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ON THE NUMBER OF REFUSALS IN A BUSY PERIOD

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Formulas are derived for moments of the number of refused customers in a busy period for the $M/GI/1/n$ and the $GI/M/1/n$ queueing systems. As an interesting special case for the $M/GI/1/n$ system, we note that the mean number is 1 when the mean interarrival time equals the mean service time. This provides a more direct argument for a result given in Abramov [1].

1. INTRODUCTION

In a recent paper, Abramov [1] studies properties of L_n, the number of lost customers during a busy period for an $M/GI/1/n$ queueing system. This system has Poisson arrivals with rate λ, an arbitrary service distribution X with mean $1/\mu$, and a waiting space for n customers. Customers which arrive to find system full are refused and are lost. The following interesting result is given there: When $\lambda/\mu = 1$, then $E[L_n] = 1$ for all n. In this note, we provide a slightly more direct argument for this fact, which has the advantage that it applies to higher moments of L_n as well as to the $GI/M/1/n$ system. As a special case, we show that, for the $M/GI/1/n$ system, when $\lambda/\mu = 1$, then $Var(L_n)$ grows linearly in n. For the $GI/M/1/n$, if the service rate λ satisfies $\phi(-\lambda) = \frac{1}{2}$ for the arrival distribution moment generating function $\phi(t) = E[e^{tX}]$, we show that $E[L_n] = 1$ for all n and give a recursion for $Var(L_n)$. We also give results for both systems in the case where $\lambda \neq \mu$.

2. RESULTS FOR THE $GI/M/1/n$ SYSTEM

Let $p = E[e^{-\lambda X}]$ be the chance that the first service of the busy period is completed after the second arrival. The following recursion holds for $\Phi_n(t) = E[e^{tL_n}]$.

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Proposition 2.1:

$$\Phi_n(t) = \frac{1-p}{1-p\Phi_{n-1}(t)}.$$

PROOF: In the case where the first service of the busy period finishes before the second arrival, $L_n = 0$. Otherwise, an arrival occurs before the service is completed and the refusals during the busy period can be divided into two categories: the refusals which occur before the first time there is only a single customer remaining in the system and the refusals which occur after that time. Due to the Markov service distribution, the number of refusals in the first category has the same distribution as the number of refusals in a busy period for a system with one less waiting space, or L_{n-1}. Since the state of the system at the service completion epochs is a Markov chain, the number in the second category has the same distribution as L_n and the counts in the two categories are independent. Thus, in this case, L_n has the distribution of $L_n * L_{n-1}$ where $*$ denotes convolution, $\Phi_n(t) = p\Phi_{n-1}(t)\Phi_n(t) + 1 - p$, and the proposition follows. ■

Taking the first and second derivatives with respect to t gives

$$\Phi_n'(t) = \frac{(1-p)p\Phi_{n-1}'(t)}{(1-p\Phi_{n-1}(t))^2}$$

and

$$\Phi_n''(t) = \frac{(1-p)p\Phi_{n-1}''(t)}{(1-p\Phi_{n-1}(t))^2} + \frac{2(1-p)(p\Phi_{n-1}'(t))^2}{(1-p\Phi_{n-1}(t))^3}.$$

When evaluated at $t = 0$ these give

$$E[L_n] = \frac{p}{1-p}E[L_{n-1}]$$

and

$$E[L_n^2] = \frac{p}{1-p}E[L_{n-1}^2] + 2\left(\frac{p}{1-p}E[L_{n-1}]\right)^2.$$

We can then use the boundary conditions $E[L_0] = p/(1-p)$ and $E[L_0^2] = p(1+p)/(1-p)^2$, which follows because $L_0 + 1$ has a geometric distribution with parameter $1-p$. This gives

$$E[L_n] = \left(\frac{p}{1-p}\right)^{n+1}$$

and

$$E[L_n^2] = \left(\frac{p}{1-p}\right)^n \left(2\sum_{i=3}^{n+2}(p/(1-p))^i + p(1+p)/(1-p)^2\right).$$

In the case where $p = 1/2$, these give

$$E[L_n] = 1$$

and

$$Var(L_n) = 3 + 2n.$$

3. RESULTS FOR THE *M/GI/1/n* SYSTEM

We assume Poisson arrivals with rate λ, and a service distribution having c.d.f. $G(t)$, mean $1/\mu$, and variance σ^2. Let Y be the number of arrivals during the first service of a busy period. Note that $P(Y = i) = p_i$ where

$$p_i = \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^i}{i!}\, dG(t).$$

The following recursion holds.

Proposition 3.1: Defining L_i for $i < 0$ to be equal to 1, we have

$$E[L_n] = \frac{1}{p_0} \sum_{j=2}^{\infty} P(Y \geq j) E[L_{n-j+1}], \tag{1}$$

and in the case where $\lambda/\mu = 1$ we have

$$E[L_n] = 1 \tag{2}$$

and

$$E[L_n^2] = \frac{1}{p_0} \sum_{j=2}^{\infty} P(Y \geq j) E[L_{n-j+1}^2] + Var(Y). \tag{3}$$

PROOF: On the event that the first service finishes before the second arrival, $L_n = 0$. And on the event that there are i arrivals during this first service, L_n has the distribution of $L_n * L_{n-1} * \cdots * L_{n-i+1}$ where $*$ denotes convolution, and where the variables L_i for $i < 0$ are defined to be deterministically equal to 1. Letting $\Phi_n(t) = E[e^{tL_n}]$, we have

$$\Phi_n(t) = \sum_{i=1}^{\infty} p_i \prod_{j=1}^{i} \Phi_{n-j+1}(t),$$

and thus

$$\Phi_n'(t) = \sum_{i=1}^{\infty} p_i \sum_{j=1}^{i} \frac{\Phi_{n-j+1}'(t)}{\Phi_{n-j+1}(t)} \prod_{k=1}^{i} \Phi_{n-k+1}(t)$$

and

$$\Phi_n''(t) = \sum_{i=1}^{\infty} p_i \sum_{j=1}^{i} \left(\frac{\Phi_{n-j+1}''(t)}{\Phi_{n-j+1}(t)} + \sum_{k:1\leq k\leq i, k\neq j} \frac{\Phi_{n-j+1}'(t)\Phi_{n-k+1}'(t)}{\Phi_{n-j+1}(t)\Phi_{n-k+1}(t)} \right) \prod_{k=1}^{i} \Phi_{n-k+1}(t).$$

When the first expression is evaluated at $t = 0$ we get

$$E[L_n] = \sum_{i=1}^{\infty} p_i E\left[\sum_{j=1}^{i} L_{n-j+1}\right]$$
$$= \sum_{j=1}^{\infty} P(Y \geq j) E[L_{n-j+1}],$$

and the first recursion, Eq. (1), of the proposition follows. Evaluating the second expression at $t = 0$ gives

$$E[L_n^2] = \sum_{i=1}^{\infty} p_i \sum_{j=1}^{i} \left(E[L_{n-j+1}^2] + \sum_{k:1 \leq k \leq i, k \neq j} E[L_{n-j+1}] E[L_{n-k+1}] \right). \qquad \textbf{(4)}$$

When $\lambda/\mu = 1$, we easily obtain, by conditioning on the service time, that $E[Y] = E[L_0] = 1$. To establish that in this case $E[L_n] = 1$ we use induction and assume that for $i < n$, $E[L_i] = 1$. The first recursion, Eq. (1), of the proposition above thus gives

$$E[L_n] = \frac{1}{p_0} \sum_{j=2}^{\infty} P(Y \geq j) = \frac{1 - P(Y \geq 1)}{p_0} = 1,$$

and establishes Eq. (2). Using this and Eq. (4), we then get

$$E[L_n^2] = \sum_{i=1}^{\infty} p_i \left(\sum_{j=1}^{i} E[L_{n-j+1}^2] + i^2 - i \right)$$
$$= \sum_{j=1}^{\infty} P(Y \geq j) E[L_{n-j+1}^2] + Var(Y),$$

and the second recursion, Eq. (3), follows. ■

Reference

1. Abramov, V. (1997). On a property of a refusal stream. *Journal of Applied Probability* 34: 800–805.