N" is nilpotent (zeros on diagonal and below; ones and zeros above diagonal).

$$A^* = \begin{bmatrix} N & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \text{ and } B^* = \begin{bmatrix} I & 0 & 0 \\ 0 & J_u & 0 \\ 0 & 0 & J_s \end{bmatrix}$$

An aside

- Solutions to |Az-B|=0 are called “generalized eigenvalues of A,B”
- Since roots of the polynomial are not affected by multiplication by arbitrary nonsingular matrices, these are the same as “generalized eigenvalues of A*,B*” i.e., the roots of |A*z-B*|=0.
- With a little work, you can see that there are only as many roots as n(Ju)+n(Js), since there are zeros on the diagonal of N. Try the case at right for intuition

$$A^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu_u & 0 \\ 0 & 0 & \mu_s \end{bmatrix}$$

Implications

- There are transformed variables which evolve according to separated equation systems

$$Y_t^* = \begin{bmatrix} i_t \\ u_t \\ s_t \end{bmatrix}$$

Some parts are just a repeat of results from nonsingular case

- Unstable canonical variables

$$E_t u_{t+1} = J_u u_t + C_{0u}^* X_t + C_{1u}^* E_t X_{t+1}$$

$$\Rightarrow u_t = -\sum_{h=0}^{\infty} [J_u^{-1}]^{h+1} E_t \{C_{0u}^* X_{t+h} + C_{1u}^* E_t X_{t+h+1}\}$$

- Stable canonical variables

$$E_t s_{t+1} = J_s s_t + C_{0s}^* X_t + C_{1s}^* E_t X_{t+1}$$

New elements

- Infinite eigenvalue canonical variables

$$N E_t i_{t+1} = i_t + C^*_{0i} X_t + C^*_{1i} E_t X_{t+1}$$

$$\Rightarrow i_t = -\sum_{h=0}^{l} N^h E_t \{ C^*_{0i} X_{t+h} + C^*_{1i} E_t X_{t+h+1} \}$$

- There is a finite forward sum because raising N to the power ℓ+1 times produces a matrix of zeros (ℓ is $\leq$ the number of rows of N)

Solving for the variables we really care about

- Partition Y into predetermined and nonpredetermined variables

$$Y_t = \begin{bmatrix} \Lambda_t \\ K_t \end{bmatrix}$$

- Group i and u variables into U

$$U_t = \begin{bmatrix} i_t \\ u_t \end{bmatrix}$$

Partition the variable transformation matrix V and its inverse R

$$\begin{bmatrix} U \\ s \end{bmatrix} = \begin{bmatrix} V_{U\Lambda} & V_{UK} \\ V_{s\Lambda} & V_{sK} \end{bmatrix} \begin{bmatrix} \Lambda \\ K \end{bmatrix}$$

$$\begin{bmatrix} \Lambda \\ K \end{bmatrix} = \begin{bmatrix} R_{\Lambda U} & R_{\Lambda s} \\ R_{KU} & R_{Ks} \end{bmatrix} \begin{bmatrix} U \\ s \end{bmatrix}$$

In a similar fashion to earlier

- We solve for nonpredetemined variables given solutions for U.

$$\Lambda_t = V_{U\Lambda}^{-1}[U_t - V_{UK}K_t]$$

- This requires a square and nonsingular matrix, as in discussion of BK (same counting rule).

Solving for predetermined variables

- Use the reverse transform, the solution for the stable variables, and the solution for the U variables (unstable and infinite cvs)

$$
\begin{aligned}
K_{t+1} &= R_{KU}E_tU_{t+1} + R_{Ks}E_ts_{t+1} \\
&= R_{KU}E_tU_{t+1} + R_{Ks}[J_ss_t + C_{0s}^*X_t + C_{1s}^*E_tX_{t+1}] \\
&= R_{KU}E_tU_{t+1} + R_{Ks}[J_s(V_{s\Lambda}\Lambda_t + V_{sK}K_t)] \\
&\quad + R_{Ks}[C_{0s}^*X_t + C_{1s}^*E_tX_{t+1}] \\
&= R_{KU}E_tU_{t+1} \\
&\quad + R_{Ks}[J_s(V_{s\Lambda}V_{U\Lambda}^{-1}[U_t - V_{UK}K_t] + V_{sK}K_t)] \\
&\quad + R_{Ks}[C_{0s}^*X_t + C_{1s}^*E_tX_{t+1}]
\end{aligned}
$$

Summary

- We now have a precise solution for a richer model, which incorporates prior work as special case.
- Solvability requires |Az-B| is nonzero for some z, which is easy to check on computer.
- Unique solvability requires that a certain matrix be square and invertible: counting rule is necessary condition.

Linking the two systems

- KW (2003) show that any uniquely solvable model has a reduced dimension nonsingular system representation

$$f_t = -Kd_t - \Psi_f(\mathbf{F})E_t X_t$$

$$E_t d_{t+1} = Wd_t - \Psi_d(\mathbf{F})E_t X_t$$

- **F** is lead operator (see homework)
- d is a vector containing all predetermined variables and some nonpredetermined variables
- There are as many “f” as “l” variables.

D. A (Recalcitrant) Example

- BK discuss an example of a model which their method cannot solve, which takes the form of

$$y_t = \theta E_{t-1} y_t + \phi x_t$$

in the notation of this lecture. We note that this model has a natural solution, unless θ=1, which is that

$$y_t = \frac{\phi}{1-\theta} E_{t-1} x_t + \phi x_t$$

but that the BK approach cannot produce it.

Casting this model in First-Order Form

- Defining $w_t = E_{t-1}y_t$, we can write this model in the standard form as

$$\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} E_t \begin{bmatrix} y_{t+1} \\ w_{t+1} \end{bmatrix} = \begin{bmatrix} -1 & \theta \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_t \\ w_t \end{bmatrix} + \begin{bmatrix} \phi \\ 0 \end{bmatrix} x_t$$

  where the first equation is the model above and the second is $w_{t+1} = E_t y_{t+1}$.
- Note that |A|=0 and |B|=0

The necessary condition

- Not easy to think about general meaning of "there exists some z for which |Az-B| isn't 0"
- But in this case it is intuitive: evaluating |Az-B| as we will on the next page says the condition is satisfied unless 1= θ in which case the model becomes degenerate in that it imposes restrictions on x but not on y!

$$y_t = E_{t-1} y_t + \phi x_t$$

$$\Rightarrow E_{t-1} y_t = E_{t-1} y_t + \phi E_{t-1} x_t$$

$$\Rightarrow 0 = \phi E_{t-1} x_t$$

Generalized eigenvalues

- Looking for finite eigenvalues:
  z=0 is a root of |Az-B|

$$0 = | Az - B | = | \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} z - \begin{bmatrix} -1 & \theta \\ 0 & 0 \end{bmatrix} |$$

$$= | \begin{bmatrix} 1 & -\theta \\ z & -z \end{bmatrix} | = -z(1-\theta)$$

- Looking for infinite eigenvalues: z=0 is a root of |Bz-A|

$$0 = | Bz - A | = | \begin{bmatrix} -1 & \theta \\ 0 & 0 \end{bmatrix} z - \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} |$$

$$= | \begin{bmatrix} -z & \theta z \\ -1 & 1 \end{bmatrix} | = -z(1-\theta)$$

General meaning

- Infinite eigenvalue: there is a dynamic identity present in the model
- Zero eigenvalue: it may not be necessary to have as many state variables as predetermined variables.
- Let's see how this can be seen in this model. We can "substitute out" the equation for y in the equation for w, producing a new version of the system which looks like that on the next page

“Reduced” System in Example

- Features
  - One identity
  - One “possibly redundant” state
- Subtlety of redundancy

$$y_t = \theta w_t + \phi x_t$$

$$w_{t+1} = 0 w_t + \frac{\phi}{1-\theta} E_t x_{t+1}$$

Comments

- Muth's model 1 was easy for him, but hard from perspective of BK: it is essentially the model that we just studied.
- Nearly every linear model that we write down is singular. For example, Muth's model 2 is singular if we do not use supply=demand to drop flow output (y), but instead want to carry it along.

Computational Topics

- A. Forecasting discounted sums
  - Suppose that we want to evaluate the stock price model under the following assumption about the driving process.

$$p_t = \sum_{j=1}^{\infty} \beta^j E_t d_{t+j}$$

$$d_t = Q\varsigma_t$$

$$\varsigma_t = \rho\varsigma_{t-1} + g e_t$$

Working out the matrix sum

$$
\begin{aligned}
p_t &= \sum_{j=1}^{\infty} \beta^j E_t d_{t+j} = \sum_{j=1}^{\infty} \beta^j Q E_t \varsigma_{t+j} \\
&= \sum_{j=1}^{\infty} \beta^j Q \rho^j \varsigma_t = Q[\sum_{j=1}^{\infty} \beta^j \rho^j] \varsigma_t \\
&= Q\beta\rho[I + (\beta\rho) + (\beta\rho)^2 + ...]\varsigma_t \\
&= Q\beta\rho[I - (\beta\rho)]^{-1} \varsigma_t
\end{aligned}
$$

In words

- Solution above tells how forward-looking asset price depends on the state variables which govern demand.
- Solution above is a very convenient formula to implement on computer, in context of empirical work or quantitative modeling
- Solution strategy generalizes naturally to evaluating forward-looking components of RE models. These are, essentially, just “lots of equations with unstable eigenvalues” although frequently the sums start with 0 rather than 1, so that the details on the prior page are slightly different.

More precisely,

- We know that solutions above include "expected distributed leads" of x.
- We can evaluate these given a forcing process like that above.
- The result is then that the SOLUTION to the RE model evolves as a state space system.

Additional detail

- Evaluating model requires we solve for

$$u_t = -\sum_{h=0}^{\infty}[J_u^{-1}]^{h+1}E_t\{C_{0u}^{*}X_{t+h} + C_{1u}^{*}E_tX_{t+h+1}\} = \Phi\varsigma_t$$

- Know all of the forecasts depend just on state, so answer must have form above.
- Algebra of working this out is similar to stock price example above (distributed lead starts at 0 here rather than 1, though).

Full solution

- Takes the form

$$Y_t = \pi S_t$$

$$S_t = MS_{t-1} + Ge_t$$

- With

$$S_t = \begin{bmatrix} K_t \\ \varsigma_t \end{bmatrix}$$

- In words: states of solved model are predetermined variables plus the forcing (driving, forecasting) variables.

Modern computational approaches make it easy to handle large linear RE models

- Approaches based on numerically desirable versions (called QZ) of the TV transformations described above
  - Klein (JEDC)
  - Sims (Computational Economics, 2003)
- Approaches based on finding a subsytem or otherwise reducing the dimension of problem
  - AIM (Anderson and Moore at FRBG)
  - King/Watson (Computational Economics, 2003)
  - Sargent and coauthors

Summary

- A. Core Ideas
  - Recursive forward solution for nonpredetermined variables
  - Recursive forward solution for predetermined variables
  - Unwinding unstable roots forward
- B. Nonsingular Systems Theory
  - Unique stable solution requires: number of predetermined = number of unstable
- C. Singular Systems Theory
  - Solvability: $|Az-B|$ nonzero plus counting rule
- D. A Singular Systems Example
  - Solvability condition interpreted
- E. Computation
  - With state space driving process, solution to model occurs in state space form
  - States are predetermined variables (the past) and driving variables (the present and future x's)