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FAIR GAMBLER'S RUIN STOCHASTICALLY MAXIMIZES PLAYING TIME

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Abstract

For the Gambler's Ruin problem with two players starting with the same amount of money, we show the playing time is stochastically maximized when the games are fair.

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1. Introduction

For a simple random walk $S_n = \sum_{i=1}^{n} X_i$, where each X_i is 1 with probability p and -1 with probability $1-p$, we are interested in $T(p)$, the time until the random walk hits either k or $-k$, where k is a specified positive integer. It should be noted that $T(p)$ is the duration of the Gambler's Ruin when both players start with k. It was recently shown, by a direct computation, in [11] that $E[T(p)]$ is maximized when $p = 1/2$. In this paper we prove the stronger result that $T(p)$ is stochastically maximized when $p = 1/2$.

The Gambler's Ruin problem is one of the oldest problems in probability. As told by [10], computing the chances each player wins was solved by Pascal and Fermat and

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first published with four other problems in [4], the first book published on probability theory. Study of the distribution of the duration of the game started with [7] and [8] and extended into modern times with [2], [5], [6] and the references therein. Applications of the problem are found in many modern settings, for example, in computer science ([3]) and business ([1]).

2. Main result

In this section we state and prove our main result.

Theorem 1. *For any* $0 < p < 1$ *we have* $T(p) \leq_{st} T(1/2)$

To prove the Theorem we need some lemmas. Let $T_0(p)$ the first time (after time 0) the walk revisits state 0, and let

$$A_p = \{T_0(p) < T(p)\}$$

be the event the walk revisits 0, starting at 0, before the game ends. We can write

$$T(p) = Z(p) + \sum_{i=1}^{N(p)-1} Y_i(p)$$

where

$$Z(p) \sim (T(p)|A_p^c), \qquad Y_i(p) \sim (T_0(p)|A_p), \qquad N(p) \sim \text{Geometric}(P(A_p^c))$$

are all independent variables. We will show the following lemma, from which Theorem 1 follows immediately.

Lemma 1. *For any* $0 < p < 1$ *we have* $Y_i(p) \leq_{st} Y_i(1/2), Z(p) \leq_{st} Z(1/2), N(p) \leq_{st} N(1/2)$.

To prove this lemma, let $u_i(p)$ be the chance the walk goes up next when it is at level i given it returns to 0 before the game ends, and let $v_i(p)$ be the chance the walk goes up next when it is at level i given the game ends before the walk returns to zero.

Lemma 2. *For* $0 < p < 1$ *and* $1 \leq i \leq k-2$ *we have* $u_i(p) \leq u_i(\frac{1}{2})$, $v_i(p) \geq v_i(\frac{1}{2})$ *and* $P(A_p^c) \geq P(A_{1/2}^c)$.

Proof of Lemma 2. Letting

$$P_i(p) = \begin{cases} \frac{1-\left(\frac{1-p}{p}\right)^i}{1-\left(\frac{1-p}{p}\right)^k} & p \neq \frac{1}{2} \\ i/k & p = \frac{1}{2} \end{cases}$$

from [9][(4.14), p. 235] be the chance a gambler reaches k before 0 starting with i, we have

$$u_i(p) = \frac{p(1 - P_{i+1}(p))}{1 - P_i(p)}.$$

Making the substitution $n = k - i$ we have

$$u_i(1/2) = \frac{n-1}{2n}$$

and for $p \neq 1/2$ when we make the substitution $x = (1-p)/p$ we have

$$1 - P_i(p) = 1 - \frac{1 - x^i}{1 - x^k} = \frac{x^i - x^k}{1 - x^k}$$

and $p = 1/(1+x)$ and we get

$$u_i(p) = \frac{1}{1+x}\frac{x^{i+1} - x^k}{x^i - x^k} = \frac{x^n - x}{(1+x)(x^n - 1)}.$$

We will thus have $u_i(p) \leq u_i(1/2)$ whenever

$$f(x) \equiv x^n - x - \frac{n-1}{2n}(x+1)(x^n - 1) \begin{cases} \leq 0 & \text{if } x > 1 \\ \geq 0 & \text{if } x < 1 \end{cases} \tag{1}$$

Since

$$f'(x) = -\frac{(n+1)(x - x^{n+1} + n(x-1)x^n)}{2nx}$$

implies $f'(1) = 0$ and

$$f''(x) = \frac{1}{2}(n-1)(n+1)(1-x)x^{n-2} \begin{cases} < 0 & \text{if } x > 1 \\ > 0 & \text{if } x < 1 \end{cases}$$

this means $f(x)$ is monotone decreasing with $f(1) = 0$ and therefore (1) holds, implying $u_i(p) \leq u_i(1/2)$.

Next, looking at the walk reflected across the origin gives $v_i(p) = 1 - u_{k-i}(1-p)$ and thus by the reasoning above $v_i(p) \geq v_i(1/2)$.

Finally, we have

$$P(A_p^c) = pP_1(p) + (1-p)P_1(1-p)$$

and thus

$$P(A^c_{1/2}) = 1/k,$$

and for $p \neq 1/2$ making the substitution $x = (1-p)/p$ gives

$$P(A^c_p) = \frac{1}{1+x}\frac{1-x}{1-x^k} + \frac{x}{1+x}\frac{1-x^{-1}}{1-x^{-k}} = \frac{(x-1)(x^k+1)}{(x+1)(x^k-1)}$$

and we will have $P(A^c_p) \geq 1/k$ whenever

$$f(x) \equiv k(x-1)(x^k+1) - (x+1)(x^k-1) \begin{cases} \geq 0 & \text{if } x > 1 \\ \leq 0 & \text{if } x < 1 \end{cases} \tag{2}$$

Since

$$f'(x) = -\frac{(k+1)((k(x-1)-x)x^k + x)}{x}$$

implies $f'(1) = 0$ and

$$f''(x) = k(k-1)(k+1)(x-1)x^{k-2} \begin{cases} > 0 & \text{if } x > 1 \\ < 0 & \text{if } x < 1 \end{cases}$$

this means $f(x)$ is monotone increasing with $f(1) = 0$ and therefore (2) holds, implying $P(A^c_p) \geq P(A^c_{1/2})$.

□

Proof of Lemma 1. We first construct two coupled random walks, both conditional on a revisit to the origin before the game ends, but where the first walk uses any $0 < p < 1$ and the second uses $p = 1/2$. We will let $Y(p)$ and $Y(1/2)$ denote the first time the origin is revisited respectively for the two conditioned walks. We let the first walk take the first step and, since the second walk is symmetric and has the same length distribution regardless of its first step, we let its first step be the same as for the first walk. Suppose both the walks first step above the origin. Whenever the walks are at different levels, we let them step independently. When the walks are at the same level, by Lemma 2 the chance of a step away from the origin is at least as large for the second walk as it is for the first walk, so we can couple the steps so the second walk ends up always no closer to the origin than the first walk. This means in this case we will have $Y(p) \leq Y(1/2)$ almost surely. In the event that the walks first step below the origin, the same reasoning applies because the distance to the origin for the first walk is the same as if it was above the origin but with $1 - p$ substituted for p. This shows $Y(p) \leq_{st} Y(1/2)$.

A similar construction using Lemma 2 gives $Z(p) \leq_{st} Z(1/2)$, and $N(p) \leq_{st} N(1/2)$ follows from Lemma 2 since $P(A_p^c) \geq P(A_{1/2}^c)$. $\square$

Proof of Theorem 1. By Lemma 1, we can couple $T(1/2)$ together with $T(p)$ on the same probability space so that $T(1/2) \geq T(p)$ almost surely, and the Theorem follows. $\square$

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