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SOME RESULTS FOR SKIP-FREE RANDOM WALK

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A random walk that is skip-free to the left can only move down one level at a time but can skip up several levels. Such random walk features prominently in many places in applied probability including queuing theory and the theory of branching processes. This article exploits the special structure in this class of random walk to obtain a number of simplified derivations for results that are much more difficult in general cases. Although some of the results in this article have appeared elsewhere, our proof approach is different.

1. INTRODUCTION

A well-known but peculiar result (see [5,11]) is that a simple random walk started above the origin with probability $p > 1/2$ of going up and probability $q = 1 - p$ of going down behaves, conditional on eventually returning to the origin, like an unconditioned random walk for which q is now the probability of going up and p the

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probability of going down. It is perhaps less well known (see [4] for a discussion in the setting of a branching process) that a general random walk on the integers with step-size distribution X having $E[X] > 0, P(X \geq -1) = 1$, and $P(X < 0) > 0$ also behaves, conditional on eventually returning to the origin or below, like another unconditioned random walk. Because at any transition this random walk cannot decrease by more than 1, it is said to be skip-free to the left. To see why this latter result is true, notice that the skip-free property implies that the probability of eventually returning to the origin starting from $m > 0$ equals c^m, where c is the probability of eventually returning to the origin starting from 1. From this it follows that the conditional probability of taking a step of size i starting from state $m > 0$ given the random walk eventually returns to the origin equals $P(X = i)c^{m+i}/c^m = c^i P(X = i)$ independent of m and thus implying that the conditioned process behaves like an unconditioned random walk. Another elementary argument can show that this behavior does not occur for general random walk that are not not skip-free to the left.

Skip-free to the left random walk is an example of what is called a Markov chain of $M/G/1$ type (see [9, p. 267]), a chain that can only move down one level at a time but can skip up several levels. The name comes from the fact that an example is the $M/G/1$ queue length at the times when customers depart. The transition probability matrixes for such Markov chains have a special structure that can be exploited in matrix analytical methods for efficient computation (see [9,10]). Such chains have been studied in many other places—for example [2] and [6]. Other examples of such a Markov chain include the size of the current population in a Galton, Watson branching process in which the offspring for an individual happen simultaneously but for siblings occur sequentially, and various casino games such as roulette, keno, and slots, where each Independent and identically distributed (i.i.d) bet puts only one unit at risk.

The organization of this article is as follows. In Section 2 we consider skip-free to the left random walk with a downward drift and compute moments of some first and last hitting times (conditional on being finite) and relate them to hitting probabilities. We also give a derivation of the distribution of the first positive value of the random walk, given that there is one, that simplifies the generating-function approach given in Feller [3] and Asmussen [1]. Finally, we compute the mean and variance of the maximum of the random walk using elementary methods. In Section 3 we employ a duality between the conditioned and unconditioned random walk to compute similar quantities for skip-free to the left random walk with an upward drift. Finally, in Section 4 we give some results for the number of visits to a given state.

2. RANDOM WALK WITH A DOWNWARD DRIFT

Let X_i be (i.i.d.) integer-valued with $P(X_i \geq -1) = 1$ and define the random walk process

$$S_n = S_0 + \sum_{i=1}^{n} X_i, \qquad n \geq 0,$$

where, unless otherwise noted, we will take $S_0 = 0$. Because at any transition this random walk cannot decrease by more than 1, it is said to be skip-free (to the left). Let X be a generic random variable having the distribution of X_i and let $q = P(X = -1)$, $v = -E[X]$, and $\sigma^2 = \mathrm{Var}(X)$. We assume throughout that $q > 0$.

Assume throughout this section that $E[X_i] < 0$. Let $T_r = \inf(n : S_n = -r)$.

Although the results of Lemma 1 and Proposition 1 are known (see, for instance, [1]), we include them for completeness.

LEMMA 1: *For $r > 0$,*

$$E[T_r] = r/v$$

and

$$\mathrm{Var}(T_r) = \frac{r\sigma^2}{v^3}.$$

PROOF: The skip-free property yields that

$$E[T_1|X_1] = 1 + (X_1 + 1)E[T_1],$$

which yields that $E[T_1] = 1/v$, with the more general result again following from the skip-free property. Similarly, the skip-free property yields that

$$\mathrm{Var}(T_1|X_1) = (X_1 + 1)\,\mathrm{Var}(T_1)$$

and the conditional variance formula gives

$$\mathrm{Var}(T_1) = (E[X_1] + 1)\,\mathrm{Var}(T_1) + (E[T_1])^2\,\mathrm{Var}(X_1)$$

or

$$\mathrm{Var}(T_1) = \frac{(E[T_1])^2\sigma^2}{v},$$

which, using the previously defined expression for $E[T_1]$, proves the result when $r = 1$. The general result now follows from the skip-free property. ■

Now, let α be the probability that the random walk never returns to state 0; that is,

$$\alpha = P(S_n \neq 0 \text{ for all } n > 0).$$

PROPOSITION 1:

$$\alpha = v.$$

PROOF: Using that $X_1, \ldots, X_n$ has the same joint distribution as does $X_n, \ldots, X_1$ gives that

$$\begin{aligned} P(S_n < S_j, j = 0, \ldots, n-1) &= P(X_n < 0, X_n + X_{n-1} < 0, X_n + \cdots + X_1 < 0) \\ &= P(S_1 < 0, S_2 < 0, \ldots, S_n < 0). \end{aligned}$$

Hence,

$$\alpha = \lim_n P(S_n < S_j, j = 0, \dots, n-1).$$

Noting that every new low is a renewal, it now follows from the elementary renewal theorem ([13], p. 171) that

$$\alpha = 1/E[T_1] = v. \qquad \blacksquare$$

COROLLARY 1:

$$P(S_n \leq 0 \textit{ for all } n) = v/q.$$

PROOF:

$$\begin{aligned} v = \alpha &= qP(\text{never returns to } 0 | S_1 = -1) \\ &= qP(\text{never visits } 0 | S_0 = -1) \\ &= qP(\text{never positive} | S_0 = 0). \end{aligned}$$

For $r \geq 0$, let

$$L_r = \max(k \geq 0 : S_k = -r)$$

be the last time that the random walk is in state $-r$. $\blacksquare$

PROPOSITION 2: *For* $r \geq 0$,

$$E[L_r] = \frac{r}{v} + \frac{\sigma^2}{v^2}.$$

To prove the preceding we will need the following result, known as the hitting time theorem (HTT). (For a proof of the HTT, which does not require that $E[X_i] < 0$, see [7].)

HITTING TIME THEOREM:

$$P(T_r = n) = \frac{r}{n} P(S_n = -r).$$

PROOF OF PROPOSITION 2:

$$P(L_1 = n) = P(S_n = -1)\alpha.$$

Hence,

$$\begin{aligned} E[L_1] &= v \sum_n nP(S_n = -1) \qquad \text{(by Proposition 1)} \\ &= v \sum_n n^2 P(T_1 = n) \qquad \text{(by HTT)} \\ &= vE[T_1^2] \\ &= \frac{\sigma^2}{v^2} + \frac{1}{v} \qquad \text{(by Lemma 1).} \end{aligned}$$

Because

$$E[L_r] = E[T_r] + E[L_0]$$

it follows, by taking $r = 1$, that

$$E[L_0] = \frac{\sigma^2}{v^2}$$

and, thus,

$$E[L_r] = \frac{r}{v} + \frac{\sigma^2}{v^2}. \qquad \blacksquare$$

PROPOSITION 3: *Let*

$$F_0 = \min(k \geq 1 : S_k = 0)$$

be the time of the first return to 0 (*and let it be infinite if there is no return*). *Then*

$$E[F_0|F_0 < \infty] = \frac{\sigma^2}{v(1-v)}.$$

PROOF: Conditioning on whether the random walk ever returns to state 0 yields

$$E[L_0] = (1-\alpha)(E[F_0|F_0 < \infty] + E[L_0]).$$

Using that $\alpha = v$, the result follows from Proposition 2. $\blacksquare$

PROPOSITION 4: *If F_1 is the first time that state* 1 *is visited*, *Then*

$$E[F_1|F_1 < \infty] = \frac{E[X^2] - q}{v(q-v)}.$$

PROOF: Using that $P(F_0 < \infty) = 1 - v$, note first that, with $p_k = P(X = k)$,

$$P(X_1 = k|F_0 < \infty) = \frac{p_k}{1-v}, \qquad k \geq 0,$$

$$P(X_1 = -1|F_0 < \infty) = \frac{q-v}{1-v}.$$

Hence, letting $p_k = P(X = k)$ and using Proposition 3 gives

$$\begin{aligned}
\frac{\sigma^2}{v(1-v)} &= E[F_0|F_0 < \infty] \\
&= \sum_{k \geq -1} E[F_0|F_0 < \infty, X_1 = k] P(X_1 = k|F_0 < \infty) \\
&= 1 + E[F_1|F_1 < \infty] \frac{q-v}{1-v} + \sum_{k \geq 1} E[T_k] \frac{p_k}{1-v} \\
&= 1 + E[F_1|F_1 < \infty] \frac{q-v}{1-v} + \frac{1}{v(1-v)} \sum_{k \geq 1} k p_k
\end{aligned}$$

$$= 1 + E[F_1|F_1 < \infty]\frac{q-v}{1-v} + \frac{q-v}{v(1-v)},$$

which proves the result. ■

Let

$$T = \min(n : S_n > 0)$$

and let it be infinite if S_n is never positive. Additionally, let $\lambda_k = P(T < \infty, S_T = k)$ be the probability that k is the first positive value of the random walk.

PROPOSITION 5:

$$\lambda_k = P(X \geq k)/q.$$

PROOF: Let $p_j = P(X = j)$. Conditioning on X_1 yields

$$\lambda_k = p_0\lambda_k + p_k + q(\lambda_1\lambda_k + \lambda_{k+1}), \tag{1}$$

where $\lambda_1\lambda_k$ is the probability, starting at -1, that the first state larger than -1 is 0 and the first one larger than 0 is k, and λ_{k+1} is the probability that the first value larger than -1 is k. Hence,

$$\sum_{k\geq 1}\lambda_k = 1 - q - p_0 + (p_0 + q\lambda_1)\sum_{k\geq 1}\lambda_k + q\sum_{k\geq 1}\lambda_{k+1}.$$

However, Corollary 1 gives that $\sum_{k\geq 1}\lambda_k = 1 - v/q$. Consequently,

$$1 - v/q = 1 - q - p_0 + (p_0 + q\lambda_1)(1 - v/q) + q(1 - v/q - \lambda_1).$$

Solving for λ_1 yields

$$\lambda_1 = \frac{1 - q - p_0}{q} = P(X \geq 1)/q.$$

Substituting this back in (1) gives

$$\lambda_k = p_0\lambda_k + p_k + (1 - q - p_0)\lambda_k + q\lambda_{k+1} = p_k + (1-q)\lambda_k + q\lambda_{k+1}.$$

Hence,

$$\lambda_k - \lambda_{k+1} = p_k/q,$$

giving that

$$\lambda_2 = \lambda_1 - p_1/q = P(X \geq 2)/q$$

and so on. ■

Remark: Proposition 5 was originally proven by Feller [3, p. 425] by using generating functions to solve (1). Another proof is given in Asmussen [1].

PROPOSITION 6: *With* $M = \max_n S_n$,

$$E[M] = \frac{E[X^2] - v}{2v}$$

and

$$\mathrm{Var}(M) = \left(\frac{E[X(X+1)]}{2v}\right)^2 + \frac{E[X(X+1)(2X+1)]}{6v}.$$

PROOF: Let N be the number of new highs that are attained. Since $1 - v/q$ is the probability, starting with a high, of ever getting another new high, it follows that

$$P(N = j) = (1 - v/q)^j \, v/q, \qquad j \geq 0$$

(i.e., $N + 1$ is geometric with parameter v/q). Letting Y_i be the amount added by the ith new high, it follows that

$$M = \sum_{i=1}^{N} Y_i,$$

where Y_i, $i \geq 1$, are i.i.d. distributed as a jump from a previous high conditional on the event that there is a new high; that is,

$$P(Y_i = k) = \frac{\lambda_k}{1 - v/q}, \qquad k \geq 1.$$

Moreover, N is independent of the sequence Y_i.

Because

$$\frac{P(X \geq k | X > 0)}{E[X|X > 0]} = \frac{P(X \geq k)}{E[X] + q} = P(Y_i = k),$$

it follows that Y_i is distributed according to the stationary renewal distribution of X given that $X > 0$. Thus,

$$\begin{aligned} E[Y] &= \frac{E[X(X+1)|X > 0]}{2E[X|X > 0]} \\ &= \frac{E[X(X+1)I(X > 0)]}{2E[XI(X > 0)]} \\ &= \frac{E[X(X+1)]}{2(q - v)}. \end{aligned}$$

Similarly,

$$E[Y^2] = \frac{E[X(X+1)(2X+1)|X > 0]}{6E[X|X > 0]} = \frac{E[X(X+1)(2X+1)]}{6(q - v)}.$$

Using the formulas for the mean and variance of a compound random variable (see [12]), the preceding yields that

$$E[M] = E[N]E[Y] = \frac{q-v}{v}E[Y] = \frac{E[X(X+1)]}{2v}$$

and

$$\mathrm{Var}(M) = \left(\frac{E[X(X+1)]}{2v}\right)^2 + \frac{E[X(X+1)(2X+1)]}{6v}. \quad \blacksquare$$

Remark: The distribution of M is derived in [1].

Roulette Example: Suppose

$$P(X = \frac{36}{j} - 1) = j/38, \qquad q = P(X = -1) = 1 - j/38;$$

that is, at each stage, you place a 1-unit bet that the roulette ball will land on one of a set of j numbers. If you win your bet, you win $36/j - 1$; otherwise you lose 1. Thus, $v = 2/38$. Note that

$$\lambda_k = \frac{j}{38-j}, \qquad k \leq j,$$

which shows that, conditional on a high being hit, its value is equally likely to be any of $1, \ldots, r$, where $r = (36-j)/j$. Hence,

$$E[Y] = \frac{r+1}{2}, \qquad \mathrm{Var}(Y) = \frac{(r+1)(2r+1)}{6} - \frac{(r+1)^2}{4}.$$

Because $N+1$ is geometric with parameter $v/q = 2/(38-j)$, we have

$$E[N] = \frac{36-j}{2}, \quad \mathrm{Var}(N) = \frac{(36-j)(38-j)}{4}.$$

Thus,

$$E[M] = \frac{(36-j)(r+1)}{4} = 9\left(\frac{36-j}{j}\right).$$

and, after some computations,

$$\mathrm{Var}(M) = \frac{12(36-j)(261-7j)}{j^2}.$$

With P_j and E_j being probabilities as a function of j, we have

$$P_1(S_n \leq 0, \text{ all } n) = \frac{2}{37}, \qquad E_1[L_0] = 11{,}988, \qquad E_1[M] = 315;$$

$$P_{12}(S_n \leq 0, \text{ all } n) = \frac{1}{13}, \qquad E_{12}[L_0] = 702, \qquad E_{12}[M] = 18;$$

$$P_{18}(S_n \leq 0, \text{ all } n) = \frac{1}{10}, \qquad E_{18}[L_0] = 360, \qquad E_{12}[M] = 9.$$

3. RANDOM WALK WITH AN UPWARD DRIFT

Assume throughout this section that $E[X] > 0$. Let $p_j = P(X = j)$, with $q = p_{-1}$. As earlier, let $T_r = \inf(n : S_n = -r)$. Additionally, for $s \geq 0$, let

$$g(s) = E[s^X] = \sum_{j \geq -1} s^j p_j.$$

LEMMA 2: *$c \equiv P(T_1 < \infty)$ is the unique value $s \in (0, 1)$ with $g(s) = 1$. Moreover,* $g'(c) < 0$.

PROOF: The skip-free property yields

$$\begin{aligned} c &= \sum_{r \geq -1} P(T_1 < \infty | X_1 = j) p_j \\ &= q + \sum_{r \geq 0} c^{j+1} p_j. \end{aligned}$$

Dividing by c shows that $g(c) = 1$. Because $E[X] > 0$ and $P(X = -1) > 0$, it follows that $0 < c < 1$. Additionally,

$$g''(s) = \frac{2q}{s^3} + \sum_{j \geq 2} j(j-1) s^{j-2} p_j > 0.$$

Hence, $g'(s)$ is strictly increasing for $s \geq 0$. Moreover,

$$g'(s) = \frac{-q}{s^2} + \sum_{j \geq 1} j s^{j-1} p_j,$$

showing that $g'(0+) = -\infty$, and $\lim_{s \to \infty} g'(s) > 0$. Consequently, there is a unique value s^* with $g'(s^*) = 0$, and $g(s^*)$ is the global minimum of $g(s)$. Since $g(c) = g(1) = 1$, it follows that $g(s^*) < 1$ and that $c < s^* < 1$. Hence, $g'(c) < g'(s^*) = 0$. ■

Remark: That c satisfies $g(c) = 0$ was also proven in [1].

Now, define a dual random walk

$$S'_n = \sum_{i=1}^{n} X'_i,$$

where the X'_i are i.i.d. according to X', whose mass function is

$$P(X' = j) = c^j P(X = j), \qquad j \geq -1.$$

We call $S'_n, n \geq 0$, the dual random walk of $S_n, n \geq 0$. Let $q' = q/c = P(X' = -1)$.

LEMMA 3:

$$E[X'] < 0.$$

PROOF: This follows from Lemma 2 by noting that

$$0 > g'(c) = E[Xc^{X-1}] = c^{-1}E[X']. \qquad \blacksquare$$

Results corresponding to those of the previous section, which assumed a negative mean, can be obtained by making use of the duality relationship. Let

$$\nu' = -E[X']$$

and

$$\sigma'^2 = \text{Var}(X');$$

additionally, let $T_r' = \inf(n : S_n' = -r)$.

PROPOSITION 7:

$$E[T_r | T_r < \infty] = \frac{r}{\nu'},$$

$$\text{Var}(T_r | T_r < \infty) = \frac{r\sigma'^2}{\nu'^3}.$$

PROOF: For any set of outcomes $X_i = x_i, i = 1, \ldots, n$, that result in $T_r = n$, we have that

$$P(X_i = x_i, i = 1, \ldots, n) = c^r P(X_i' = x_i, i = 1, \ldots, n),$$

which implies that

$$P(T_r = n) = c^r P(T_r' = n).$$

Summing over n gives that

$$P(T_r < \infty) = c^r. \tag{2}$$

Now,

$$\begin{aligned} E[T_r^k | T_r < \infty] &= \sum_n n^k P(T_r = n | T_r < \infty) \\ &= \sum_n n^k P(T_r = n)/P(T_r < \infty) \\ &= \sum_n n^k P(T_r' = n) \\ &= E[(T_r')^k]. \end{aligned}$$

Hence, from Lemma 1,

$$E[T_r | T_r < \infty] = \frac{r}{\nu'}$$

and

$$E[T_r^2 | T_r < \infty] = \left(\frac{r}{\nu'}\right)^2 + \frac{r\sigma'^2}{\nu'^3},$$

which yields the result. ■

Now, let F_0 and F_0' denote the times of the first return to state 0 for the random walk process and for its dual.

PROPOSITION 8:

(*a*)

$$P(S_n \neq 0, n \geq 1) = \nu';$$

(*b*)

$$E[F_0 | F_0 < \infty] = \frac{\sigma'^2}{\nu'(1 - \nu')}.$$

PROOF: For any set of outcomes $X_i = x_i, i = 1, \ldots, n$, that result in $F_0 = n$, we have that

$$P(X_i = x_i, i = 1, \ldots, n) = P(X_i' = x_i, i = 1, \ldots, n),$$

showing that

$$P(F_0 = n) = P(F_0' = n).$$

Summing the preceding over all n shows that

$$P(F_0 < \infty) = P(F_0' < \infty).$$

or

$$P(S_n \neq 0, n \geq 1) = P(S_n' \neq 0, n \geq 1),$$

which, by Proposition 1, yields (a). The preceding also implies that

$$E[F_0 | F_0 < \infty] = E[F_0' | F_0' < \infty]$$

and so (b) follows from Proposition 5. ■

Now, let L_r and L_r' denote the times of the last return to state $-r$ for the random walk process and for its dual.

PROPOSITION 9:

$$E[L_r | T_r < \infty] = \frac{r}{\nu'} + \frac{\sigma'^2}{\nu'^2}, \qquad r \geq 0.$$

PROOF: For any set of outcomes $X_i = x_i, i = 1, \ldots, n$, that result in $S_n = r$, we have that

$$P(X_i = x_i, i = 1, \ldots, n) = c^r P(X_i' = x_i, i = 1, \ldots, n).$$

Thus, using Proposition 8,

$$P(L_r = n) = P(S_n = r)v' = c^r P(S_n' = r)v' = c^r P(L_r' = n).$$

Thus,

$$E[L_r | T_r < \infty] = \sum_n nP(L_r = n)/c^r = E[L_r']$$

and the result follows from Proposition 4. ■

PROPOSITION 10: *With λ_k being the probability that k is the first positive value of the random walk*,

$$\lambda_k = \frac{1}{q} \sum_{r=k}^{\infty} c^{r-k+1} P(X = r).$$

PROOF: With λ_k' being the corresponding probability for the dual process, duality gives that

$$\lambda_k = c^{-k} \lambda_k'.$$

Hence, from Proposition 5,

$$\begin{aligned} \lambda_k &= \frac{c^{-k} P(X' \geq k)}{q'} \\ &= \frac{c^{-k+1} P(X' \geq k)}{q} \\ &= \frac{1}{q} \sum_{r=k}^{\infty} c^{r-k+1} P(X = r). \end{aligned}$$

■

COROLLARY 2: *If $T = \min(n : S_n > 0)$ is the first time that the random walk becomes positive, then*

$$E[T] = \frac{c}{q(1-c)}.$$

PROOF: With Λ equal to the first positive value of the random walk,

$$
\begin{aligned}
E[\Lambda] &= \sum_{k=1}^{\infty} k\lambda_k \\
&= \sum_{k=1}^{\infty} k \frac{1}{q} \sum_{r=k}^{\infty} c^{r-k+1} P(X = r) \\
&= \frac{1}{q} \sum_{r=1}^{\infty} c^r P(X = r) \sum_{k=1}^{r} k c^{-k+1} \\
&= \frac{1}{q} \sum_{r=1}^{\infty} c^r P(X = r) \frac{1 + r/c^{r+1} - (r+1)/c^r}{(1 - 1/c)^2} \\
&= \frac{1}{q} \left(\frac{c}{1-c}\right)^2 \sum_{r=1}^{\infty} P(X = r) \left(\frac{c^r + r}{c - (r+1)}\right) \\
&= \frac{1}{q} \left(\frac{c}{1-c}\right)^2 \left[\sum_{r=1}^{\infty} P(X' = r) + \left(\frac{1-c}{c}\right)(E[X] + q) - \sum_{r=1}^{\infty} P(X = r)\right] \\
&= \frac{1}{q} \left(\frac{c}{1-c}\right)^2 \left[1 - P(X = 0) - \frac{q}{c} + \left(\frac{1-c}{c}\right)\right. \\
&\qquad \left. \times\ (E[X] + q) - 1 + P(X = 0) + q\right] \\
&= \frac{cE[X]}{q(1-c)}.
\end{aligned}
$$

The result now follows upon applying Wald's equation to the identity

$$\Lambda = \sum_{i=1}^{T} X_i.$$

■

Examples:

- Suppose $X = Y - 1$, where Y is Poisson with mean $\lambda > 1$. Then c would be the smallest positive solution of

$$e^{\lambda(c-1)} = c.$$

For instance, suppose $\lambda = 2$. Then $c \approx 0.20319$. Since $q = e^{-2}$,

$$E[T] = e^2 \frac{c}{1-c} \approx 1.8842.$$

- Suppose $X = Y - 1$, where Y is geometric with parameter $p^* < 1/2$. Then $c = p^*/(1 - p^*)$ and, because $q = p^*$,

$$E[T] = \frac{1}{1 - 2p^*}.$$

- If $P(X = -1) = .7 = 1 - P(X = 3)$, then $c \approx 0.8795$ and

$$E[T] = \frac{10}{7} \frac{c}{1 - c} \approx 10.4268.$$

PROPOSITION 11:

$$(a) \qquad P(S_n \geq 0, \textit{ for all } n \geq 1) = 1 - c;$$

$$(b) \qquad P(S_n \geq 1, \textit{ for all } n \geq 1) = q\left(\frac{1 - c}{c}\right).$$

PROOF: Duality shows that

$$P(S_n = -1 \text{ for some } n) = cP(S_n' = -1 \text{ for some } n) = c$$

which proves (a). To prove (b), condition on X_1 and use (a):

$$\begin{aligned} &P(S_n \geq 1, \text{ for all } n \geq 1) \\ &\quad = \sum_{k>0} P(X = k)(1 - c^k) \\ &\quad = \sum_{k>0} P(X = k) - \sum_{k>0} P(X' = k) \\ &\quad = 1 - q - P(X = 0) - (1 - q' - P(X' = 0)) \\ &\quad = q' - q = q\left(\frac{1}{c} - 1\right). \end{aligned}$$

■

COROLLARY 3: *If* $W = \min(k > 0 : S_k \geq 0)$, *then*

$$E[W] = \frac{1}{1 - c} = \frac{1}{P(S_n \geq 0 \textit{ for all } n)}.$$

PROOF: With T equal to the time to reach a positive value, conditioning on X_1 yields

$$E[W] = 1 + qE[T]$$

and the result follows from Corollary 2 and Proposition 11. ■

4. ADDITIONAL RESULTS

In this section we again suppose that $P(X \geq -1) = 1$ and that $P(X = -1) > 0$, but we put no restrictions on $E[X]$ (which can be zero, positive, negative, or plus infinity).

Let $N_i(n)$ be the number of time periods spent in state i during times $0, \ldots, n$. (So, $N_0(n)$ is 1 plus the number of returns to state 0 in the first n transitions.) Our main result in this section is that, conditional on $S_n = -j$, the random variables $N_i(n), 0 \leq i \leq j$, all have the same distribution. Indeed, if we let $T_j(m)$ denote the time of the mth visit to state $-j$ and let it be ∞ if the process spends less than m periods in state $-j$, then we have the following theorem.

THEOREM 1: *For $0 \leq i \leq j$ and $n \geq j$,*

$$P(N_i(n) = m | S_n = -j) = \frac{P(T_j(m) = n)}{P(S_n = -j)}.$$

To prove Theorem 1, we need some lemmas.

LEMMA 4: *For $j > 0$,*

$$P(S_k < 0, k = 1, \ldots, n-1,\ S_n = -j) = P(T_j = n).$$

PROOF:

$$\begin{aligned} &P(S_k < 0,\ k = 1, \ldots, n-1,\ S_n = -j) \\ &= P(S_n - S_k > -j,\ k = 1, \ldots, n-1,\ S_n = -j) \\ &= P(X_{k+1} + \cdots + X_n > -j,\ k = 1, \ldots, n-1,\ S_n = -j) \\ &= P(X_1 + \cdots + X_{n-k} > -j,\ k = 1, \ldots, n-1,\ S_n = -j) \\ &= P(T_j = n), \end{aligned}$$

where the third equality used that $X_1, \ldots, X_n$ and $X_n, \ldots, X_1$ have the same joint distribution. ■

LEMMA 5: *With $p(m, r) = P(S_r = 0, N_0(r-1) = m-1)$, we have*

$$P(T_j(m) = n) = \sum_r P(T_j = n - r) p(m, r).$$

PROOF: This follows immediately upon conditioning on the time of the first visit to state j. ■

PROOF OF THEOREM 1: We need to verify that

$$P(N_i(n) = m,\ S_n = -j) = P(T_j(m) = n). \tag{3}$$

Case 1: $1 \leq i < j$: Conditioning on T_i and $T_i(m)$ gives that

$$\begin{aligned}
P(N_i(n) = m,\ S_n = -j) \\
&= \sum_{t,r} P(N_i(n) = m,\ S_n = -j | T_i = t, T_i(m) = t + r) \\
&\quad \times P(T_i = t, T_i(m) = t + r) \\
&= \sum_{t,r} P(N_i(n) = m,\ S_n = -j | T_i = t, T_i(m) = t + r) P(T_i = t) p(m, r) \\
&= \sum_{t,r} P(S_k < 0, k = 1, \ldots, n - r - t,\ S_{n-r-t} = i - j) P(T_i = t) p(m, r) \\
&= \sum_{t,r} P(T_{j-i} = n - r - t) P(T_i = t) p(m, r) \qquad \text{(by Lemma 4)} \\
&= \sum_{r} p(m, r) \sum_{t} P(T_{j-i} = n - r - t) P(T_i = t) \\
&= \sum_{r} p(m, r) P(T_j = n - r) \\
&= P(T_j(m) = n),
\end{aligned}$$

where the final equality used Lemma 5.

Case 2: $i = 0 < j$: Conditioning on $T_0(m)$, the time of the mth visit to 0 and then using Lemma 4 yields

$$\begin{aligned}
P(N_0(n) = m, S_n = -j) \\
&= \sum_{r} P(N_0(n) = m, S_n = -j | T_0(m) = r) P(T_0(m) = r) \\
&= \sum_{r} P(S_k < 0, k = 1, \ldots, n - r - 1, S_{n-r} = -j) p(m, r) \\
&= \sum_{r} P(T_j = n - r) p(m, r) \qquad \text{(by Lemma 4)} \\
&= P(T_j(m) = n).
\end{aligned}$$

Case 2: $i = j > 0$: In this case, it follows by definition that

$$P(N_i(n) = m,\ S_n = -j) = P(T_j(m) = n),$$

which completes the proof of Theorem 1. ■

COROLLARY 4: *For* $0 \leq i \leq j$,

$$P(N_i(n) = 1 | S_n = j) = j/n.$$

PROOF: The proof follows from Theorem 1 upon applying the HTT. ■

The next result generalizes a well-known result for the symmetric random walk.

THEOREM 2: *Let N_j denote the number of visits to state $-j$ prior to returning to zero. (If the walk never returns to zero, then N_j is just the number of visits to $-j$.) Then, for $j > 0$.*

$$E[N_j] = 1 \quad \text{if } E[X] \leq 0,$$

$$E[N_j] = c^j \quad \text{if } E[X] > 0,$$

where c is such that $0 < c < 1$ and $E[c^X] = 1$.

PROOF: It follows from Lemma 4 that

$$E[N] = \sum_{n=1}^{\infty} P(T_j = n) = P(T_j < \infty)$$

and the result follows Lemma 2. ■

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