

The Effect of Uncertainty on Saving Decisions

1. INTRODUCTION

How does increased uncertainty about the future affect the consumer's choice between saving and immediate consumption? This question has received considerable attention in the literature, although not often of a formal character. Thus, Alfred Marshall [8, p. 226] wrote:

“ The thriftlessness of early times was in great measure due to the want of security that those who made provision for the future would enjoy it: only those who were already wealthy were strong enough to hold what they had saved; the laborious and self-denying peasant who had heaped up a little store of wealth only to see it taken from him by a stronger hand, was a constant warning to his neighbours to enjoy their pleasure and their rest when they could ”.

In a more recent discussion of the problem Boulding [2, p. 535] writes:

“ Other things being equal, we should expect a man with a safe job to save less than a man with an uncertain job ”.

At first glance these two statements may seem inconsistent. But closer inspection reveals that Marshall and Boulding do not really discuss the same kind of uncertainty. While Boulding is concerned with uncertainty concerning *future non-capital income*, Marshall analyzes the effect of an uncertain *yield on capital investment*. The role of saving in the two cases is fundamentally different. In Boulding's case of *income risk*, the role of accumulated savings is that of a buffer providing a guarantee that future consumption will not fall below some minimum level. In other words, accumulated savings is the certain component of total resources available for future consumption. In the Marshallian case of *capital risk*, however, the more one saves, the more one stands to lose. Giving up a dollar's worth of certain present consumption does not result in a certain increase in future consumption. It is by no means obvious that these two types of uncertainty affect saving decisions in the same manner so that it may still be possible to reconcile the statements of Marshall and Boulding, both of which appear to have considerable intuitive appeal.

This paper is an attempt to analyze the effects of the two types of uncertainty in a systematic manner. The vehicle of analysis is the familiar two-period model of consumption and investment. The two-period framework is hardly very restrictive for a study of the problems that concern us here. Indeed, recent work by Fama [4] shows that under very general conditions, the empirically observable implications that can be derived from a multiperiod model of saving and consumption are indistinguishable from those implied by a two-period model.¹

¹ There are not many examples of formal treatments of saving decisions under uncertainty. The approach adopted in the present paper is similar to that of Drèze and Modigliani [3], Leland [6] and Sandmo [12], all of whom work within a two-period framework without assuming additivity of the utility function. Additive utility functions are assumed by, e.g., Phelps [10], Hakansson [5] and Mirrlees [9], who work with *n*-period or infinite-horizon models.

2. THE RISK AVERSION FUNCTION

Important contributions to the theory of choice under uncertainty have recently been made by Arrow [1] and Pratt [11], who have introduced the concept of a risk aversion function. Arrow and Pratt are concerned with preferences over probability distributions of final wealth only, expressed in terms of a concave utility function $W(Z)$, where Z is final wealth. If the risk premium is defined as the actuarial value of an uncertain prospect minus its certainty equivalent, it can be shown that this risk premium is proportional to the function $-W''(Z)/W'(Z)$, which Arrow [1] calls absolute risk aversion.¹ It seems reasonable to assume that the risk premium should be decreasing with wealth, because "it seems likely that many decision makers would feel they ought to pay less for insurance against a given risk the greater their assets" [11, p. 123]. We shall now introduce a risk aversion function for temporal risks, i.e. for prospects the outcomes of which will not be known until after the saving-consumption decision has been made, and present a temporal version of the hypothesis of decreasing risk aversion.²

The consumer is assumed to have a preference ordering over present and future consumption (C_1, C_2) which can be represented by a continuous, cardinal utility function,

$$U = U(C_1, C_2), \quad C_1, C_2 \geq 0, \quad \dots(1)$$

which is further assumed to possess continuous derivatives of first, second and third order with first-order derivatives everywhere positive.

Suppose now that a consumer is offered a gamble with vectors of present and future consumption as outcomes. Let there be two possible outcomes, ($C_1, C_2 - h$) and ($C_1, C_2 + h$) occurring with equal probability.³ The expected utility of the gamble is then

$$\frac{1}{2}U(C_1, C_2 + h) + \frac{1}{2}U(C_1, C_2 - h),$$

while the utility of the expected outcome is

$$U(C_1, C_2).$$

Let the risk premium, p , be defined by the equation

$$U(C_1, C_2 - p) = \frac{1}{2}U(C_1, C_2 + h) + \frac{1}{2}U(C_1, C_2 - h).$$

By some manipulation it can now be shown that

$$\frac{2}{h^2} p = - \frac{U_{22}(C_1, C_2)}{U_2(C_1, C_2)}. \quad \dots(2)$$

The right-hand side is the risk aversion function, which is twice the risk premium per unit of variance for infinitesimal risks. In order to have risk aversion ($p > 0$), we must require that $U_{22} < 0$.

The chief complexity introduced by the risk aversion function (2) as compared with that of Arrow and Pratt, is that it is a function of two variables, so that there is no obvious candidate for the concept of decreasing risk aversion. In [12] it has been suggested that the risk aversion function is decreasing in C_2 and increasing in C_1 ; this hypothesis was shown to lead to sensible results. We now observe that this implies knowledge of the behaviour of the risk aversion function for opposite movements in C_1 and C_2 . Diagrammatically it means that, starting from any point in the indifference map, the risk aversion function decreases with movements in the NW direction and increases with movements in the SE direction. We shall refer to this assumption as the hypothesis of *decreasing temporal risk aversion*.

¹ We shall not here be concerned with the relative risk aversion function, which is defined as $-W''(Z)Z/W'(Z)$.

² For details of the derivation of the risk aversion function the reader is referred to [12].

³ h is taken to be a small number, so that this gamble conforms to Pratt's definition of an infinitesimal risk.

It should be stressed that this hypothesis about the risk aversion function is a restriction on the utility function and should be interpreted solely in terms of properties of the preference ordering, independently of the budget constraint of the particular problem discussed here. The interpretation is as follows: Suppose a consumer "owns" a consumption vector $c = \{C_1, C_2\}$ and is offered a gamble where the two possible outcomes are $-h$ and h of future consumption. He is asked to give the odds on which he will accept the gamble. Under risk aversion we know that the odds will be "better than fair"; thus, if $\pi(h)$ is the probability of a gain of h , we know that the consumer will demand $\pi(h) > \frac{1}{2}$ in order to accept the gamble. It is reasonable to assume that $\pi(h)$ will be lower, the higher is C_2 ; this suggests itself as a natural extension of the Arrow-Pratt assumption. Likewise, it seems attractive to assume that $\pi(h)$ will be higher, the higher is C_1 ; a higher level of present consumption makes the consumer less inclined to gamble on the value of future consumption. *A fortiori* it follows that $\pi(h)$ will fall with a simultaneous increase in C_2 and decrease in C_1 , and that it will rise with a simultaneous decrease in C_2 and increase in C_1 .¹

It is easy to see that if the utility function is additive, the risk aversion function will depend on C_2 only, and the assumption that risk aversion is a decreasing function of C_2 is then sufficient to establish the results derived in the following sections.

3. INCOME RISK

In this section we shall discuss the effects of increased riskiness of future income on present consumption. The first-period budget constraint facing the consumer is

$$Y_1 = C_1 + S_1, \quad \dots(3)$$

where Y_1 is income in the first period, assumed to be known with certainty, and S_1 is saving. Future consumption is given by

$$C_2 = Y_2 + S_1(1+r), \quad \dots(4)$$

where r is the rate of interest, which is assumed to be known in this case of pure income risk, and Y_2 is future income, which is not known in period 1. The consumer's beliefs about the value of future income can be summarized in a subjective probability density function $f(Y_2)$ with mean ξ ; on the basis of this the consumer maximizes expected utility in the von Neumann-Morgenstern sense.

Combining (3) and (4) we can write

$$C_2 = Y_2 + (Y_1 - C_1)(1+r). \quad \dots(5)$$

Expected utility can then be written as

$$E[U] = \int U(C_1, Y_2 + (Y_1 - C_1)(1+r)) f(Y_2) dY_2,$$

where integration is over the range of Y_2 . Maximizing with respect to C_1 , we obtain the first-order condition

$$E[U_1 - (1+r)U_2] = 0, \quad \dots(6)$$

and the second-order condition

$$D = E[U_{11} - 2(1+r)U_{12} + (1+r)^2 U_{22}] < 0.$$

¹ An alternative interpretation of the hypothesis, which will emerge from the discussion below, is the following: For any consumption vector $\{C_1, C_2\}$ we may compute its expected present value as

$$E[C_1 + (1+r)^{-1}C_2].$$

If C_2 is increased and C_1 is decreased so as to hold the present value constant, the risk aversion function will decrease. The hypothesis of decreasing temporal risk aversion implies that this will be true for all values of r . Following this interpretation, the hypothesis might alternatively have been denoted "decreasing risk aversion along a budget line".

Leland's hypothesis [6] is that the risk aversion function decreases with movements to the NW along an indifference curve. In the neighbourhood of the optimum these measures will be approximately the same. Indeed, Leland relies on a Taylor expansion to establish his result, which is consistent with the one obtained in section 3 of the present paper.

The effect of an increase in income (Y_1) can be found by implicit differentiation in (6):

$$\partial C_1 / \partial Y_1 = -(1+r)E[U_{12} - (1+r)U_{22}] / D.$$

The sign of this derivative cannot be determined *a priori*, but in the following we shall assume that it is always positive, both under certainty and uncertainty, which implies that

$$U_{12} - (1+r)U_{22} > 0, \quad E[U_{12} - (1+r)U_{22}] > 0. \quad \dots(7)$$

We now wish to examine the effect on present consumption of an increase in the degree of risk concerning future income. This raises the problem of how to measure the "degree of risk" without adopting the rather restrictive mean-variance approach. One solution to this problem, used by Leland [6], is to expand (6) around (Y_1, ξ); one then obtains an expression containing the variance of Y_2 . Here we shall take a more direct approach.

One can examine two kinds of shift in the probability distribution of Y_2 . One is an additive shift, which is equivalent to an increase in the mean with all other moments constant. The other is a multiplicative shift, by which the distribution is "stretched" around zero.¹ A pure increase in dispersion can be defined as a stretching of the distribution around a constant mean. This is equivalent to a combination of additive and multiplicative parameter changes.

Let us write future income as

$$\gamma Y_2 + \theta, \quad \dots(8)$$

the expected value of which is

$$E[\gamma Y_2 + \theta].$$

Here γ is the multiplicative shift parameter, and θ is the additive one. Because of the non-negativity of Y_2 a multiplicative shift around zero will increase the mean. It must, therefore, be counteracted by an additive shift in the negative direction, so that the expected value is held constant. Taking the differential, the requirement is that

$$dE[\gamma Y_2 + \theta] = E[Y_2 d\gamma + d\theta] = 0,$$

which implies that

$$d\theta/d\gamma = -E[Y_2] = -\xi.^2$$

We can now substitute (8) into the first-order condition (6) and differentiate with respect to γ . We then obtain

$$(\partial C_1 / \partial \gamma)_{\partial \theta / \partial \gamma} = -\xi = -(1/D)E[(U_{12} - (1+r)U_{22})(Y_2 - \xi)]. \quad \dots(9)$$

It can be shown that decreasing temporal risk aversion is a sufficient condition for this derivative to be negative, so that *increased uncertainty about future income decreases consumption* (increases saving). The proof of this result is set out in the appendix. Our analysis thus confirms Boulding's conjecture that increased uncertainty about future income leads to more saving. In the final section we present a few brief comments on the relationship of this result to empirical studies of saving behaviour.

¹ Since Y_2 is most naturally interpreted as a non-negative number, the distribution will really be stretched only on the right side of zero.

² A numerical illustration is perhaps in order at this point. Let there be

Y_2^1	Y_2^2	Mean	Variance
10	20	15	25
12	24	18	36
9	21	15	36

two possible values of future income, Y_2^1 and Y_2^2 , occurring with equal probability. Initially we have $Y_2^1 = 10$ and $Y_2^2 = 20$ with mean and variance as given in the first line of the table. Multiplying Y_2^1 by 1.2 increases the variance, but it also increases the mean, as shown in the second line. We can now restore the mean to its original value by subtracting 3 from each Y_2^i in the second line. By a combination of a positive multiplicative shift and a negative additive shift, we have obtained an increase in the variance with the mean constant.

4. CAPITAL RISK

We now turn to a stylized version of Marshall's "laborious and self-denying peasant". In the first period he can allocate his resources (Y_1) between present consumption (C_1) and capital investment (K):

$$Y_1 = C_1 + K.$$

In general, capital investment is transformed into resources available for future consumption by means of a transformation function $F(K, x)$, where x is a stochastic parameter. We shall assume that the transformation function is of the following simple form:

$$C_2 = K(1+x), \quad 1+x \geq 0,$$

with x as the random rate of return on capital.

Combining these two equations, we have that

$$C_2 = (Y_1 - C_1)(1+x).$$

Expected utility is then

$$E[U] = \int U(C_1, (Y_1 - C_1)(1+x))g(x)dx,$$

where $g(x)$ is the subjective density function of x and integration is over the range of x .

Necessary and sufficient conditions for a maximum of $E[U]$ are

$$E[U_1 - (1+x)U_2] = 0, \quad \dots(10)$$

$$H = E[U_{11} - 2(1+x)U_{12} + (1+x)^2 U_{22}] < 0. \quad \dots(11)$$

To examine the effect of a pure increase in risk, we proceed exactly as in the preceding section. Writing the yield on capital as $\gamma x + \theta$, we find that for a multiplicative shift around zero to keep the mean constant, we must have

$$dE[\gamma x + \theta] = 0,$$

i.e.

$$d\theta/d\gamma = -\mu,$$

where $\mu = E[x]$.

Differentiating (10) with respect to γ and evaluating the derivative at $(\gamma = 1, \theta = 0)$ we obtain

$$[\partial C_1 / \partial \gamma]_{\partial \theta / \partial \gamma = -\mu} = -(1/H)KE[(U_{12} - (1+x)U_{22})(x - \mu)] + (1/H)E[U_2(x - \mu)]. \quad \dots(12)$$

It is natural to call the first term the income effect because of its similarity to (9); the second term we shall refer to as the substitution effect.

It can be shown that the *existence of risk aversion is a necessary and sufficient condition for the substitution effect to be positive. The additional assumption of decreasing temporal risk aversion is sufficient for the income effect to be negative.* The total effect cannot be determined without additional assumptions. For proofs the reader is again referred to the appendix.

The crucial distinction between income risk and capital risk is now clear. Under income risk, increased saving raises the expected value of future consumption, while leaving higher moments unaffected. Hence the consumer reacts to increased riskiness by raising his level of saving, so that the increased variance (and higher moments) of future consumption is compensated by a higher expected value. But under capital risk this is only part of the story, since increases in saving will now increase both the mean and the variance of future consumption; hence the conflicting tendencies of the income and substitution effects.

A more intuitive interpretation of the result is as follows: An increase in the degree of risk makes the consumer less inclined to expose his resources to the possibility of loss; hence the positive substitution effect on consumption. On the other hand, higher riskiness makes it necessary to save more in order to protect oneself against very low levels of future

consumption. This explains the negative income effect on consumption. It may be noted that the quotation from Marshall in section 1 can be interpreted as a statement on the substitution effect only, neglecting the income effect.

After the present paper was completed I read the recent article by Levhari and Srinivasan [7]. In section III of their paper they discuss the effect of increased capital risk in a model where the objective is to maximize

$$E \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right]$$

with

$$u(c) = \frac{1}{1-\alpha} c^{1-\alpha},$$

so that $-u''(c)c/u'(c) = \alpha$, i.e. the elasticity of marginal utility is a constant. For the case of normal distribution of the rate of return they then prove that increased variance leads to increased consumption for $\alpha < 1$ and to decreased consumption for $\alpha > 1$. The economic interpretation of this condition is far from obvious.

The two-period equivalent of the Levhari-Srinivasan utility function is

$$U(C_1, C_2) = u(C_1) + \beta u(C_2).$$

For the constant elasticity case the RHS of our equation (12) reduces to

$$(1/H)E[\beta u'(C_2)(x-u)](1-\alpha).$$

By the method used in the appendix it can be shown that $u''(C_2) < 0$ implies

$$E[u'(C_2)(x-u)] < 0.$$

Since $H < 0$, it then follows that increased capital risk will increase, leave constant, or decrease consumption according as $\alpha \leq 1$. This means that $\alpha < 1$ corresponds to the case where the substitution effect dominates, while $\alpha > 1$ means that the income effect dominates. This result seems to offer an economic interpretation of the importance of the Levhari-Srinivasan condition. It is also consistent with the results obtained in [9] and [10].

In analyzing the effect of capital risk it is sometimes desirable to allow for asset choice, so that the consumer may react to change in riskiness by a reallocation among assets. A model along these lines has been studied in [12]. However, the present analysis is not necessarily a step backward. The one-asset model may be of considerable relevance for many real-world problems, since many types of increase in riskiness will apply to the yield on all assets, so that the possibility of hedging against risk by portfolio rearrangements is limited. Moreover, for society as a whole, real capital constitutes the only form that saving can take (at least in a closed economy); the present model may, therefore, be seen as a simplified analysis of optimal growth under uncertainty.

5. A CONCLUDING COMMENT

It has often been observed that there is a significant difference in saving behaviour between wage and salary earners on the one hand, and self-employed persons on the other. Moreover, it is generally accepted that the latter group, farmers and businessmen, have more variable incomes than the former. On the reasonable assumption that *ex post* variability goes together with *ex ante* uncertainty, theoretical considerations should lead us to expect the self-employed group to save more, and this conclusion appears in fact to be supported by empirical research.

However, some care should be taken in identifying empirical and theoretical results at this point. As far as reactions to income uncertainty is concerned, comparison should be restricted to consumers with incomes that are exogenous, i.e. independent of their own

saving behaviour. As regards self-employed persons, however, their future income may depend in an essential way on how much they save in the present, so that a comparison between these two groups would rather constitute a test of the effect of capital uncertainty. But as regards that effect, theory does not offer any clearcut hypothesis.

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APPENDIX

The differential of the risk aversion function is:

$$d \left\{ -\frac{U_{22}}{U_2} \right\} = \frac{\partial}{\partial C_1} \left\{ -\frac{U_{22}}{U_2} \right\} dC_1 + \frac{\partial}{\partial C_2} \left\{ -\frac{U_{22}}{U_2} \right\} dC_2.$$

This is negative if C_2 increases and C_1 decreases, e.g. so that (from (5))

$$dC_2 = -(1+r)dC_1, \quad 1+r \geq 0.$$

Substituting for dC_1 and dividing by dC_2 we then have

$$\frac{d}{dC_2} \left\{ -\frac{U_{22}}{U_2} \right\} = -\frac{\partial}{\partial C_1} \left\{ -\frac{U_{22}}{U_2} \right\} + (1+r) \frac{\partial}{\partial C_2} \left\{ -\frac{U_{22}}{U_2} \right\} < 0.$$

We now observe that under our continuity assumption the following holds as an identity.

$$\frac{\partial}{\partial C_1} \left\{ -\frac{U_{22}}{U_2} \right\} = \frac{\partial}{\partial C_2} \left\{ -\frac{U_{12}}{U_2} \right\}.$$

The above inequality can now be written as

$$\frac{\partial}{\partial C_2} \left\{ \frac{U_{12} - (1+r)U_{22}}{U_2} \right\} < 0.$$

We now wish to prove that the hypothesis of decreasing temporal risk aversion implies that the derivative (9) is negative.

We first define

$$\bar{C}_2 = (Y_1 - C_1)(1+r) + \xi.$$

From (5) we have that

$$C_2 = \bar{C}_2 + Y_2 - \xi.$$

Because $(U_{12} - (1+r)U_{22})/U_2$ is decreasing in C_2 , we must have that

$$\frac{U_{12} - (1+r)U_{22}}{U_2} \leq \left\{ \frac{U_{12} - (1+r)U_{22}}{U_2} \right\}_{\xi} \quad \text{if } Y_2 \geq \xi. \quad \dots(A.1)$$

The right side of this inequality is evaluated at $C_2 = \bar{C}_2$ and is not a random variable.

Obviously

$$U_2(Y_2 - \xi) \geq 0 \quad \text{if } Y_2 \geq \xi. \quad \dots(A.2)$$

We now multiply both sides of (A.1) by $U_2(Y_2 - \xi)$. We then obtain

$$(U_{12} - (1+r)U_{22})(Y_2 - \xi) \leq \left\{ \frac{U_{12} - (1+r)U_{22}}{U_2} \right\}_{\xi} \cdot U_2(Y_2 - \xi) \quad \text{if } Y_2 \geq \xi.$$

Taking expected values on both sides we have that

$$E[(U_{12} - (1+r)U_{22})(Y_2 - \xi)] \leq \left\{ \frac{U_{12} - (1+r)U_{22}}{U_2} \right\}_{\xi} E[U_2(Y_2 - \xi)]. \quad \dots(A.3)$$

We now observe that if $Y_2 \leq \xi$, inequalities (A.1) and (A.2) will both be reversed, so that (A.3) holds for all Y_2 .

To prove that the left side of (A.3) is negative, it is sufficient to show that the right side is negative. From (7) the expression in braces is positive, so that we have to show that $E(U_2(Y_2 - \xi)) \leq 0$. Since $U_{22} < 0$, we must have

$$U_2 \leq (U_2)_\xi \quad \text{if} \quad Y_2 \geq \xi. \quad \dots(A.4)$$

Trivially,

$$Y_2 - \xi \geq 0 \quad \text{if} \quad Y_2 \geq \xi. \quad \dots(A.5)$$

Multiplying in (A.4) by $(Y_2 - \xi)$ we can write

$$U_2(Y_2 - \xi) \leq (U_2)_\xi(Y_2 - \xi).$$

This holds for all Y_2 , since inequalities (A.4) and (A.5) are both reversed if $Y_2 \leq \xi$. Taking expectations, we obtain

$$E[U_2(Y_2 - \xi)] \leq (U_2)_\xi E[Y_2 - \xi] = 0,$$

which implies

$$E[(U_{12} - (1+r)U_{22})(Y_2 - \xi)] \leq 0.$$

Therefore, since $D < 0$, it follows that the derivative (9) is negative. Q.E.D.¹

The proofs for the signs of the income and substitution effects in (12) follow immediately. In particular, to prove that the income effect is negative one has to write down a set of inequalities similar to (A.1)-(A.3). To prove that the substitution effect is positive, inequalities similar to (A.4)-(A.5) can be used.

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¹ It may be of interest to record that in the case of the quadratic utility function

$$k_1 C_1 + k_2 C_2 + k_{12} C_1 C_2 + k_{11} C_1^2 + k_{22} C_2^2$$

present consumption is independent of the variance of future income. This function can easily be shown to display increasing temporal risk aversion.